Constrained Equations for Completion and the Like

Progress Report

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Seminar 3
Computational Logic Group

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Knuth-Bendix Completion

$$\underset{\text{reduction ordering}}{\succ} + \underset{\text{equations}}{\mathcal{E}} \longrightarrow_{KB} \underset{\text{rewrite system}}{\mathcal{R}}$$

$\mathcal{R}$ is confluent, terminating, reduced and $\approx_{\mathcal{E}} = \leftrightarrow^*_{\mathcal{R}}$

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Example (Group Theory)

$$\mathcal{E} \qquad \begin{aligned} \mathsf{e} \cdot x &\approx x \\ x^- \cdot x &\approx \mathsf{e} \\ (x \cdot y) \cdot z &\approx x \cdot (y \cdot z) \end{aligned}$$

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$$\mathcal{R} \quad \begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

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Example (Group Theory)

$$(x^{-} \cdot x)^{-} \cdot (\mathsf{e} \cdot (y \cdot \mathsf{e})) \overset{?}{\approx} y \cdot \mathsf{e}$$

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$$\mathcal{R} \qquad \begin{aligned} \mathsf{e} \cdot x &\to x \\ x^{-} \cdot x &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) \\ \mathsf{e}^{-} &\to \mathsf{e} \\ x^{-} \cdot (x \cdot y) &\to y \end{aligned} \qquad \begin{aligned} x \cdot \mathsf{e} &\to x \\ x \cdot x^{-} &\to \mathsf{e} \\ x^{--} &\to x \\ (x \cdot y)^{-} &\to y^{-} \cdot x^{-} \\ x \cdot (x^{-} \cdot y) &\to y \end{aligned}$$

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$$y \leftarrow^{!}_{\mathcal{R}} (x^- \cdot x)^- \cdot (\mathsf{e} \cdot (y \cdot \mathsf{e})) \approx y \cdot \mathsf{e} \rightarrow^{!}_{\mathcal{R}} y$$

$\mathcal{E}$

$$\mathsf{e} \cdot x \approx x$$
$$x^- \cdot x \approx \mathsf{e}$$
$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$

$\mathcal{R}$

$$\mathsf{e} \cdot x \to x$$
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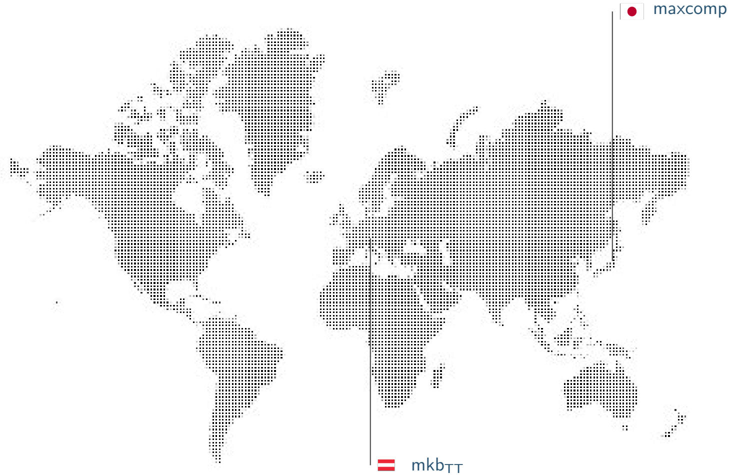
Comparison of Completion Tools



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Aim 1: Combined Approach

mkb_{TT}	maxcomp
uses selection heuristic to advance one branch	can advance several branches at once

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Aim 2: Extensions

$\mathsf{mkb_{TT}}$ approach was extended to ordered completion, AC-completion – how about maxcomp?

Outline

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Definition (termination constraint)

$$C ::= \ell \to r \mid \top \mid \bot \mid \neg C \mid C \vee C \mid C \wedge C$$

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$\mathcal{R} \models \ell \to r$ iff $\ell \to r \in \mathcal{R}$	$\mathcal{R} \models \top$
$\mathcal{R} \not\models \bot$	$\mathcal{R} \models C_1 \vee C_2$ iff $\mathcal{R} \models C_1$ or $\mathcal{R} \models C_2$
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Definition (constrained equalities)

- constrained equality $(s \approx t, C)$ is pair of equality $s \approx t$ and termination constraint C
- constrained equation system (CES) $\mathbb{C}$ is set of constrained equalities

Notation

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$$\mathbb{C} = \{(1, \mathsf{s}(\mathsf{p}(x)) \to x), (2, \mathsf{s}(x) + y \to \mathsf{s}(x + y)), (3, \mathsf{s}(x + y) \to \mathsf{s}(x) + y)\}$$

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$$\mathbb{C} = \{(1,1),(2,3),(3,3')\}$$

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$$\mathcal{E}^\top \ominus \mathcal{R} = \{(1, \neg(1 \wedge 3')), (2, \neg(1 \wedge 3')), (3, \neg(1 \wedge 3'))\}$$

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mapping S from CESs to CESs is (ground) reduction if $\forall$ CES $\mathbb{C}$, TRS $\mathcal{R}$

$$S(\mathbb{C})[\![\mathcal{R}]\!] \text{ (ground) } \mathcal{R}\text{-joinable} \Longrightarrow \mathbb{C}[\![\mathcal{R}]\!] \text{ (ground) } \mathcal{R}\text{-joinable}$$

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Lemma

S is a (ground) reduction.

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Theorem

If $\mathbb{C} = S^{n}_{KB}(\mathcal{E}^{\top})$ *and* $\mathbb{C}[\![\mathcal{R}]\!] = \varnothing$ *for* $\mathcal{R} \in \mathfrak{R}(\mathbb{C})$ *then* $\mathcal{R}$ *is convergent for* $\mathcal{E}$*.*

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Proof.

- $n = 0$: if $\mathbb{C}[\![\mathcal{R}]\!] = \varnothing$ then $\mathcal{E} = \varnothing$, thus $\mathcal{R} = \varnothing$

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$\mathbb{C}_3[\![\mathcal{R}_3]\!] = \varnothing$, so $\mathcal{R}_3$ convergent for $\mathcal{E}$

Ordered Completion

Definition

- $\mathcal{R}$ is ground convergent if $\mathcal{R}$ is terminating and for all ground terms $s \leftrightarrow_{\mathcal{R}}^* t$ there is some v such that $s \rightarrow_{\mathcal{R}}^* v \leftarrow_{\mathcal{R}}^* t$

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set of $\succ$-oriented ground instances of $\mathcal{E}$

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Outline

Inductionless Induction

Definition (inductive theory)

- $\mathcal{R}_0 \vdash_i s \approx t$ if $s\sigma \leftrightarrow^*_{\mathcal{R}_0} t\sigma$ for all ground substitutions σ

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Inductionless Induction

Definition (inductive theory)

- $\mathcal{R}_0 \vdash_i s \approx t$ if $s\sigma \leftrightarrow^*_{\mathcal{R}_0} t\sigma$ for all ground substitutions σ
- $\mathcal{R}_0 \vdash_i \mathcal{H}$ if $\mathcal{R}_0 \vdash_i s \approx t$ for all $s \approx t$ in $\mathcal{H}$

Example

For $\mathcal{R}_0$ being

$$0 - x \to x \qquad \mathsf{s}(x) - 0 \to \mathsf{s}(x) \qquad \mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$\mathsf{p}(0) \to 0 \qquad \mathsf{p}(\mathsf{s}(x)) \to x$$

we have $\mathcal{R}_0 \vdash_i \mathsf{p}(x - y) \approx \mathsf{p}(x) - y$

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we have $\mathcal{R}_0 \vdash_i \mathsf{p}(x - y) \approx \mathsf{p}(x) - y$ (but not $\mathsf{p}(x - y) \leftrightarrow^*_{\mathcal{R}_0} \mathsf{p}(x) - y$)

Definition

- term t is $\mathcal{R}_0$-inductively reducible if
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Lemma (Gramlich 90)

If $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{H}$ is terminating and left-$\mathcal{R}_0$-inductively reducible TRS and $\mathrm{CP}(\mathcal{R}_0, \mathcal{H}) \subseteq \downarrow_{\mathcal{R}}$ *then* $\mathcal{R}_0 \vdash_i \mathcal{H}$

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For fixed $\mathcal{R}_0$ and $\mathcal{E}$

$$S_I(\mathbb{C}) = (\mathbb{C} \ominus \mathcal{R}) \ \cup \ \mathbb{C}{\downarrow_{\mathcal{R}}} \ \cup \ F(\mathcal{R})^{\top}{\downarrow_{\mathcal{R}}}$$

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Theorem

If $\mathbb{C} = S_I^n(\mathcal{E}^{\top})$ *and* $\mathbb{C}[\![\mathcal{R}]\!] = \varnothing$ *for* $\mathcal{R} \in \mathfrak{R}(\mathbb{C})$ *then* $\mathcal{R}_0 \vdash_i \mathcal{E}$

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$\mathbb{C}_2[\![\mathcal{R}_2]\!] = \varnothing$, so $\mathcal{R}_0 \vdash_i \mathsf{p}(x - y) \approx \mathsf{p}(x) - y$

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Rewriting Induction

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Given TRS $\mathcal{R}_0$,

- defined symbols $\mathcal{D} = \{f \mid f \text{ is root symbol of } \ell \text{ for } \ell \to r \in \mathcal{R}_0\}$

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- $\mathsf{Expd}(\mathcal{R}_0, \mathcal{R})$ is set of CPs from overlaps $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu$ where $\ell_1 \to r_1 \in \mathcal{R}_0$, $\ell_2 \to r_2 \in \mathcal{R}$, and p is basic in ℓ_2

Definition

For fixed $\mathcal{R}_0$ and $\mathcal{E}$

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If $\mathbb{C} = S^n_{RI}(\mathcal{E}^{\top})$ *and* $\mathbb{C}[\![\mathcal{R}]\!] = \varnothing$ *for* $\mathcal{R} \in \mathfrak{R}(\mathbb{C})$ *then* $\mathcal{R}_0 \vdash_i \mathcal{E}$

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Example

$\mathcal{R}_0$: 1 : $x + 0 \to x$ 2 : $x + \mathsf{s}(y) \to \mathsf{s}(x + y)$

$\mathcal{E}$: 3 : $(x + y) + z \approx x + (y + z)$

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$\mathbb{C}_2[\![\mathcal{R}_2]\!] = \varnothing$, so $\mathcal{R}_0 \vdash_i (x + y) + z \approx x + (y + z)$

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Outline

Completion

Definition

$$S_{\mathcal{R}}(\mathbb{C}) = (\mathbb{C} \ominus \mathcal{R}) \ \cup \ \mathbb{C}{\downarrow_{\mathcal{R}}} \ \cup \ F(\mathcal{R})^{\top}{\downarrow_{\mathcal{R}}}$$

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$$\mathbb{C}_0 = \mathcal{E}^{\top} \qquad \mathbb{C}_1 = S_{\mathcal{R}_1}(\mathbb{C}_0) \qquad \mathbb{C}_2 = S_{\mathcal{R}_2}(\mathbb{C}_1) \qquad \mathbb{C}_3 = S_{\mathcal{R}_3}(\mathbb{C}_2) \qquad \ldots$$

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Procedure

how to find $\mathcal{R}_1, \mathcal{R}_2.\mathcal{R}_3, \ldots$?

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Maximal Completion Approach

$$\bigvee_{(s \approx t, C) \in \mathbb{C}_k} (\neg C \vee \ulcorner s > t \urcorner \vee \ulcorner t > s \urcorner)$$

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SAT/SMT encoding of $>_{\mathsf{kbo}}$, $>_{\mathsf{lpo}}$ or $>_{\mathsf{mpo}}$

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maximize $\displaystyle \bigvee_{(s \approx t, C) \in \mathbb{C}_k} (\neg C \vee \ulcorner s > t \urcorner \vee \ulcorner t > s \urcorner)$

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to obtain assignment α

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and let $\mathcal{R}_k = \{s \to t \mid (s \simeq t, C) \in \mathbb{C}_k \text{ and } \alpha \models \ulcorner s > t \urcorner\}$

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is maximization appropriate?

maximize $\displaystyle\bigvee_{(s \approx t, C) \in \mathbb{C}_k} (\neg C \vee \ulcorner s > t \urcorner \vee \ulcorner t > s \urcorner)$ *subject to* $\displaystyle\bigwedge_{i=1}^{k} \neg \bigwedge \mathcal{R}_i$

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Preliminary Results

Completion

115 systems in $\mathrm{mkb}_{\mathrm{TT}}$ distribution

	LPO	
	Maxcomp	Constraints
completed	86	51
failure	6	0
timeout	23	64

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Rewriting Induction

103 systems from *Dream Corpus of Inductive Conjectures*

	LPO
success	30
timeout	73

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- constrained equation framework adds rewriting to maximal completion approach
- constrained equation framework allows for simple correctness proofs
- maximal completion was extended to ordered completion and inductive theorem proving

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Further Work

- cover AC/normalized completion
- implement approach for ordered and AC completion
- automation of theorem proving: what to maximize?
- can completeness be expressed in framework?