

The SPASS Handbook

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Chapter 1

Foundations

A *multiset* over a set A is a function M from A to the natural numbers. Intuitively, $M(a)$ specifies the number of occurrences of a in M . We say that a is an element of M if $M(a) > 0$. The union, intersection, and difference of multisets are defined by the identities $(M_1 \cup M_2)(x) = M_1(x) + M_2(x)$, $(M_1 \cap M_2)(x) = \min(M_1(x), M_2(x))$, and $(M_1 \setminus M_2)(x) = \max(0, M_1(x) - M_2(x))$. We use a set-like notation to describe multisets.

A first-order language is constructed over a signature $\Sigma = (\mathcal{F}, \mathcal{R})$, where \mathcal{F} and \mathcal{R} are non-empty, disjoint, in general infinite sets of function and predicate symbols, respectively. Every function or predicate symbol has some fixed arity. Function and predicate symbols with arity one are called *monadic*. In addition to these sets that are specific for a first-order language, we assume a further, infinite set \mathcal{X} of variable symbols disjoint from the symbols in Σ . Then the set of all *terms* $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is recursively defined by: (i) every function symbol $c \in \mathcal{F}$ with arity zero (a *constant*) is a term, (ii) every variable $x \in \mathcal{X}$ is a term and (iii) whenever t_1, \dots, t_n are terms and $f \in \mathcal{F}$ is a function symbol with arity n , then $f(t_1, \dots, t_n)$ is a term. A term not containing a variable is a *ground term*. If t_1, \dots, t_n are terms and $R \in \mathcal{R}$ is a predicate symbol with arity n , then $R(t_1, \dots, t_n)$ is an *atom*. An atom or the negation of an atom is called a *literal*. Disjunctions of literals are *clauses* where all variables are implicitly universally quantified. Clauses are often denoted by their respective multisets of literals where we write multisets in usual set notation. A clause consisting of exactly one literal is called a *unit*.

The set of *free* variables of an atom (term) ϕ denoted by $\text{vars}(\phi)$ is defined as follows: $\text{vars}(P(t_1, \dots, t_n)) = \cup_i \text{vars}(t_i)$ and $\text{vars}(f(t_1, \dots, t_n)) = \cup_i \text{vars}(t_i)$, $\text{vars}(x) = \{x\}$. The function naturally extends to literals, clauses and (multi)sets of terms (literals, clauses).

A *substitution* σ is a mapping from the set of variables to the set of terms such that $x\sigma \neq x$ for only finitely many $x \in \mathcal{X}$. We define the *domain* of σ to be $\text{dom}(\sigma) = \{x \mid x\sigma \neq x\}$ and the co-domain of σ to be $\text{cdom}(\sigma) = \{x\sigma \mid x\sigma \neq x\}$. Hence, we can denote a substitution σ by the finite set $\{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$ where $x_i\sigma = t_i$ and $\text{dom}(\sigma) = \{x_1, \dots, x_n\}$. A *ground substitution* σ has no variable occurrences in its co-domain, $\text{vars}(\text{cdom}(\sigma)) = \emptyset$. An injective substitution σ where $\text{cdom}(\sigma) \subset \mathcal{X}$ is called a *variable renaming*. The application of substitutions to terms is given by $f(t_1, \dots, t_n)\sigma = f(t_1\sigma, \dots, t_n\sigma)$ for all $f \in \mathcal{F}$ with arity n . We extend the application of substitutions to literals and clauses as usual: $P(t_1, \dots, t_n)\sigma = P(t_1\sigma, \dots, t_n\sigma)$ (accordingly for literals) and $\{L_1, \dots, L_n\}\sigma = \{L_1\sigma, \dots, L_n\sigma\}$.

Given two terms (atoms) s, t , a substitution σ is called a *unifier* for s and t if $s\sigma = t\sigma$. It is called a *most general unifier* (mgu) if for any other unifier τ of s, t there exists a substitution λ with $\sigma\lambda = \tau$. A substitution σ is called a *matcher* from s to t if $s\sigma = t$. The notion of a mgu is extended to atoms, literals in the obvious way. We say that σ is a unifier for a sequence of terms (atoms, literals) t_1, \dots, t_n if $t_i\sigma = t_j\sigma$ for all $1 \leq i, j \leq n$ and σ is a mgu if in addition for any other unifier τ of t_1, \dots, t_n , there exists a substitution λ with $\sigma\lambda = \tau$.

A *position* is a word over the natural numbers. The set $\text{pos}(f(t_1, \dots, t_n))$ of positions of a given term $f(t_1, \dots, t_n)$ is defined as follows: (i) the empty word ϵ is a position in any term t and $t|_\epsilon = t$, (ii) if $t|_\pi = f(t_1, \dots, t_n)$, then $\pi.i$ is a position in t for all $i = 1, \dots, n$, and $t|_{\pi.i} = t_i$. We write $t[s]_\pi$ for $t|_\pi = s$. With $t[\pi/s]$, where $\pi \in \text{pos}(t)$, we denote the term (atom) obtained by replacing $t|_\pi$ by s at position π in

t . The *length* of a position π is defined by $length(\epsilon) = 0$ and $length(i.\tau) = 1 + length(\tau)$. The notion of a position can be extended to atoms, literals and even formulae in the obvious way.

As an alternative to the already mentioned multiset notation of clauses, we also write clauses in the form $\Theta \parallel \Gamma \rightarrow \Delta$ where Θ is a multiset of monadic atoms¹ and Γ, Δ are multisets containing arbitrary atoms. Logically, the atoms in Θ and Γ denote negative literals while the atoms in Δ denote the positive literals in the clause. The empty clause \square denotes \perp (falsity). The multiset Θ is called the *sort constraint* of $\Theta \parallel \Gamma \rightarrow \Delta$. A sort constraint Θ is *solved* in a clause $\Theta \parallel \Gamma \rightarrow \Delta$ if it does not contain non-variable terms and $vars(\Theta) \subseteq vars(\Gamma \cup \Delta)$. If the clause is determined by the context, we simply say that a sort constraint is solved. In case we are not interested in a separation of the negative literals in a clause, we write clauses in the form $\Gamma \rightarrow \Delta$. We often abbreviate disjoint set union with sequencing, e.g., we write $\Theta \parallel \Gamma \rightarrow \Delta, R(t_1, \dots, t_n)$ for $\Theta \parallel \Gamma \rightarrow \Delta \cup \{R(t_1, \dots, t_n)\}$. Equality atoms are written $l \approx r$ and are mostly distinguished from non-equality atoms. The latter are named A, B . In case we don't want to distinguish these two different kinds of atoms we use the letter E (possibly indexed) to denote an arbitrary atom. Inferences and reductions where equations are involved are applied with respect to the symmetry of \approx .

A clause $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1$ *subsumes* a clause $\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2$ if $\Theta_1 \sigma \subseteq \Theta_2, \Gamma_1 \sigma \subseteq \Gamma_2$ and $\Delta_1 \sigma \subseteq \Delta_2$ for some matcher σ . The relation "is subsumed by" between clauses is a quasi-ordering on clauses. Please recall that we consider clauses to be multisets. Hence, e.g., the clause $\{P(x), P(y)\}$ (also possibly written $\rightarrow P(x), P(y)$) does not subsume the clause $\{P(x)\}$ (possibly written $\rightarrow P(x)$).

The function *size* maps terms, atoms, literals to the number of symbols they are built from, e.g., $size(t) = |pos(t)|$. In case of a literal, we don't consider the negation symbol for its size. The *depth* of a term, literal is the maximal length of a position in the term, literal, e.g., $depth(t) = \max(\{length(\pi) \mid \pi \in pos(t)\})$. The depth of a clause is the maximal depth of its literals. The size of clause is the sum of its literal sizes.

For the definition of our inference/reduction rules we shall often need the notion of an ordering to compare terms. This notion is then lifted to tuples, sets, clauses and (multi)sets of clauses. A *partial order* is a reflexive, transitive and antisymmetric relation. A *strict order* is a transitive and irreflexive relation. Every partial order \succeq induces a strict order \succ by $t \succ s$ iff $t \succeq s$ and $t \neq s$. The lexicographic extension \succ^{lex} on tuples of some strict order \succ is defined by $(t_1, \dots, t_n) \succ^{lex} (s_1, \dots, s_n)$ if for some $1 \leq i \leq n$ we have $t_i \succ s_i$ and for all $1 \leq j < i$ it is the case that $t_j = s_j$. The multiset extension \succ^{mul} is defined by $M \succ^{mul} N$ if $N \neq M$ and for all $n \in N \setminus M$ there exists an $m \in M \setminus N$ with $m \succ n$. A *reduction ordering* \succ is a well-founded, transitive relation satisfying for all terms t, s, l , positions $p \in pos(l)$ and substitutions σ that whenever $s \succ t$ then $l[p/s\sigma] \succ l[p/t\sigma]$. For the purpose of this article, we are mainly interested in reduction orderings that are total on ground terms, possibly up to some congruence on the ground terms. Any (reduction) ordering \succ on terms (atoms) can be extended to clauses in the following way. We consider clauses as multisets of *occurrences* of equations and atoms. The occurrence of an equation $s \approx t$ in the antecedent is identified with the multiset $\{\{s, t\}\}$, the occurrence of an atom A in the antecedent is identified with the multiset $\{\{A, \top\}\}$, the occurrence of an equation in the succedent is identified with the multiset $\{\{s\}, \{t\}\}$ and the occurrence of an atom in the succedent is identified with the multiset $\{\{A\}, \{\top\}\}$. We always assume that \top is the minimal constant with respect to \succ . Now we overload \succ on literal occurrences to be the twofold multiset extension of \succ on terms (atoms) and \succ on clauses to be the multiset extension of \succ on literal occurrences. If \succ is well-founded (total) on terms (atoms), so are the multiset extensions on literals and clauses.

Observe that an occurrence of an equation $s \approx t$ (an atom) in the antecedent is strictly bigger than an occurrence of $s \approx t$ in the succedent. The atoms in the sort constraint will not be subject to ordering restrictions but will be processed by specific inference/reduction rules.

An antecedent or succedent occurrence of an equation $s \approx t$ (an atom A) is *maximal* in a clause $\Theta \parallel \Lambda \rightarrow \Pi$ if there is no occurrence of an equation or atom in $\Lambda \rightarrow \Pi$ that is strictly greater than the occurrence $s \approx t$ (the atom A) with respect to \succ . An antecedent or succedent occurrence of an equation $s \approx t$ is *strictly maximal* in a clause $\Theta \parallel \Lambda \rightarrow \Pi$ if there is no occurrence of an equation in $\Lambda \rightarrow \Pi$ that is greater or equal than the occurrence $s \approx t$ with respect to \succ . A clause $\Theta \parallel \Lambda \rightarrow \Pi, s \approx t$ (clause $\Theta \parallel \Lambda \rightarrow \Pi, A$) is *reductive* for the equation $s \approx t$ (the atom A), if $s \approx t$ (the atom A) is a strictly maximal

¹These are atoms with a monadic predicate as their top symbol that form the sort constraint.

occurrence of an equation (atom) and $t \neq s$.

For the specific sort constraint approach introduced here, monadic Horn theories are of particular importance. Such theories provide a natural representation of sort/type information (see Section 2.3.2). A *Horn clause* is a clause with at most one positive literal. A *monadic Horn theory* is a set of Horn clauses where all occurring predicates are monadic. A *declaration* is a clause $S_1(x_1), \dots, S_n(x_n) \rightarrow S(t)$ with $\{x_1, \dots, x_n\} \subseteq \text{vars}(t)$. It is called a *term declaration* if t is not a variable and a *subsort declaration* otherwise. A subsort declaration is called *trivial* if $n = 0$. A term t is called *shallow* if t is a variable or is of the form $f(x_1, \dots, x_n)$ where the x_i are not necessarily different variables. A term t is called *linear* if every variable occurs at most once in t . It is called *semi-linear* if it is a variable or of the form $f(t_1, \dots, t_n)$ such that every t_i is semi-linear and whenever $\text{vars}(t_i) \cap \text{vars}(t_j) \neq \emptyset$ we have $t_i = t_j$ for all i, j . A term declaration is called *shallow (linear, semi-linear)* if t is shallow (linear, semi-linear). Note that shallow term declarations don't include arbitrary ground terms. However, any ground term declaration can be equivalently represented, with respect to the minimal model semantics, by finitely many shallow term declarations. For example, the ground term declaration $\rightarrow S(f(a))$ can be represented by the shallow declarations $T(x) \rightarrow S(f(x)), \rightarrow T(a)$. A *sort theory* is a finite set of declarations. It is called *shallow (linear, semi-linear)* if all term declarations are shallow (linear, semi-linear).

A *clause store* is a multiset of clauses. A *clause store collection* is a multiset of clause stores. The inference and reduction rules discussed in this chapter operate on clauses occurring in a clause store of a clause store collection. There are inference rules

$$\mathcal{I} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1 \quad \dots \quad \Theta_n \parallel \Gamma_n \rightarrow \Delta_n}{\Psi \parallel \Pi \rightarrow \Lambda}$$

reduction rules

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1 \quad \dots \quad \Theta_n \parallel \Gamma_n \rightarrow \Delta_n}{\Psi_1 \parallel \Pi_1 \rightarrow \Lambda_1}$$

$$\vdots$$

$$\Psi_k \parallel \Pi_k \rightarrow \Lambda_k$$

and splitting rules.

$$\mathcal{S} \frac{\Theta \parallel \Gamma \rightarrow \Delta}{\begin{array}{c|c} \Psi_{1,1} \parallel \Pi_{1,1} \rightarrow \Lambda_{1,1} & \Psi_{1,2} \parallel \Pi_{1,2} \rightarrow \Lambda_{1,2} \\ \vdots & \vdots \\ \Psi_{n,1} \parallel \Pi_{n,1} \rightarrow \Lambda_{n,1} & \Psi_{m,2} \parallel \Pi_{m,2} \rightarrow \Lambda_{m,2} \end{array}}$$

The clauses $\Theta_i \parallel \Gamma_i \rightarrow \Delta_i$ are called the *parent clauses* or *premises* of the splitting (reduction, inference) rule and the clauses $\Psi_{i,(j)} \parallel \Pi_{i,(j)} \rightarrow \Lambda_{i,(j)}$ the *conclusions*. A rule is applied to a clause store collection P by selecting a clause store N out of P such that the premises of an inference (reduction, splitting) rule are contained in N . In this case, N is called the *current* clause store. If an inference is performed, the conclusion of the inference is *added* to N . If a reduction is performed, the premises are *replaced* in N by the conclusions. As a special case, if no conclusion is present, the premises are deleted from N . If a splitting rule is applied, the current store N is *replaced* in P by two stores

$$N \setminus \{\Theta \parallel \Gamma \rightarrow \Delta\} \cup \{\Psi_{j,1} \parallel \Pi_{j,1} \rightarrow \Lambda_{j,1} \mid 1 \leq j \leq n\}$$

$$N \setminus \{\Theta \parallel \Gamma \rightarrow \Delta\} \cup \{\Psi_{j,2} \parallel \Pi_{j,2} \rightarrow \Lambda_{j,2} \mid 1 \leq j \leq m\}$$

One can think of more general splitting rules but the above schema is sufficient for a general understanding of the implementation consequences caused by such a rule and is actually implemented in SPASS (see Section 2.3.5). Semantically, clause stores represent conjunctions of their clauses whilst clause store collections represent disjunctions of their contained clause stores. So a clause store collection P represents a disjunction (clause stores) of conjunctions (of universally quantified clauses) of disjunctions (of literals).

A clause store N is *saturated* with respect to a set of inference and reduction rules (no splitting rules), if any conclusion of an inference rule application to N yields a clause that can eventually be deleted by a sequence of reduction rule applications. This definition of saturation provides an operational point of view.

Chapter 2

SPASS

2.1 What This Chapter is (not) About

This article is about the design, the implementation and the use of SPASS Version 2.0 [54], a saturation-based automated theorem prover for first-order logic with equality. SPASS is unique due to the combination of the superposition calculus with specific inference/reduction rules for sorts (types) and a splitting rule for case analysis motivated by the β -rule of analytic tableau and the case analysis employed in the Davis-Putnam procedure [10]. Furthermore, SPASS provides a sophisticated clause normal form translation [37, 38]. This chapter is not about completeness/soundness proofs for saturation-based first-order logic calculi. For this we refer to [4], [35] and the corresponding references in this article. Nevertheless, this documentation introduces a variety of inference/reduction rules that are implemented in SPASS and that form a basis for various first-order calculi.

At the heart of SPASS is a first-order calculus. It consists of inference rules that generate new clauses and reduction rules that reduce the number of clauses or transform clauses into simpler ones. In SPASS we introduced a great variety of clause set¹ based inference and reduction rules that can be composed to various sound and complete first-order calculi. The clause store data structure together with such a calculus are the basis for most of today's theorem proving systems, like Otter, E or Vampire (see Appendix 2.7). SPASS goes one step further by introducing a splitting rule that supports explicit case analysis. This generalizes the standard clause store based approach to a clause store collection² approach where different clause stores represent the different cases. Therefore, the splitting rule introduces a second dimension in saturation-based automated theorem proving.

The third dimension we consider in SPASS are constraints, extra information attached to a clause restricting its semantics and/or usage with respect to the calculus. Well-known constraints are ordering constraints, forcing substituted terms to satisfy the attached ordering restrictions, basicness constraints, forbidding paramodulation inferences on certain terms in the clause, or type constraints guaranteeing that instantiations for variables conform to the attached type of the variable. From an abstract implementation point of view the handling of constraints is always the same. The information is attached to a clause, it is maintained during the inference/reduction application process and it is exploited by constraint specific algorithms/deduction mechanisms to restrict inferences/reductions or to even eventually delete a clause. We implemented sort constraints, specific type constraints for variables where the type (sort) theory is itself expressed by clauses.

A software project like SPASS is always a compromise between different goals like maintainability, efficiency, flexibility, readability, short development time, modularity, etc. For SPASS, the most important goals are maintainability, flexibility, readability and modularity of the design (code). This does not mean that SPASS is inefficient, but whenever there is a conflict between efficiency and, e.g., a modular design, we prefer the latter. Best evidence that SPASS really meets its design goals is the fact that its code is used by several research groups as a basis for code development and that the project has at the time of this

¹From an implementation point of view we consider clause multisets, called *clause stores*.

²A *clause store collection* is a multiset of clause stores.

writing run successfully for eight years. We also view this prover as a tool(box), so even for users that don't want to spend effort in implementation work it offers great flexibility, rich documentation and a number of indispensable extra tools like syntax translators or a proof checker.

We believe that a sophisticated calculus, a “good theory”, has the highest impact on the performance of a prover. Therefore, we won't study the implementation of provers at the level of data structures, object hierarchies or module design. Instead, we will discuss the needs for an efficient implementation of the various inference/reduction rules and the impacts that the top-level search algorithms have on an actual implementation. This together with a specific design goal decision can then lead to a design concept for a real prover like SPASS.

Heuristics are also not in focus of SPASS, although they can play an important rôle in automated theorem proving. For example, the heuristic that chooses the next clause for inferences inside a typical “main loop” of a saturation based prover (see Table 2.1 on page 12) can have a great impact on the success/non-success of a search attempt for a proof. However, it is the nature of heuristics that they are sometimes useful and sometimes make things even worse. In the context of automated theorem proving, it is often not predictable what will be the case as long as we don't restrict our attention to specific problems (problem classes). Therefore again, the main focus of SPASS is on inference/simplification/reduction techniques. For these techniques we know, e.g., that they can be composed to decision procedures for a variety of syntactically identifiable subclasses of first-order logic [5, 34, 27, 53]. Our level of abstraction is often lower compared to papers that solely are concerned with theory, because we want to emphasize on the implementation relevant aspects of inference/simplification/reduction techniques. Hence, we always refrain from “more elegant” formulations in order to make the consequences for an (efficient) implementation more explicit.

The design concepts introduced in SPASS and discussed here are not necessarily original contributions of the author. For example, the combination of saturation and splitting is original, but the use of indexing techniques [22] is a widely used method. Many of the design ideas introduced in SPASS are “common knowledge” among the developers of first-order saturation based theorem provers and are regularly discussed among these. Thus it is hard to say where the origin of some idea comes from and I refer to my colleagues listed in the acknowledgments.

In this chapter I frequently use the notion *in practice* to argue for design decisions. This refers to the problem domains we have been interested in so far: Problems resulting from the analysis/verification of software [16], from the area of automatic type inference [17, 8], from the analysis of security protocols [24, 53], planning problems [29], modal logic problems [26], and problems from the TPTP problem library [48]. If we say that some technique/design/calculus is preferred over some other technique/design/calculus *in practice*, this is always meant with respect to the above mentioned problem domains.

SPASS is freely available from the SPASS homepage at

<http://spass.mpi-sb.mpg.de/>

After a section on notation and notions (Chapter 1), an introduction to major design aspects of saturation-based provers (Section 2.2), we discuss the inference/reduction rules (Section 2.3) of SPASS. For each rule we provide a formal definition and explain specific aspects of its pragmatics and implementation. In Section 2.4 we evolve the global design of a prover from all these rules. Finally, the appendix establishes links between all mentioned design concepts, inference/reduction rules and the user interface of SPASS as well as its source code.

2.2 A First Simple Prover

In this section, we discuss the implementation of a simple resolution based calculus. Although the calculi implemented by SPASS are much more sophisticated than the simple resolution calculus considered here, some important design decisions can already be explained on the basis of such a simple example. The resolution calculus consists of the inference rules resolution, factoring and the reduction rules subsumption deletion and tautology deletion

$$\begin{array}{l}
\text{Resolution} \\
\mathcal{I} \frac{\Gamma_1, A \rightarrow \Delta_1 \quad \Gamma_2 \rightarrow \Delta_2, B}{(\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2)\sigma} \\
\\
\text{Subsumption Deletion} \\
\mathcal{R} \frac{\Gamma_1 \rightarrow \Delta_1 \quad \Gamma_2 \rightarrow \Delta_2}{\Gamma_1 \rightarrow \Delta_1} \\
\\
\text{Tautology Deletion} \\
\mathcal{R} \frac{\Gamma, A \rightarrow \Delta, A}{\Gamma, A \rightarrow \Delta, A}
\end{array}
\qquad
\begin{array}{l}
\text{Factoring Right} \\
\mathcal{I} \frac{\Gamma \rightarrow \Delta, A, B}{(\Gamma \rightarrow \Delta, A)\sigma} \\
\\
\text{Factoring Left} \\
\mathcal{I} \frac{\Gamma, A, B \rightarrow \Delta}{(\Gamma, A \rightarrow \Delta)\sigma}
\end{array}$$

where σ is a most general unifier (mgu) of the atoms A and B for the rules resolution, factoring and in order to apply subsumption, the clause $\Gamma_1 \rightarrow \Delta_1$ must subsume the clause $\Gamma_2 \rightarrow \Delta_2$.

For the resolution rule to be complete, it is required that the parent clauses $\Gamma_1, A \rightarrow \Delta_1$ and $\Gamma_2 \rightarrow \Delta_2, B$ have no variables in common. Actual implementations of the rule satisfy this requirement in different ways. They all have in common that variables are represented by (natural) numbers, so this is our assumption for the rest of this paragraph. The first solution explicitly renames the clauses such that they have no variables in common. The second solution accepts clauses that share variables, but when running the unification algorithm the variables are separated by adding an offset to the variables of one clause.³ A typical offset is the value of the maximal, with respect to number greater, variable of the other clause. The third solution also accepts clauses that share variables and solves the problem by employing two substitutions, one for each clause. This requires some modifications to the standard unification algorithms, because the terms of the different atoms need to be explicitly separated. This is the solution implemented in SPASS. In order to test applicability of the resolution rule, it is sufficient to explicitly or implicitly rename the variables of the considered atoms, not the overall clause.

For the factoring rule there is an extra variant for positive (Factoring Right) and negative literals (Factoring Left). We could have presented both variants in one rule, by denoting clauses as disjunctions of literals. However, our representation is closer to the implementation of the rule. All clause data structures used in well-known provers explicitly separate positive from negative literals. The reason is efficiency and already becomes obvious for factoring: Whenever we search for a partner literal for a positive literal it does not make sense to consider negative literals at all. Similar situations arise for other inference/reduction rules. Therefore, the decision in this article is always to distinguish positive and negative literals when presenting inference/reduction rules.

Now let us compose the inference/reduction rules to an actual prover. The input of the prover is a clause store containing clauses without equality and the output on termination is a proof or a saturated clause store. The above resolution calculus is complete, so we also want our search procedure to be complete in the sense that if resources don't matter and our procedure is called with an unsatisfiable clause store then it will eventually find a proof (the empty clause). In order to achieve this goal, we have to guarantee that the considered clause set is saturated in the limit. This includes that all inferences between clauses have been performed. An easy way to remember which inferences have already been performed is to split the input clause store in a set Wo of clauses (Worked off clauses) where all inferences between clauses in this set already took place and a set Us of clauses (Usable clauses) which still have to be considered for inferences. Then a main loop iteration of the prover consists of selecting a clause from the Us set, moving it to the Wo set and then adding all inferences between the selected clause and the clauses in Wo to the Us set. If the selection is fair, i.e., no clause stays arbitrarily long in the Us set without being selected, this results in a complete procedure. It remains to build reductions into this loop. The idea for this loop is due to the Otter theorem prover and its predecessors [32].

The reduction rules tautology deletion and subsumption deletion decrease the number of clauses in the clause store while the inference rules increase the number of clauses. Hence, exhaustive application of the reduction rules terminates and produces smaller clause stores. In practice, small clause sets are preferred over large ones, hence reductions are preferred over inferences. This consideration together with the idea of the main-loop introduced above leads to *ResolutionProver1* depicted in Table 2.1. Note that subsumption

³Please recall that we assume variables to be represented by naturals.

1	<i>ResolutionProver1</i> (<i>N</i>)
2	<i>Wo</i> := \emptyset ;
3	<i>Us</i> := <i>taut</i> (<i>sub</i> (<i>N</i>));
4	While (<i>Us</i> $\neq \emptyset$ and $\square \notin Us$) {
5	(<i>Given</i> , <i>Us</i>) := <i>choose</i> (<i>Us</i>);
6	<i>Wo</i> := <i>Wo</i> \cup { <i>Given</i> };
7	<i>New</i> := <i>res</i> (<i>Given</i> , <i>Wo</i>) \cup <i>fac</i> (<i>Given</i>);
8	<i>New</i> := <i>taut</i> (<i>sub</i> (<i>New</i>));
9	<i>New</i> := <i>sub</i> (<i>sub</i> (<i>New</i> , <i>Wo</i>), <i>Us</i>);
10	<i>Wo</i> := <i>sub</i> (<i>Wo</i> , <i>New</i>);
11	<i>Us</i> := <i>sub</i> (<i>Us</i> , <i>New</i>) \cup <i>New</i> ;
12	}
13	If (<i>Us</i> = \emptyset) then print “Completion Found”;
14	If ($\square \in Us$) then print “Proof Found”;

Table 2.1: A First Resolution Based Prover

and tautology deletion are independent in the sense that once all tautologies have been removed, subsumption does not generate new tautologies. Analyzing such dependencies between reductions is one key for an efficient implementation.

For the description of theorem proving procedures we use the following abbreviations: *fac*(*C*) is the set of all factoring inference conclusions (left and right) from the clause *C*, *res*(*C*, *D*) is the set of all resolution inference conclusions between two clauses *C* and *D*, *taut*(*N*) is the set *N* after exhaustive application of tautology deletion and *sub*(*N*, *M*) is the set of all clauses from *N* that are not subsumed by a clause in *M*. We overload *sub* for one argument, where *sub*(*N*) denotes the set *N* after exhaustive application of subsumption deletion to the clauses in *N*. We overload *res* by defining *res*(*C*, *N*) to be the set of all resolution inferences between the clause *C* and a clause in *N*. The function *choose* selects and removes a clause from its argument clause store and returns the selected clause as well as the updated argument clause store.

As already motivated, the procedure *ResolutionProver1* operates on two clause stores: *Wo* and *Us*. The store *Wo* holds all clauses that have already been selected for inferences, while the store *Us* contains all candidate clauses to generate inferences. The prover *ResolutionProver1* is called with a finite clause store *N* and tests those for unsatisfiability. Lines 2 and 3 initialize the sets *Wo* and *Us*. Note that *Us* is not initialized with *N*, but its completely inter-reduced equivalent. This step is called *input reduction*. The search for the empty clause (a saturation) is implemented by the lines 4–12. The while-loop starting at line 4 terminates if the empty clause is found or the set *Us* is empty. We will argue below that this implies that the set *Wo* is saturated. If *Us* is not empty and the body of the while-loop is entered, the function *choose* selects at line 5 a clause out of the usable set. The function is fair, if no clause stays in *Us* for an infinite number of iterations through the while loop. A widely used, fair implementation (heuristic) of *choose* is to select a lightest clause that is a clause of smallest size. This selection function is fair, because there are only finitely many different clauses with respect to subsumption having less than *k* symbols, for any constant *k*.⁴ Many refinements of the *choose* function are possible: using different weights for variable and signature symbols, preferring clauses with more/fewer variables, preferring clauses that contain certain atoms/term structures or considering in addition the depth of a clause in the search space. The depth of a clause in the search space is zero for all input clauses and every conclusion of an inference has the maximal

⁴Note that since the input set *N* is finite, the relevant signature is finite, too.

depth of their parent clauses plus one. Many provers use a combination of weight and depth selection, e.g., choosing four times clauses by minimal weight and every fifth time by minimal depth. This combination again goes back to Otter where the ratio can be controlled by the *pick-given* ratio parameter. The parameter is also available in SPASS.

Then the clause *Given* is selected, removed from *Us* and added to *Wo* (lines 5, 6). Next (line 7) all resolution inference conclusions between *Given* and *Wo* and all factoring inference conclusions from *Given* are stored in *New*. Note that since *Given* is already contained in *Wo* these inferences include self resolution inferences. The clauses generated so far are called *derived* clauses. The lines 8–11 are devoted to reduction. First, all tautologies and subsumed clauses are removed from *New*. Then all clauses that are subsumed by a clause in *Wo* or *Us* are deleted from *New*. This operation is called *forward subsumption*. Clauses remaining in *New* are then used for *backward subsumption*, the subsumption of clauses in the sets *Wo* and *Us* by clauses from *New*. Finally, the clauses from *New* are added to *Us*. These clauses are usually called *kept* clauses.

There are two invariants that hold each time line 4 is executed:

- Any resolution inference conclusion from two clauses in *Wo* (factoring inference conclusion from a clause in *Wo*) is either contained in *Wo*, *Us* or is subsumed by a clause in *Wo*, *Us* or is a tautology.
- The sets *Wo* and *Us* are completely inter-reduced:
 $Wo \cup Us = \text{taut}(Wo \cup Us)$ and
 $Wo \cup Us = \text{sub}(Wo \cup Us)$.

A consequence of these invariants to hold is that if the procedure stops then the set *Wo* is saturated. Furthermore, if the function *choose* is fair, then the *ResolutionProver1* is complete.

In case that for the set *N* a satisfiable subset *N'* is known, e.g., if the clauses represent a proof attempt of a conjecture with respect to some theory that is known to be satisfiable, we could also initialize the sets by $Wo := N'$ and $Us := (N \setminus N')$, obtaining the so called set of support (SOS) strategy [55]. The SOS strategy preserves completeness.

Many other saturation based provers (e.g., Otter, SPASS, Waldmeister, see Appendix 2.7) have a search algorithm based on two sets of clauses.⁵ SPASS implements *ResolutionProver1* on a Unix system by the shell invocation

```
SPASS -Auto=0 -ISRe -ISFc -RTaut -RFSub -RSub <file>
```

where the option `-Auto=0` turns off the automatic mode of SPASS. In this mode SPASS decides on the basis of the input problem the set of inference and reduction rules. If the automatic mode is turned off, no inference/reduction rules are activated. All options starting with an **I** (de)activate inference rules, options starting with an **R** (de)activate reduction rules. So the above call to SPASS activates the inference rules standard resolution (`-ISRe`), standard factoring (`-ISFc`) and the reduction rules tautology deletion (`-RTaut`), forward subsumption (`-RFSub`) and backward subsumption (`-RSub`). An inference/reduction rule option is activated by setting it to 1 (the default) and deactivated by setting it to 0. For further details consider Appendix 2.5.

For example, we simulate a run of *ResolutionProver1* on the clauses

```
1:                               → P(f(a))
2:      P(f(x)) → P(x)
3:  P(f(a)), P(f(x)) →
```

shown in Table 2.2. For each while-loop iteration, we show the content of the *Wo* and *Us* set at line 4, the selected *Given* clause and the content of *New* before execution of line 8. Newly generated clauses are printed in full detail while we refer to a clause in the sets *Wo* and *Us* only by its unique clause number. The function *choose* selects lightest clauses.

Every box in Table 2.2 represents one while-loop iteration. For newly generated clauses we also show the applied inference rule and parent clauses/literals. Here *Res* indicates a resolution inference, *Fac* a factoring inference and the notion *n.m* refers to literal *m* of clause *n*. So, for example, clause 7 is generated

⁵However, they use different names for the sets. So don't be confused.

Iteration 1	
$W_0 = \emptyset$	$U_s = \{1, 2, 3\}$
$Given = 1: \rightarrow P(f(a))$	
$New = \emptyset$	
↓	
Iteration 2	
$W_0 = \{1\}$	$U_s = \{2, 3\}$
$Given = 2: P(f(x)) \rightarrow P(x)$	
$New = \{4:[Res:1.1,2.1] \rightarrow P(a),$	
$5:[Res:2.1,2.2] P(f(f(x))) \rightarrow P(x)\}$	
↓	
Iteration 3	
$W_0 = \{1, 2\}$	$U_s = \{3, 4, 5\}$
$Given = 4: \rightarrow P(a)$	
$New = \emptyset$	
↓	
Iteration 4	
$W_0 = \{1, 2, 4\}$	$U_s = \{3, 5\}$
$Given = 3: P(f(a)), P(f(x)) \rightarrow$	
$New = \{ 6:[Res:1.1,3.1] P(f(x)) \rightarrow,$	
$7:[Res:1.1,3.2] P(f(a)) \rightarrow,$	
$8:[Res:2.2,3.1] P(f(f(a))), P(f(x)) \rightarrow,$	
$9:[Res:2.2,3.2] P(f(a)), P(f(f(x))) \rightarrow,$	
$10:[Fac:3.1,3.2] P(f(a)) \rightarrow \}$	
↓	
Iteration 5	
$W_0 = \{1, 4\}$	$U_s = \{6\}$
$Given = 6: P(f(x)) \rightarrow$	
$New = \{11:[Res:1.1,6.1] \square \}$	

Table 2.2: A Run of *ResolutionProver1*

1	<i>ResolutionProver2</i> (<i>N</i>)
2	<i>Wo</i> := \emptyset ;
3	<i>Us</i> := <i>taut</i> (<i>sub</i> (<i>N</i>));
4	While (<i>Us</i> $\neq \emptyset$ and $\square \notin Us$) {
5	(<i>Given</i> , <i>Us</i>):= <i>choose</i> (<i>Us</i>);
6	if (<i>sub</i> ({ <i>Given</i> }, <i>Wo</i>) $\neq \emptyset$) {
7	<i>Wo</i> := <i>sub</i> (<i>Wo</i> , { <i>Given</i> });
8	<i>Wo</i> := <i>Wo</i> \cup { <i>Given</i> };
9	<i>New</i> := <i>res</i> (<i>Given</i> , <i>Wo</i>) \cup <i>fac</i> (<i>Given</i>);
10	<i>New</i> := <i>taut</i> (<i>sub</i> (<i>New</i>));
11	<i>New</i> := <i>sub</i> (<i>New</i> , <i>Wo</i>);
12	<i>Us</i> := <i>Us</i> \cup <i>New</i> ;
13	}
14	}
15	If (<i>Us</i> = \emptyset) then print “Completion Found”;
16	If ($\square \in Us$) then print “Proof Found”;

Table 2.3: A Second Resolution Based Prover

by a resolution inference between the first literal of clause 1 and the second literal of clause 3 where literals are counted from left to right. Iteration 4 shows already some common phenomena of saturation based calculi. First, these calculi are typically redundant in the sense that the very same clause can be generated in various, different ways. For example, clause 7 and clause 10 are logically identical, although the former is generated by a resolution inference while the latter is the result of a factoring application. As a consequence, subsumption is indispensable for saturation based calculi to cut down the number of kept clauses. The situation gets even more dramatic in the context of equality, where a single loop iteration can already cause an explosion in the number of newly generated clauses. This will be discussed in more detail in Section 2.3. Coming back to our run, note that in the reduction part of while-loop iteration 4, the clauses 2, 3, 5, 7–10 are all subsumed by clause 6. Second, even for this simple example, it happened that the selection of the *Given* clause is not always unique when choosing lightest clauses. During iteration 4, the clauses 3 and 5 have both size 6, but choosing clause 5 instead of clause 3 would have caused an additional while loop iteration before the empty clause is derived. Of course, the function *choose* could be refined and we will in fact discuss such refinements, but in practice it happens (and must happen) frequently that several clauses have the same precedence with respect to *choose*. Then selecting the right clause (by accident) can enable a prover to find a proof where it gets lost in the search space by selecting a different one. This phenomenon is common to all theorem provers and can be observed at the yearly CADE CASC system competitions (e.g., see [49]), where the performance of provers varies depending on the ordering of the input problem clauses.

If *ResolutionProver1* is ran on non-trivial examples, the *Us* set rapidly gets much larger than the *Wo* set. It easily happens that after some iterations the size increases by a factor of 1000. In particular, it is common in the context of problems containing equality. Therefore, at least with respect to the number of clauses that have to be considered, the subsumption tests with respect to the *Us* set are the most expensive parts of the algorithm. Typical runs of *ResolutionProver1* show a behavior where more than 95% of the overall time is spent for subsumption checks. This motivates the design of *ResolutionProver2* shown in Table 2.3.

ResolutionProver2(N) does not perform any subsumption tests with respect to the *Us* set and back subsumption is only performed with respect to the actually selected given clause. The two invariants for *ResolutionProver2* are

- Any resolution inference conclusion from two clauses in *Wo* (factoring inference conclusion from a clause in *Wo*) is either contained in *Wo*, *Us* or is subsumed by a clause in *Wo* or is a tautology.
- The set *Wo* is completely inter-reduced:
 $Wo = \text{taut}(Wo)$ and
 $Wo = \text{sub}(Wo)$.

These two invariants are still strong enough to guarantee that if the while loop terminates, the *Wo* set is saturated. Note that although *New* is always reduced with respect to *Wo* at line 11, the set *Us* is in general not reduced with respect to *Wo*, i.e., $Us \neq \text{sub}(Us, Wo)$.

If we assume that *choose* selects light clauses there is a further motivation to leave out subsumption tests with respect to the *Us* set. If a clause *C* subsumes a clause *D*, then $\text{size}(C) \leq \text{size}(D)$. So small clauses have a higher probability to subsume other clauses than larger clauses. Therefore, because we always select the lightest given clause, the hope is that not too many clauses that could have been subsumed stay in the *Us* set. In practice, *ResolutionProver2* saves about 10% of the time spent for reductions (subsumption) compared to *ResolutionProver1*. For the simple resolution calculus we studied so far, *ResolutionProver2* is mostly in favor of *ResolutionProver1* when run in practice. As soon as our reduction techniques include rules that produce lighter clauses (see Section 2.3) the choice is no longer obvious in general. There are examples where an overall interreduction easily yields the empty clause, but for a *ResolutionProver2* style algorithm sophisticated heuristics are needed to still find a proof.

Running *ResolutionProver2* on the example clause store, the result is similar to the run of *ResolutionProver1* (Table 2.2). The first three iterations are identical, but at iteration 4, the clauses 2, 3, 5, 7–10 are not subsumed but stay in their respective sets. Then, in iteration 5, where clause 6 is selected as given clause, the clauses 2, 3 are removed from the *Wo* set (line 7 of *ResolutionProver2*, Table 2.3) and the empty clause is derived.

SPASS implements *ResolutionProver2* by the call

```
SPASS -Auto=0 -FullRed=0 -ISRe -ISFc -RTaut -RFSub -RSub <file>
```

where the option `-FullRed=0` deactivates reduction with respect to the *Us* set and modifies the algorithm accordingly. For further details consider Appendix 2.5.

There are many possible alternatives, variations, refinements for the two loops suggested here. Let us discuss some aspects. First, concerning factoring, any clause store can be finitely saturated with respect to factoring, since a factor has strictly fewer literals than its parent. So one could get the idea to keep the *Wo* set always saturated with respect to factoring. The disadvantage of this approach is that the number of factors that can be generated out of one clause grows worst case exponentially in the number of literals. The prover Bliksem allows a user to prefer factors (see Appendix 2.7).

Second, concerning resolution and the selection of the given clause, we could also a priori built for each loop iteration all one step resolvents between the clauses in the *Us* set and between one parent from the *Us* set and one parent from the *Wo* set. Then instead of picking a *Given* clause, we pick one resolvent, use it for (back and/or forward) reduction and finally add it to the *Wo* set. This approach results in a more fine grained development of the search space. This design for a proof search is closely related to clause graph resolution [15].

Third, on the implementation side, if we once decide to implement *ResolutionProver2*, the only information we need for the clauses in *Us* are their properties with respect to the *choose* function and how these clauses can be generated. For all clauses except the input clauses it suffices to store references for the parents and the used inference. This way it is possible to store all *Us* clauses in a compact way. In practice constant space suffices for any clause. This dramatically decreases memory consumption and results in an extra speed up. The necessary regeneration of clauses once they are selected, plays no rôle concerning performance. The Waldmeister prover follows this approach. Fourth, another way to keep the *Us* set small is to throw away clauses with respect to certain weight or complexity restrictions on the newly generated clauses. Either these clauses are just thrown away resulting in an incomplete procedure, this is supported

by Otter, SPASS and Vampire (see Appendix 2.7), or the restrictions can be set in a way such that only finitely many clauses can pass the restriction test and once the search results in such a saturated set, the restrictions are adjusted and the search is restarted. This design is supported by Bliksem, SPASS and Fiesta. In SPASS the resource restriction strategy is controlled by the flags `BoundMode` specifying the resource type where 0 means no resource restriction, 1 means clause size restriction and 2 means clause depth restriction, the flag `BoundStart` specifying the initial start value to restrict the selected resource type and `BoundLoops` determines how often a saturation caused by resource restrictions is restarted with adjusted restrictions. So the call

```
SPASS -BoundMode=1 -BoundStart=5 -BoundLoops=3 <file>
```

causes SPASS to throw away all clauses that have a weight greater 5. If this leads to an empty Us set without finding the empty clause, the bound is increased to the smallest size greater 5 that caused a clause to be deleted. This process is repeated at most 3 times, then any weight restrictions are discarded. Such an exploration of the search space can be particularly useful in the context of unit equational problems.

2.3 Inference and Reduction Rules

In this section we describe a variety of inference/reduction rules. For every rule, we start with a formal definition of the rule and then, if necessary, discuss aspects of its pragmatics, complexity, interaction with other rules or design concepts and its implementation and usage. Some rules are stated in a general, possibly non-effective form (e.g., see the conflict rule, Definition 2.3.19). In this case we also discuss effective instantiations. The rules don't form a particular calculus, instead several well-known calculi can be implemented by forming appropriate groups of rules. An example is the simple resolution calculus considered in Section 2.2.

Many reduction rules can be simulated by one or several inference rule applications followed by a (trivial) subsumption step. As long as the inference rule set is complete this observation is not too surprising, since we require all our rules to be sound. So one might think that the sophisticated reduction machinery introduced in this section is not really necessary but just a waste of resources when implemented. However, it is just the other way round. Reduction rules always lead to "simpler" clause stores by deleting some clause or by replacing a clause by a "simpler" one. This often ensures the termination of exhaustive application of (groups of) such rules and enables application of these rules to *all* clauses. Therefore, in the context of an implementation, reduction rules cannot be simulated by inference rule applications since those don't terminate when applied exhaustively. Inference rules are only applied to some selected *Given* clause. Reduction rules should be viewed as restricted inference rules that eventually lead to simpler clause stores and help to explore the "easy" parts of the search space (problem). They replace search space exploration by (efficient) calculation. In fact, some of the reduction rules introduced in this section are motivated by decidability results for various first-order logic fragments.

2.3.1 Reduction Orderings

For many of the inference/reduction rules defined in the sequel, maximality restrictions on literals, terms play an important rôle. The two most popular orderings are the Knuth-Bendix ordering (KBO) [30, 39] and the recursive path ordering with status (RPOS) [11]. For a broad introduction to orderings, consider the article by [12] and the more recent book by [2]. The definitions below differ in some details from other definitions found in the literature, but reflect implementation experience.

Nearly all orderings used in today's provers are variations of the KBO and the RPOS. In particular, weaker versions the orderings are often used. For example, purely weight based orderings or variants of the RPOS without recursive consideration of subterms. These weaker versions have the advantage of cheaper computation and when used to restrict inference rules (see Section 2.3.3) of a broader exploration of the search space. This can be useful for the search of short proofs. We describe KBO and RPOS exactly the way they are implemented in SPASS.

Let $>$ be a strict order on the set of signature symbols (functions, predicates), called a *precedence*. Let *weight* be a mapping from the set of signature symbols into the non-negative integers. We call a weight

function *admissible* for some precedence if for every unary function symbol f with $weight(f) = 0$, the function f is maximal in the precedence, i.e., $f \geq g$ for all other function symbols g . The function *weight* is extended to a weight function for terms (atoms) as follows: (i) if t is a variable, then $weight(t) = k$, where k is the minimum weight of any constant and (ii) if $t = f(t_1, \dots, t_n)$, then $weight(t) = weight(f) + \sum_i weight(t_i)$. Let *occ* be a function returning the number of occurrences $occ(s, t)$ of a term s in a term t , defined by $occ(s, t) = |\{p \in pos(t) \mid t|_p = s\}|$ and let *status* be a mapping from the signature symbols to the set $\{left, right, mul\}$.

Definition 2.3.1 (KBO)

If t, s are terms, then $t \succ_{kbo} s$ if $occ(x, t) \geq occ(x, s)$ for every variable $x \in (vars(t) \cup vars(s))$ and

- (1) $weight(t) > weight(s)$ or
- (2) $weight(t) = weight(s)$ and $t = f(t_1, \dots, t_k)$ and $s = g(s_1, \dots, s_l)$ and
 - (2a) $f > g$ in the precedence or
 - (2b) $f = g$ and
 - (2b1) $status(f) = left$ and $(t_1, \dots, t_k) \succ_{kbo}^{lex} (s_1, \dots, s_l)$ or
 - (2b2) $status(f) = right$ and $(t_k, t_{k-1}, \dots, t_1) \succ_{kbo}^{lex} (s_l, s_{l-1}, \dots, s_1)$

Note that in case (2b) the condition $f = g$ implies $k = l$. Multiset status for function symbols can also be defined but does not pay off in practice for the KBO. If the weight function is admissible for the precedence, then the KBO is a reduction ordering [2]. If the precedence $>$ is total, then the KBO is total on ground terms (atoms). For some finite set of signature symbols⁶ and two terms s, t with $s \succ_{kbo} t$, there are finitely many terms s' with $s \succ_{kbo} s' \succ_{kbo} t$.

The motivation to consider unary function symbols with weight zero comes in particular from group theory. The standard group axioms can be turned into a convergent system [2] using the KBO with precedence $i > f > e$ and weights $weight(i) = 0$, $weight(f) = weight(e) = 1$ where i is the inverse function, f denotes group multiplication and e represents the neutral element. During the saturation (completion) process it is crucial to orient the derived equation $i(f(x, y)) \approx f(i(y), i(x))$ from left to right, for otherwise the saturation process won't terminate. The only way to achieve $i(f(x, y)) \succ f(i(y), i(x))$ is to assign weight 0 to the function symbol i .

Implementation of the KBO can be done straightforward from the definition. For the RPOS we also assume $>$ to be a strict order (precedence) on the set of signature symbols (functions, predicates).

Definition 2.3.2 (RPOS)

If t, s are terms, then $t \succ_{rpos} s$ if

- (1) $s \in vars(t)$ and $t \neq s$ or
- (2) $t = f(t_1, \dots, t_k)$ and $s = g(s_1, \dots, s_l)$ and
 - (2a) $t_i \succeq_{rpos} s$ for some $1 \leq i \leq k$ or
 - (2b) $f > g$ and $t \succ_{rpos} s_j$ for all $1 \leq j \leq l$ or
 - (2c) $f = g$ and
 - (2c1) $status(f) = left$ and $(t_1, \dots, t_k) \succ_{rpos}^{lex} (s_1, \dots, s_l)$ and $t \succ_{rpos} s_j$ for all $1 \leq j \leq l$ or
 - (2c2) $status(f) = right$ and $(t_k, t_{k-1}, \dots, t_1) \succ_{rpos}^{lex} (s_l, s_{l-1}, \dots, s_1)$ and $t \succ_{rpos} s_j$ for all $1 \leq j \leq l$ or
 - (2c3) $status(f) = mul$ and $\{t_1, \dots, t_k\} \succ_{rpos}^{mul} \{s_1, \dots, s_l\}$

⁶For an infinite set the condition does obviously not hold.

The RPOS is a reduction ordering as well and if the precedence $>$ is total RPOS is also total on ground terms (atoms), up to the congruence relation $=_{mul}$ generated from the symbols with multiset status. If f is a function symbol with $status(f) = mul$ then $f(t_1, \dots, t_n) =_{mul} f(s_1, \dots, s_n)$ if $\{t_1, \dots, t_n\} =_{mul} \{s_1, \dots, s_n\}$, for example $f(a, f(a, b)) =_{mul} f(f(b, a), a)$. Even for some finite set of signature symbols and two terms s, t with $s \succ_{rpos} t$, there are in general infinitely many terms s' with $s \succ_{rpos} s' \succ_{rpos} t$.

The RPOS can for example be used to orient distributivity the “right way”. If $f > g$, then the equation $f(x, g(y, z)) \approx g(f(x, y), f(x, z))$ is oriented by RPOS from left to right. Note that KBO cannot orient the equation from left to right, because the right hand side has one more occurrence of the variable x .

Given a specific theorem proving problem, the relevant signature is finite and fixed. In this case it can be useful to further refine an ordering by defining $t \succ s$ if $t\sigma \succ s\sigma$ for all ground substitutions σ where $(vars(t) \cup vars(s)) \subseteq dom(\sigma)$. Following this idea, RPOS can be instantiated to an ordering that totally orders all atoms by predicate symbols and only in second place considers possible argument terms, independently from variable occurrences! This can be achieved by making all (some) predicate symbols larger in the precedence than all function symbols. For example, with respect to the above suggested lifting and a signature with predicate symbols P, Q and function symbols f, a where $P > Q > f > a$ it holds that $P(x) \succ_{rpos} Q(f(x, y))$ because P is greater in the precedence than Q, f, a and hence any ground term that can be substituted for x or y . Predicates can be declared to be superior over function symbols, by a declaration

$$\text{set_DomPred}(< \text{predicate sequence} >).$$

in the SPASS settings section of an input file [23]. Such an application of RPOS can, e.g., be useful to make literals built from newly introduced formula renaming predicates minimal. This prevents the generation of the standard CNF via ordered resolution [38].

Straightforward recursive implementation of RPOS following the definition results in an algorithm with worst case exponential complexity. Using a dynamic programming idea, a polynomial algorithm can be devised [45]. However, in practice, it turns out that the straightforward implementation is superior to the dynamic programming approach, if the following filter is added. Whenever we test $t \succ_{rpos} s$ for two terms s, t , we first check $vars(s) \subseteq vars(t)$.

2.3.2 Sorts

The motivation for sorts comes from programming languages, where one likes to catch as many errors at compile time as possible. For example, if the addition function is only defined for number sorts (types) but used in a program with a list type, the compiler can complain about such a statement by exploiting the sort information. Of course, the sort checking must be tractable, i.e., it should at least be decidable and/or show acceptable performance for real world programs. A prerequisite for the sort information to be checked at compile time, is that the sort information is separated from the program and it is typically included in an extra declaration part.

Here we generalize this situation. The sort information is not separated from the first-order problem as, e.g., done in algebraic specification languages, but part of the problem itself. Therefore, we cannot check sort information at compile time, after or while reading the problem. Instead the sort information is used at run time, during proof search, to detect ill-sorted and therefore redundant clauses and to simplify the sort information contained in the clauses by specific algorithms. These algorithms exploit the sort information in a much more efficient way than their standard first-order reduction rule counterparts.

Sorts are activated in SPASS by the `-Sorts` option. If SPASS is called with option `-Sorts=1` all negative monadic literals with a variable argument are considered for the initial sort constraints, whereas `-Sorts=2` causes SPASS to consider all negative monadic literals for the initial sort constraints. The latter choice can affect completeness, because of the basicness restriction on the sort constraint.

Definition 2.3.3 (Sort Constraint Resolution)

The inference

$$\mathcal{I} \frac{T_1(t), \dots, T_n(t), \Psi \parallel \Gamma \rightarrow \Delta \quad \Theta_i \parallel \Gamma_i \rightarrow \Delta_i, T_i(s_i) \quad (1 \leq i \leq n)}{(\Theta_1, \dots, \Theta_n, \Psi \parallel \Gamma_1, \dots, \Gamma_n, \Gamma \rightarrow \Delta_1, \dots, \Delta_n, \Delta)\sigma}$$

where (i) σ is the simultaneous mgu of t, s_1, \dots, s_n , (ii) t is a non-variable term and there is no further literal $S(t) \in \Psi$, (iii) all Θ_i are solved, (iv) all $T_i(s_i)\sigma$ are reductive for $(\Theta_i \parallel \Gamma_i \rightarrow \Delta_i, T_i(s_i))\sigma$ is a *sort constraint resolution* inference.

Sort constraint resolution is a hyper resolution (see Definition 2.3.14) like inference rule. It simulates the rule weakening of sorted unification [52] on the relativization of sorted variables represented by the sort constraint. Sort constraint resolution is activated by the `-ISO` option.

Definition 2.3.4 (Empty Sort)

The inference

$$\mathcal{I} \frac{T_1(x), \dots, T_n(x), \Psi \parallel \Gamma \rightarrow \Delta \quad \Theta_i \parallel \Gamma_i \rightarrow \Delta_i, T_i(s_i) \quad (1 \leq i \leq n)}{(\Theta_1, \dots, \Theta_n, \Psi \parallel \Gamma_1, \dots, \Gamma_n, \Gamma \rightarrow \Delta_1, \dots, \Delta_n, \Delta)\sigma}$$

where (i) σ is the simultaneous mgu of s_1, \dots, s_n , (ii) $x \notin \text{vars}(\Gamma \cup \Delta \cup \Psi)$ and no non-variable term occurs in Ψ , (iii) all Θ_i are solved, (iv) all $T_i(s_i)\sigma$ are reductive for $(\Theta_i \parallel \Gamma_i \rightarrow \Delta_i, T_i(s_i))\sigma$ is an *empty sort* inference.

Empty sort is similar to sort resolution and, in fact, in some of our papers (e.g., [27]) we unified both rules into one inference rule. For the purpose of the decidability results presented in these papers this is appropriate. It makes sense to distinguish these rules, because the eventual success of empty sort, i.e., we are able to show that some sort is non-empty, does not rely on the particular sort constraint, but only on the set of monadic (sort) symbols that share their variable argument. We check emptiness of an intersection of sort symbols. Since there are only finitely many different such sorts with respect to some finite clause store, it may make sense to store constraints that resulted in successful non-emptiness proofs and to reuse them. One application domain are the proofs required in the context of static soft typing (Definition 2.3.6). Empty sort is activated by the `-IEms` option.

Definition 2.3.5 (Sort Simplification)

Let N be the current clause store and $N' \subseteq N$ be exactly the set of all declarations in N . The reduction

$$\mathcal{R} \frac{S(t), \Theta \parallel \Gamma \rightarrow \Delta}{\Theta \parallel \Gamma \rightarrow \Delta}$$

where $N' \models \forall x_1, \dots, x_n [S_1(x_1), \dots, S_n(x_n) \supset S(t)]$ and $\{S_1(x_1), \dots, S_n(x_n)\} \subseteq \Theta$ is the maximal subset of Θ for which $\{x_1, \dots, x_n\} \subseteq \text{vars}(t)$ is called *sort simplification*.

Given an arbitrary sort theory N' , the relation

$$N' \models \forall x_1, \dots, x_n [S_1(x_1), \dots, S_n(x_n) \supset S(t)]$$

is always decidable in polynomial time. In terms of sorted unification the problem means deciding well-sortedness [52]. A bottom-up algorithm based on dynamic programming yields the polynomial complexity whereas a simple top down approach results in an exponential procedure. The latter procedure would correspond to solve the problem with ordered resolution and an SOS strategy. The former algorithm is implemented in SPASS. Sort simplification is activated by the `-RSSi` option.

Sort simplification is one important reason why it makes sense to treat particular occurrences of monadic predicates in a special way. Sort simplification cannot be simulated via other standard reduction techniques like matching replacement resolution (see Definition 2.3.20) and cannot be extended to non-monic predicates. For example, for binary relations, the undecidable problem whether two ground terms are contained in a transitive binary relation generated by some positive unit clauses [43] can be reduced to deciding applicability of an extended sort simplification rule for binary relations. So without further restrictions, sort simplification cannot be effectively used for other n -ary relations.

Definition 2.3.6 (Static Soft Typing)

Let N be the current clause store over some fixed signature Σ and M be a sort theory such that $N \models S(t)$ implies $M \models S(t)$ for any ground monadic atom $S(t)$ over Σ where S occurs in some sort constraint in N . The reduction

$$\mathcal{R} \frac{\Theta \parallel \Gamma \rightarrow \Delta}{\quad}$$

where $M \not\models \exists x_1, \dots, x_n \Theta$ with $\text{vars}(\Theta) = \{x_1, \dots, x_n\}$ is called *static soft typing*.

The above definition of static soft typing is not effective. The problem $M \not\models \exists x_1, \dots, x_n \Theta$ is not decidable for arbitrary sort theories M and sort constraints Θ . It includes the general problem of sorted unification [52] that is well-known to be undecidable, in general. Furthermore, it is not obvious how the sort theory M can be constructed out of N such that it meets the requirements of Definition 2.3.6. A solution to all these problems is the following. First, all clauses that contain positive monadic atoms are safely approximated and restricted to the sort information they contain:

$$\mathcal{R} \frac{\Theta \parallel \Gamma \rightarrow \Delta, S_1(t_1), \dots, S_n(t_n)}{\Theta_1 \parallel \rightarrow S_1(t_1)} \\ \vdots \\ \Theta_n \parallel \rightarrow S_n(t_n)$$

where no monadic atom occurs in Δ , Θ is solved and $\Theta_i = \{S(x) \mid S(x) \in \Theta \text{ and } x \in \text{vars}(S_i(t_i))\}$ for $1 \leq i \leq n$. By construction all Θ_i are solved and the rule does not modify declarations. If the initial clause set N does not contain positive equations, then the sort theory N' obtained by a fix-point computation of the above reduction on N approximates N in the desired way (see Definition 2.3.6). Second, the sort theory N' is approximated to a sort theory N'' such that satisfiability of sort constraints in N'' gets decidable.

$$\mathcal{R} \frac{\Theta \parallel \rightarrow S(f(t_1, \dots, t_n))}{\Theta_1, T(x) \parallel \rightarrow S(f(s_1, \dots, s_n))} \\ \Theta_2 \parallel \rightarrow T(t_i)$$

where t_i is not a variable and for all $1 \leq j \leq n$ we define $s_j = x$ if $t_j = t_i$ and $s_j = t_j$ otherwise. Furthermore, $\Theta_1 = \{S(y) \mid S(y) \in \Theta \text{ and } y \in \text{vars}(S(f(s_1, \dots, s_n)))\}$ and Θ_2 is the restriction of Θ to atoms with argument $x \in \text{vars}(t_i)$.

By construction, the derived clauses have a solved sort constraint and N'' approximates N' as desired. The sort theory N'' is shallow and satisfiability of sort constraints with respect to shallow sort theories is decidable by the inference rules sort resolution, empty sort and the reduction rules sort simplification, subsumption deletion (Definition 2.3.16) and condensation (Definition 2.3.17) [27, 53]. Hence, this instance of static soft typing is effective.

So if we start with a clause store N that does not contain positive equations, we construct once the approximated sort theory N'' . If this theory is not trivial, i.e., there is at least one monadic predicate S with $N'' \not\models \forall x S(x)$, the sort theory N'' is stored and static soft typing is applied to any input or derived clause. Since N'' is only approximated once, typically at the beginning of the inference process, the rule is called static soft typing. If in the input clause store all sort constraints are solved and there are no positive equations, static soft typing preserves completeness [51, 18].

If equations occur in a clause store a dynamic soft typing approach seems to be more suitable. Consider [18] and [33] for details. These techniques are not implemented in SPAS Version 2.0 but are an option for later releases. Static soft typing is activated by the `-RRSST` option.

2.3.3 Inference Rules

The introduced inference rules can be composed to a variety of (well-known) calculi. The calculi range from the ordinary resolution calculus investigated in Section 2.2 to a superposition calculus with selection, splitting and sort constraints that are subject to the basicness restriction. To cover all these cases, the rules defined here are given in generic way such that each definition covers several variants of the rule.

In particular, all rules are available with a selection restriction of negative literals that does not destroy completeness [3]. For any clause we can select some negative literals with the effect that all inference rule applications taking this clause as a parent clause must involve the selected literals. For example, if we select the literal $R(x, y)$ in the clause $\| R(x, y), f(g(x), y) \approx f(y, z) \rightarrow$ then no equality resolution inference (see below) is possible from this clause.

Definition 2.3.7 (Equality/Reflexivity Resolution)

The inference

$$\mathcal{I} \frac{\Theta \| l \approx r, \Gamma \rightarrow \Delta}{(\Theta \| \Gamma \rightarrow \Delta)\sigma}$$

where (i) σ is the mgu of l and r , (ii) Θ is solved, (iii) $l \approx r$ is selected or $(l \approx r)\sigma$ is maximal in $(\Theta \| l \approx r, \Gamma \rightarrow \Delta)\sigma$ and no literal is selected in Γ is called an *equality resolution* inference. If condition (iii) is replaced by $l \approx r$ is selected or no literal is selected in Γ , the inference is called *reflexivity resolution*.

Equality resolution is activated by the $-\text{IEQR}$ option. Reflexivity resolution is activated by the $-\text{IERR}$ option.

Definition 2.3.8 ((Ordered) Paramodulation/Superposition Left)

The inferences

$$\mathcal{I} \frac{\Theta_1 \| \Gamma_1 \rightarrow \Delta_1, l \approx r \quad \Theta_2 \| s[l']_p \approx t, \Gamma_2 \rightarrow \Delta_2}{(\Theta_1, \Theta_2 \| s[p/r] \approx t, \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2)\sigma}$$

and

$$\mathcal{I} \frac{\Theta_1 \| \Gamma_1 \rightarrow \Delta_1, l \approx r \quad \Theta_2 \| A[l']_p, \Gamma_2 \rightarrow \Delta_2}{(\Theta_1, \Theta_2 \| A[p/r], \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2)\sigma}$$

where (i) σ is the mgu of l' and l , (ii) l' is not a variable, (iii) Θ_1 and Θ_2 are solved (iv) no literal in Γ_1 is selected, (v) $s \approx t$ (the atom A) is selected or no literal in Γ_2 is selected, is called a *paramodulation left* inference. If, in addition, $r\sigma \neq l\sigma$ the inference is an *ordered paramodulation left* inference. If, in addition, $l\sigma \approx r\sigma$ is reductive for $(\Theta_1 \| \Gamma_1 \rightarrow \Delta_1, l \approx r)\sigma$, (v) is replaced by $s\sigma \approx t\sigma$ (the atom $A\sigma$) is selected or it is maximal in $(\Theta_2 \| s \approx t, \Gamma_2 \rightarrow \Delta_2)\sigma$ (in $(\Theta_2 \| A, \Gamma_2 \rightarrow \Delta_2)\sigma$) and no literal in Γ_2 is selected and $t\sigma \neq s\sigma$ then the inference is called a *superposition left* inference.

Standard paramodulation is activated by the $-\text{ISPM}$ option. Ordered paramodulation is activated by the $-\text{IOPM}$ option. Superposition left is activated by the $-\text{ISPL}$ option.

Note that no paramodulation/superposition inference is performed into the sort constraint. Hence, the sort constraint is subject to the basicness restriction. In case all sort constraints of an initial clause store were solved, the basicness restriction preserves completeness.

Definition 2.3.9 ((Ordered) Paramodulation/Superposition Right)

The inferences

$$\mathcal{I} \frac{\Theta_1 \| \Gamma_1 \rightarrow \Delta_1, l \approx r \quad \Theta_2 \| \Gamma_2 \rightarrow \Delta_2, s[l']_p \approx t}{(\Theta_1, \Theta_2 \| \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, s[p/r] \approx t)\sigma}$$

and

$$\mathcal{I} \frac{\Theta_1 \| \Gamma_1 \rightarrow \Delta_1, l \approx r \quad \Theta_2 \| \Gamma_2 \rightarrow \Delta_2, A[l']_p}{(\Theta_1, \Theta_2 \| \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, A[p/r])\sigma}$$

where (i) σ is the mgu of l' and l , (ii) l' is not a variable, (iii) Θ_1 and Θ_2 are solved (iv) no literal in Γ_1, Γ_2 is selected is a *paramodulation right* inference. If, in addition, $r\sigma \neq l\sigma$ the inference is an *ordered paramodulation right* inference. If, in addition, $l\sigma \approx r\sigma$ is reductive for $(\Theta_1 \| \Gamma_1 \rightarrow \Delta_1, l \approx r)\sigma$, $s\sigma \approx t\sigma$ ($A\sigma$) is reductive for $(\Theta_2 \| \Gamma_2 \rightarrow \Delta_2, s \approx t)\sigma$ ($\Theta_2 \| \Gamma_2 \rightarrow \Delta_2, A$) σ the inference is called a *superposition right* inference.

Superposition right is activated by the $-ISP_R$ option.

In SPASS the parallel extensions of the above defined paramodulation/superposition left/right inferences [6] are preferred. Whenever such an inference rule is applicable, we don't only replace the initially found occurrence of $l\sigma$ in the second clause by $r\sigma$, but all occurrences. On the ground level the parallel replacement corresponds to an application of the inference rules exactly the way they are defined above plus exhaustive application of non-unit rewriting with the left premise (Definition 2.3.21) on the conclusion.

Note that the ordering conditions of the above inference rules as well as the ordering conditions of the inference rules defined below, are checked with respect to the found unifier. This is called the a posteriori ordering check. For all inference rules that have ordering restrictions, SPASS first orders the clauses as they are and only searches for inferences with respect to the found candidates (maximal literals, maximal sides of equations). This is called the a priori ordering check. Then, after having found a second candidate clause together with a unifier, the a posteriori check is evaluated. This second check is more expensive than the first, because it has to be dynamically computed with respect to any found unifier. However, since most of the time in a saturation prover is spent with reduction (see Section 2.2), the extra time for the a posteriori check does not matter, but needs some effort for an (efficient) implementation.

The following example shows that the a posteriori check can in fact prevent the generation of extra clauses. Consider the two clauses

$$\begin{aligned} &\rightarrow f(x, y) \approx f(y, x) \\ &\rightarrow P(f(a, b)) \end{aligned}$$

The equation $f(x, y) \approx f(y, x)$ cannot be oriented by any reduction ordering. So without an a posteriori ordering check, we can derive the clause $\rightarrow P(f(b, a))$ by a superposition right inference. Now consider the very same example where we use an RPOS with precedence $f > b > a$ and $status(f) = left$. This implies $f(b, a) \succ_{rpos} f(a, b)$ and therefore the a posteriori ordering check for the potential superposition right inference conclusion $\rightarrow P(f(b, a))$ fails. No inference is possible between the above two clauses.

Next we define three factoring rules, namely (ordered) factoring, equality factoring and merging paramodulation. The different rules are needed to obtain completeness results with respect to different inference rule sets. For the standard resolution/paramodulation calculus [42, 41, 7, 39] the factoring rule without ordering restrictions suffices for completeness. For the ordered resolution/superposition calculus, ordered factoring has to be combined with either equality factoring or merging paramodulation to obtain completeness [3].

Definition 2.3.10 ((Ordered) Factoring)

The inferences

$$\mathcal{I} \frac{\Theta \parallel \Gamma \rightarrow \Delta, E_1, E_2}{(\Theta \parallel \Gamma \rightarrow \Delta, E_1)\sigma}$$

and

$$\mathcal{I} \frac{\Theta \parallel \Gamma, E_1, E_2 \rightarrow \Delta}{(\Theta \parallel \Gamma, E_1 \rightarrow \Delta)\sigma}$$

where (i) σ is the mgu of E_1 and E_2 , (ii) Θ is solved (iii) (E_1, E_2 occur positively, E_1 is maximal and no literal in Γ is selected) or (E_1, E_2 occur negatively, E_1 is maximal and no literal in Γ is selected or E_1 is selected) are called *ordered factoring right* and *ordered factoring left*, respectively. If condition (iii) is replaced by (E_1, E_2 occur positively and no literal in Γ is selected) or (E_1, E_2 occur negatively, E_1 is selected or no literal in Γ is selected) the inferences are called *factoring right* and *factoring left*, respectively.

Standard factoring is activated by the $-ISFC$ option. Ordered factoring is activated by the $-IOFC$ option.

There is an overlap between Ordered Factoring defined above and Equality Factoring defined below, because the rule ordered factoring also considers equations. We did so because for the ordered paramodulation calculus with respect to our definitions Equality Factoring is not needed for completeness. The rule Ordered Factoring suffices for completeness.

Definition 2.3.11 (Equality Factoring)

The inference

$$\mathcal{I} \frac{\Theta \parallel \Gamma \rightarrow \Delta, l \approx r, l' \approx r'}{(\Theta \parallel \Gamma, r \approx r' \rightarrow \Delta, l' \approx r')\sigma}$$

where (i) σ is the mgu of l' and l , (ii) $r\sigma \not\approx l\sigma$, (iii) Θ is solved, (iv) no literal in Γ is selected, (v) $l\sigma \approx r\sigma$ is a maximal occurrence in $(\Theta \parallel \Gamma \rightarrow \Delta, l \approx r, l' \approx r')\sigma$ is called an *equality factoring* inference.

Equality factoring is activated by the $-\text{IEqF}$ option.

Definition 2.3.12 (Merging Paramodulation)

The inference

$$\mathcal{I} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, l \approx r \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2, s \approx t[l']_p, s' \approx t'}{(\Theta_1, \Theta_2 \parallel \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, s \approx t[p/r], s \approx t')\sigma}$$

where (i) σ is the composition of the mgu τ of l and l' and the mgu λ of $s\tau$ and $s'\tau$, (ii) the clause $(\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, l \approx r)\sigma$ is reductive for $l\sigma \approx r\sigma$, (iii) Θ_1 and Θ_2 are solved, (iv) no literal in Γ_1, Γ_2 is selected, (v) the clause $(\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2, s \approx t, s' \approx t')\sigma$ is reductive for $s\sigma \approx t\sigma$, (vi) $s\tau \succ t\tau$, (vii) l' is not a variable is called a *merging paramodulation* inference.

Merging paramodulation is activated by the $-\text{IMPm}$ option.

Definition 2.3.13 ((Ordered) Resolution)

The inference

$$\mathcal{I} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, E_1 \quad \Theta_2 \parallel E_2, \Gamma_2 \rightarrow \Delta_2}{(\Theta_1, \Theta_2 \parallel \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2)\sigma}$$

where (i) σ is the mgu of E_1 and E_2 , (ii) Θ_1 and Θ_2 are solved, (iii) no literal in Γ_1 is selected, (iv) $E_1\sigma$ is strictly maximal in $(\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, E_1)\sigma$, (v) the atom $E_2\sigma$ is selected or it is maximal in $(\Theta_2 \parallel E_2, \Gamma_2 \rightarrow \Delta_2)\sigma$ and no literal in Γ_2 is selected is called *ordered resolution*. If conditions (iv), (v) are replaced by E_2 is selected or no literal is selected in Γ_2 , the inference is called *resolution*.

Standard resolution is activated by the $-\text{ISRe}$ option. Ordered resolution is activated by the $-\text{IORe}$ option. If any of the options is set to 2, equations are also considered for the inferences.

If, in Definition 2.3.13, one of the parent clauses of the inference is a unit, the inference is called *(ordered) unit resolution*. The standard resolution rule is an instance of this rule if we omit the conditions (ii)–(iv) and restrict our attention to non-equational atoms.

Definition 2.3.14 ((Ordered) Hyper Resolution)

The inference

$$\mathcal{I} \frac{\Theta \parallel E_1, \dots, E_n \rightarrow \Delta \quad \Theta_i \parallel \rightarrow \Delta_i, E'_i \quad (1 \leq i \leq n)}{(\Theta, \Theta_1, \dots, \Theta_n \parallel \rightarrow \Delta, \Delta_1, \dots, \Delta_n)\sigma}$$

(i) σ is the simultaneous mgu of $E_1, \dots, E_n, E'_1, \dots, E'_n$, (ii) Θ as well as all Θ_i are solved, (iii) all $E'_i\sigma$ are strictly maximal in $(\Theta_i \parallel \Gamma_i \rightarrow \Delta_i, E'_i)\sigma$ is called an *ordered hyper resolution* inference. If condition (iii) is dropped, the inference is called a *hyper resolution* inference.

Standard hyper resolution is activated by the $-\text{ISHy}$ option. Ordered hyper resolution is activated by the $-\text{IOHy}$ option.

In the application of the inference rule hyper resolution as well as the inference rules sort resolution (Definition 2.3.3) and empty sort (Definition 2.3.4) more than two parent clauses are involved, in general. So the search for candidate clauses gets more complicated. In particular, an appropriate ordering of the literals E_1, \dots, E_n for searching partner clauses can be indispensable for efficiency reasons. For example, if we search partners for the literals $P(x), Q(a, f(x))$ it may be the case that we find thousand potential partners for $P(x)$ (all clauses with a positive (maximal) literal $P(t)$) but only a few for $Q(a, f(x))$ (only clauses with a positive (maximal) literal $Q(a, f(t))$ or with variable occurrences at the positions of $a, f(t)$).

So starting with $Q(a, f(x))$ for partner search is the more efficient way, since it will potentially provide instantiation of x when we subsequently search for partners of $P(x)$. So a good heuristic is to proceed at any time of the partner search with the literals that has a maximal number of symbols with respect to the already established partial unifier. Nevertheless, please note that the number of hyper resolvents grows in the worst case exponentially in n .

2.3.4 Reduction Rules

Our philosophy is that reduction rules are at the heart of successful automated theorem proving. The aim of reduction rules is to transform clauses (or even formulas see [38]) in simpler ones. So whereas inference rules are at the search side of automated theorem proving, reduction rules are at the computation side.

Definition 2.3.15 (Duplicate/Trivial Literal Elimination)

The reductions

$$\mathcal{R} \frac{\Theta \parallel \Gamma \rightarrow \Delta, E, E}{\Theta \parallel \Gamma \rightarrow \Delta, E}$$

and

$$\mathcal{R} \frac{\Theta \parallel \Gamma, E, E \rightarrow \Delta}{\Theta \parallel \Gamma, E \rightarrow \Delta}$$

and

$$\mathcal{R} \frac{\Theta, A, A \parallel \Gamma \rightarrow \Delta}{\Theta, A \parallel \Gamma \rightarrow \Delta}$$

are called *duplicate literal eliminations*. The reductions

$$\mathcal{R} \frac{\Theta \parallel \Gamma, t \approx t \rightarrow \Delta}{\Theta \parallel \Gamma \rightarrow \Delta}$$

and

$$\mathcal{R} \frac{\parallel t \approx s \rightarrow}{\square}$$

where for the final variant we assume that t and s are unifiable, are called *trivial literal eliminations*.

Duplicate/trivial literal elimination are both activated by the `-RObV` option.

Please recall that although trivial literal elimination can be simulated by equality resolution or factoring, for these inference rules to apply a clause must first be selected as *Given* clause. Reduction rules like duplicate or trivial literal elimination apply to all (newly) generated clauses.

Definition 2.3.16 (Subsumption Deletion)

The reduction

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1 \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2}{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1}$$

where $\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2$ is subsumed by $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1$ is called *subsumption deletion*.

Subsumption deletion is activated by the `-RFSUB` and `-RBSUB` option for forward and backward subsumption, respectively (see Section `refsecfirstsimprover`).

Testing subsumption between two clauses is an *NP*-complete problem [19]. Nevertheless, subsumption is indispensable for saturation based theorem proving as we already discussed in Section 2.2. Hence, there exist a variety of papers presenting algorithms that show a polynomial behavior on certain subclasses of clauses (e.g., [21]) or that introduce specific data structures to speed up the subsumption test in practice (e.g., [46, 50]). Many of today's provers use a variant of the [47] algorithm for the subsumption test. Basically, the algorithm tries to find for every literal in $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1$ a different instance in $\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2$ such that all single instantiations are compatible, i.e., identical variables are mapped to identical terms. This simple version is not tractable in practice. Prefilters must be added to the algorithm that make it tractable

in practice. In SPASS we introduced two filters [37]. The first filter is based on the size of the clauses. A necessary condition for a subsumption deletion application over multisets is that $size(\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1) \leq size(\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2)$. Since the size of clauses is usually needed for selection heuristics (see the discussion on the *choose* function in Section 2.2), the size of a clause is already stored in a clause data structure and therefore this test is almost for free. For every two clauses passing this test, the second prefilter checks whether for every literal in $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1$ there exists some instance literal in $\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2$ at all. So we consider the literals in $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1$ separately and don't check compatibility between the different substitutions. This check is again a necessary condition for the subsumption test to succeed and can be done in polynomial time. Clauses passing these two tests are then subject to the Stillman algorithm. In practice, more than 95% of all subsumption tests can already be rejected by the two filters.

Note that there is a subtle difference between multiset subsumption (considered here) and set subsumption. The clause $\rightarrow Q(a, x), Q(y, b)$ subsumes the clause $\rightarrow Q(a, b)$ if we consider clauses to be sets, but does not if we consider clauses to be multisets. Therefore, in our version of the Stillman algorithm we require matched literals to be different.

When integrated into a prover, subsumption deletion is not an operation applied to two clauses but applied to two sets of clauses or a clause and a set of clauses (see Table 2.1 and Table 2.3). The former test can be reduced to the latter by considering the clauses in one set separately. So it remains to test whether some clause C subsumes some clause in a set N or is subsumed by some clause in N . We already argued that the set N (in particular the *Us* set) can become very large. Then it is in practice intractable to traverse all clauses in N and then to apply the subsumption test to each clause. An additional filter is needed: Indexing. Indexing is the data base technology of automated theorem proving. The crucial operations provided by an index of a clause store N are: compute all clauses that include an atom that is an instance/a generalization/unifiable with some query atom. Typically, the result of such a query consists of the clauses together with the found atom. So in order to test whether some clause C is subsumed by a clause in an indexed clause store N , one picks a literal from C that has a low probability of being subsumed, searches the index for generalizations of that literal and then tests C and the found clauses for subsumption. Since the query and the result literal are already found, using appropriate data structures of SPASS it is sufficient to test the clauses without these literals. We use a more general subsumption test with the possibility to hide at least one literal in each clause and we are able to keep the bindings of an indexing query result. This extended test is also needed for the reduction rules matching replacement resolution (Definition 2.3.20) and non-unit rewriting (Definition 2.3.21).

Definition 2.3.17 (Condensation)

The reduction

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1}{\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2}$$

where $\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2$ subsumes $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1$ and $\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2$ is derived from $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1$ by instantiation and (exhaustive) application of trivial literal elimination is called *condensation*.

Condensation is activated by the `-RCon` option.

In the literature condensation is often defined on the basis of factoring applications. From an implementation point of view the above definition is much sharper, because it only suggests matchers to generate duplicate literals that can be eventually removed, not unifiers as suggested by a definition based on factoring. All these candidate instantiation substitutions can be effectively computed by subsequently searching for matchers σ such that $E_1\sigma = E_2$ for $E_1, E_2 \in \Gamma_1$ (respectively for Θ_1, Δ_1) and then testing whether $(\Theta_1 \parallel \Gamma_1 \setminus \{E_2\} \rightarrow \Delta_1)\sigma$ subsumes $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1$. This idea leads to a procedure that is more efficient than the factoring based algorithm suggested by [28] and related to the techniques presented by [20].

Definition 2.3.18 (Tautology Deletion)

The reduction

$$\mathcal{R} \frac{\Theta \parallel \Gamma \rightarrow \Delta}{\quad}$$

where $\models \Theta \parallel \Gamma \rightarrow \Delta$ is called *tautology deletion*.

The above rule is sometimes also called semantic tautology deletion, since it is based on a semantic tautology test. This test corresponds to testing unsatisfiability of a set of ground literals. If we keep in mind that any literal can be coded as an (dis)equation, in order to test unsatisfiability of a set of ground literals it is sufficient to test congruence closure with respect to the positive equations. This can be done in polynomial time [14]. There are certain weaker syntactic conditions that can be checked in linear time:

$$\mathcal{R} \frac{\Theta \parallel \Gamma, E \rightarrow \Delta, E}{\quad}$$

or

$$\mathcal{R} \frac{\Theta, A \parallel \Gamma \rightarrow \Delta, A}{\quad}$$

or

$$\mathcal{R} \frac{\Theta \parallel \Gamma \rightarrow \Delta, t \approx t}{\quad}$$

In SPASS the syntactic as well as the semantic check is implemented (see Appendix 2.5). For the semantic test we adopted the algorithm presented by [14] to our data structures. These conditions are implemented by nearly all today's theorem provers. The semantic check requires appropriate data structures for an efficient implementation. It is contained in the provers E, Saturate and SPASS.

Tautology deletion is activated by the `-RTaut` option. If the option is set to 1 only syntactic tautologies are deleted and if the option is set to 2 also the semantic test is performed.

Definition 2.3.19 (Conflict)

The reduction

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1 \quad \dots \quad \Theta_n \parallel \Gamma_n \rightarrow \Delta_n}{\quad \square}$$

where $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, \dots, \Theta_n \parallel \Gamma_n \rightarrow \Delta_n \models \square$ is called *conflict*.

Even if n is fixed, the rule conflict is not effective, in general. It basically solves the general unsatisfiability problem of first-order logic. The rule sort simplification is an effective instance of this rule. Two further effective instantiations of this rule that are not related to specific theories are implemented in today's provers: *unit conflict* and the *terminator* [1].⁷ The former is the rule

$$\mathcal{R} \frac{\parallel \rightarrow E_1 \quad \parallel E_2 \rightarrow}{\quad \square}$$

such that E_1 and E_2 are unifiable. It seems that this rule is superfluous since it only detects a contradiction between two unit clauses. However, since we consider unification between E_1 and E_2 , this rule cannot be simulated by, for example, matching replacement resolution (see Definition 2.3.20), but only by a resolution step. In the context of problems where the majority of generated clauses are units (e.g., unit equational problems or condensed detachment problems [31]) the probability that both clauses are selected for inferences can become arbitrarily low. Then it can pay off to add this reduction rule that implements a (global) one step search for the empty clause.

The terminator is a generalization of unit conflict and a restriction of the general conflict rule to at most k non-unit clauses out of the n clauses, k fixed. For some given, finite set of clauses it is decidable whether we can derive the empty clause by resolution, if any derivation is restricted to contain at most k non-unit clauses. This is easy to see, since there are only finitely many different derivations using k non-unit clauses and resolving with a unit clause strictly reduces the length of the resolvent compared to the maximal length of one of its parent clauses. That's the terminator. In practice the terminator can be useful with values $k \leq 3$. Larger values rarely make sense, since the number of clauses that have to be considered for this rule grows exponentially in k times the length of the non-unit clauses. Note that if the terminator is applied to a Horn clause store without equality, it can be turned into a complete refutation procedure by subsequently increasing n .

⁷Hasta la vista baby!

As an exception from all other reduction rules, in practice the terminator is integrated in the search procedure like an inference rule, not like a reduction rule. It is too expensive to apply the terminator to all newly generated clauses and often it does not pay off. So the terminator is solely applied to the selected *Given* clauses, if it is activated.

Unit conflict is activated by the `-RUNIT` option. The terminator is activated by the `-RTer=<n>` option, where n specifies the number of considered non-unit clauses.

Definition 2.3.20 (Matching Replacement Resolution)

The reductions

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, E_1 \quad \Theta_2 \parallel \Gamma_2, E_2 \rightarrow \Delta_2}{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, E_1 \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2}$$

and

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1, E_1 \rightarrow \Delta_1 \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2, E_2}{\Theta_1 \parallel \Gamma_1, E_1 \rightarrow \Delta_1 \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2}$$

and

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, A_1 \quad \Theta_2, A_2 \parallel \Gamma_2 \rightarrow \Delta_2}{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, A_1 \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2}$$

where (i) $E_1\sigma = E_2$ ($A_1\sigma = A_2$ for the third variant), (ii) $\Theta_1\sigma \subseteq \Theta_2$, $\Gamma_1\sigma \subseteq \Gamma_2$, $\Delta_1\sigma \subseteq \Delta_2$ are called *matching replacement resolutions*.

Matching replacement resolution is activated by the `-RFMRR` and `-RBMRR` option, for the forward and backward direction, respectively.

Matching replacement resolution is a restricted variant of replacement resolution, itself a restricted form of resolution where the conclusion must subsume one of its parent clauses. For matching replacement resolution we restrict the unifier of the complementary literals computed for replacement resolution to be a matcher. This speeds up the applicability test significantly.

The third variant of the rule that applies to the sort constraint cannot be simulated by sort simplification (Definition 2.3.5), because it also considers clauses that are not declarations. On the other hand, matching replacement resolution can also not simulate sort simplification. Consider the clauses

$$\begin{array}{l} T(x), S(f(x)), \Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1 \\ R(x) \parallel \quad \rightarrow S(f(x)) \\ T(x) \parallel \quad \rightarrow R(x) \end{array}$$

The negative occurrence of $S(f(x))$ in the first clause cannot be eliminated by matching replacement resolution but by sort simplification.

Definition 2.3.21 (Non-Unit Rewriting)

The reductions

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t \quad \Theta_2 \parallel \Gamma_2, E[s']_p \rightarrow \Delta_2}{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t \quad \Theta_2 \parallel \Gamma_2, E[p/t\sigma] \rightarrow \Delta_2}$$

and

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2, E[s']_p}{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2, E[p/t\sigma]}$$

where (i) $s\sigma = s'$, (ii) $s \succ t$, (iii) $\Theta_1\sigma \subseteq \Theta_2$, $\Gamma_1\sigma \subseteq \Gamma_2$, $\Delta_1\sigma \subseteq \Delta_2$ are called *non-unit rewriting*.

Non-unit rewriting and unit rewriting (see below) are activated by the `-RFRew` and `-RBRew` option, for the forward and backward direction, respectively.

The ordering restrictions for non-unit rewriting are a priori ordering restrictions, i.e., we do not compare the terms s and t with respect to the found matcher σ . The ordering test with respect to σ is sharper, but an efficient implementation of this check is non-trivial because it requires a tight connection between indexing, ordering computation and subsumption. Therefore, SPASS uses the a priori ordering check, i.e., SPASS verifies $s \succ t$. See also the discussion on page 23.

Definition 2.3.22 (Unit Rewriting)

The reductions

$$\mathcal{R} \frac{\| \rightarrow s \approx t \quad \| E[s']_p \rightarrow}{\| \rightarrow s \approx t \quad \| E[p/t\sigma] \rightarrow}$$

and

$$\mathcal{R} \frac{\| \rightarrow s \approx t \quad \| \rightarrow E[s']_p}{\| \rightarrow s \approx t \quad \| \rightarrow E[p/t\sigma]}$$

where (i) $s\sigma = s'$, (ii) $s\sigma \succ t\sigma$, are called *unit rewriting*.

Unit rewriting is an instance of the second version of non-unit rewriting where all $\Theta_i, \Gamma_i, \Delta_i$ are empty. We mention it here explicitly, because it is the style of rewriting used in purely equational completion, a theorem proving discipline of its own. Furthermore, the a posteriori ordering check is much easier to implement, because we need no subsumption check. Actually, it is implemented in SPASS.

In practice the rewriting reductions are among the most expensive reductions. Note that any subterm of any clause has to be considered and that subsequent rewriting steps to the same clause are common. Therefore, many provers don't use the full power of non-unit rewriting, but restrict the left clause to be a positive unit equation. They reduce non-unit clauses by positive unit equations.

Even the a posteriori check, condition (ii), can be further refined. Consider an equation where the left and right hand side don't share any variables. Then the a posteriori check will typically fail but may succeed by appropriate further instantiations. For example the equation $f(x, y) \approx g(z)$ cannot be oriented and hence the equation $f(a, b) \approx a$ cannot be rewritten by unit rewriting using the first equation. Now assume a RPOS with precedence $f > g > a > b$. Then the equation $f(a, b) \approx g(z)$ (the result of matching $f(x, y)$ with $f(a, b)$) can be turned into an oriented equation by instantiating z with a or b . This enables rewriting of $f(a, b)$ to $g(a)$ or $g(b)$. In general it is sufficient to consider the minimal constant and the crucial extra variables for further instantiation. Note also that the equation $f(x, y) \approx g(z)$ subsumes an equation like $f(x, y) \approx g(y)$ that is oriented and can therefore be used for rewriting in a straightforward way. This refinement is not implemented in SPASS Version 2.0 but remains an option for further releases.

Another way to solve the problem of unorientable equations because of extra variables is to split equations. Given some equation $s \approx t$ where $\text{vars}(s) \not\subseteq \text{vars}(t)$ and $\text{vars}(t) \not\subseteq \text{vars}(s)$, we introduce a new function symbol h where the arity of h is exactly $|\text{vars}(s) \cap \text{vars}(t)|$. If $\{x_1, \dots, x_n\} = \text{vars}(s) \cap \text{vars}(t)$ then the equation $s \approx t$ is replaced by the equations $s \approx h(x_1, \dots, x_n)$ and $t \approx h(x_1, \dots, x_n)$. Given a KBO or RPOS and a precedence where the new symbol h is smaller than the top symbols of s and t , both introduced equations are oriented from left to right. In order to obtain a complete calculus that includes splitting of equations splitting must not be applied infinitely many times. Splitting equations is not implemented in SPASS Version 2.0.

Definition 2.3.23 (Contextual Rewriting)

The reductions

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t \quad \Theta_2 \parallel \Gamma_2, E[s']_p \rightarrow \Delta_2}{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t \quad \Theta_2 \parallel \Gamma_2, E[p/t\sigma] \rightarrow \Delta_2}$$

and

$$\mathcal{R} \frac{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2, E[s']_p}{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2, E[p/t\sigma]}$$

where (i) $s\sigma = s'$, (ii) $s \succ t$, (iii) $s \approx t$ is strictly maximal in $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t$, (iv) for any term t' in $\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1, s \approx t, s \succ t'$, (v) if $E[s']_p$ does not occur negatively, $s\sigma \approx t\sigma \prec E[s']_p$, (vi) $\models \text{fred}(\Theta_2 \rightarrow S(t'))$ for all sort atoms $S(t') \in \text{sortsimp}(\Theta_1\sigma)$, (vii) $\models \text{fred}(\Gamma_2 \rightarrow E)$ for all atoms $E \in \Gamma_1\sigma$, (viii) $\models \text{fred}(E \rightarrow \Delta_2)$ for all atoms $E \in \Delta_1\sigma$ are called *contextual rewriting*.

The expression $\text{sortsimp}(\Theta_1\sigma)$ denotes the sort constraint $\Theta_1\sigma$ after exhaustive application of the rule sort simplification. The function *fred*, see Table 2.5, (recursively) applies the reduction rules to the constructed subclauses before they are checked as tautologies. The performance of *fred*, i.e. which rules are actually tested/applied, determines the strength as well as the cost for testing and applying contextual rewriting. Our current idea is to use all reduction rules except contextual rewriting. We do not test/apply contextual rewriting recursively. Contextual rewriting is not implemented in SPASS Version 2.0 but will be included in the next release. We already have a prototype implementation of the rule.

The final reduction rule exploits particular equations of the form $x \approx t$ (called assignment equations), where x does not occur in t nor in the rest of the clause. Negative equations of this form can simply be removed from a clause. In order to remove positive assignment equations the domain structure shared by any model of the current clause store has to be examined. In particular, we exploit the case that the domain is non-trivial. Therefore, before this rule can be applied, certain properties of any clause store model must be checked. This can, e.g., be done by a sufficient criterion that can be tested syntactically.

Definition 2.3.24 (Assignment Equation Deletion)

Let N be the current clause store. The reductions

$$\mathcal{R} \frac{\Theta \parallel x \approx t, \Gamma \rightarrow \Delta}{\Theta \parallel \Gamma \rightarrow \Delta}$$

and

$$\mathcal{R} \frac{\Theta \parallel \Gamma \rightarrow x \approx t, \Delta}{\Theta \parallel \Gamma \rightarrow \Delta}$$

where for both variations of the rule we assume (i) $x \notin \text{vars}(t)$, (ii) $x \notin \text{vars}(\Theta \parallel \Gamma \rightarrow \Delta)$ and for the second variant, where we remove a positive equation, we assume in addition that $|\mathcal{D}| > 1$ for any interpretation \mathcal{M} with $\mathcal{M} \models N$, are called *assignment equation deletion*.

Assignment equation deletion is activated by the `-RAED` option. If `-RAED` is set to 1 only the first variant is applied, if set to 2 both variants are used.

For the elimination of the positive equation to be sound it is necessary to guarantee a non-trivial domain for any model of the current clause store. A syntactic condition is the existence of a clause $\parallel s \approx t \rightarrow$ where s and t are arbitrary. If such a clause is contained in the clause store, the domain of any model is non-trivial and we can apply the second variant of the rule.

2.3.5 Splitting

The effect of a splitting rule application is not only to extend the current clause store but also to modify and extend the current clause store collection.

Definition 2.3.25 (Splitting)

The inference

$$\mathcal{S} \frac{\Theta_1, \Theta_2 \parallel \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2}{\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1 \quad | \quad \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2}$$

where $\text{vars}(\Theta_1 \parallel \Gamma_1 \rightarrow \Delta_1) \cap \text{vars}(\Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2) = \emptyset$ and $\Delta_1 \neq \emptyset, \Delta_2 \neq \emptyset$ is called *splitting*.

Splitting is activated by the `-split s=<n>` option, where n specifies the overall number of splitting applications in a SPASS run. Unlimited application of the rule can be achieved by choosing -1 for n .

Without the condition that the two split clauses must not share variables, Splitting is very much like the β -rule of free variable tableau. Since the clauses don't share variables, the two cases are completely independent and the derived clauses can be used for simplification/reduction without any restriction. For example, both clauses subsume the parent clause.

In case the first split part is ground, i.e., $\text{vars}(\overline{S(t)} \parallel \overline{E} \rightarrow \overline{E'}) = \emptyset$, where $\overline{S(t)} = S_1(t_1), \dots, S_n(t_n)$, $\overline{E} = E_1, \dots, E_m$ and $\overline{E'} = E'_1, \dots, E'_l$ and $1 \leq i \leq n$, $1 \leq j \leq m$, $1 \leq k \leq l$, it is very useful to add the negation of the first split clause to the second part

$$S \frac{\overline{S(t)}, \Theta_2 \parallel \overline{E}, \Gamma_2 \rightarrow \overline{E'}, \Delta_2}{\overline{S(t)} \parallel \overline{E} \rightarrow \overline{E'} \quad \left| \begin{array}{l} \Theta_2 \parallel \Gamma_2 \rightarrow \Delta_2 \\ \parallel \rightarrow S_i(t_i) \\ \parallel \rightarrow E_j \\ \parallel E'_k \rightarrow \end{array} \right.}$$

All these additional unit clauses can help a lot in reducing the clause set of the second part (see matching replacement resolution, Definition 2.3.20 or non-unit rewriting, Definition 2.3.21). In a purely propositional setting, a calculus solely based on unit conflict (see Definition 2.3.19) and extended splitting can polynomially simulate truth tables, whereas a calculus based on unit conflict and the simple splitting rule cannot [9].

Of course, for any first part split clause we could add its negation to the second part. However, in general this leads to the introduction of new Skolem constants, and in practice this tends to extend the search space for the second part. Note that in this case the ground units resulting from the negated first part cannot be used for matching replacement resolutions, because the introduced Skolem constants are new. As an alternative one could also record the ground instances of the variables in the first split clause used in the refutation of the first part and then add their negation as a disjunct to the second part. But it is questionable whether such an effort pays off in practice.

Splitting itself often tends to generate a huge search tree, so additional refinements are necessary. Therefore, we required that Δ_1 and Δ_2 are non-empty in Definition 2.3.25. So we only split non-Horn clauses into clauses having strictly less positive literals. The rationale behind this comes from the propositional level. For a set of propositional Horn clauses, satisfiability can be decided in linear time [13], whereas satisfiability for arbitrary clauses is an *NP*-complete problem. The reduction rule matching replacement resolution (Definition 2.3.20) is also a decision procedure for propositional Horn clauses (although it results in a quadratic time implementation). Hence, non-Horn splitting and matching replacement resolution are a reasonable decision procedure for propositional clauses. In case a clause can be split into a propositional part (no variables) and a non-propositional one, it is very useful to split the clause that way and to add the negation of the propositional part to the second as indicated before.

An alternative to an explicit case analysis is to split clauses by the introduction of new propositional symbols. For example, the clause

$$S(x) \parallel f(x) \approx y \rightarrow Q(x, x), Q(a, z)$$

can be replaced by the clauses

$$\begin{array}{l} S(x) \parallel f(x) \approx y, A \rightarrow Q(x, x) \\ \parallel \quad \quad \quad B \rightarrow Q(a, z) \\ \parallel \quad \quad \quad \rightarrow A, B \end{array}$$

where A, B are new propositional symbols. The replacement preserves satisfiability of the current clause store and if it is only applied finitely many often during a proof attempt it also preserves completeness. If A, B are minimal in the ordering, no inference on A, B will be performed as long as other literals are contained in the respective clauses. So the different parts of the original clause don't interfere as long as they are completely resolved. This simulates splitting without the need to extend the notion of a clause store to a collection of clause stores and hence a less complicated implementation. The second advantage of this approach is that it does not introduce the inherent redundancy of an explicit splitting approach. The main disadvantage of this splitting style is that none of the generated clauses can be directly used for

1	PROVER(N)
2	$Wo := \emptyset$;
3	$Us := ired(N, N)$;
4	While ($Us \neq \emptyset$ and $\square \notin Us$) {
5	($Given, Us$) := choose(Us);
6	$Wo := Wo \cup \{Given\}$;
7	$New := inf(Given, Wo)$;
8	(New, Wo, Us) := ired(New, Wo, Us);
9	}
10	If ($Us = \emptyset$) then print “Completion Found”;
11	If ($\square \in Us$) then print “Proof Found”;

Table 2.4: The Overall Loop without Splitting

reductions, because the propositional variables A, B must be new. This splitting style is available in the provers Saturate and Vampire, explicit splitting (Definition 2.3.25) is available in SPASS.

2.4 Global Design Decisions

2.4.1 Main-Loop

The main-loop without splitting, Table 2.4, is a generalization of the main-loop introduced in Section 2.2, Table 2.1.

Compared to the simple, resolution-based prover, all inferences are computed in the extra function *inf* and (inter)reduction takes place in the function *ired*. In the automatic mode of SPASS, the inference rules applied in *inf* are chosen after an analysis of the input problem, such that the resolution calculus is sound and complete. For example, if the input problem contains no equality, the superposition/paramodulation rules are not activated or if the input problem is Horn, factoring rules are not needed. In Appendix 2.5.1 we sum up all SPASS options needed to (de)activate the inference/reduction rules introduced in Section 2.3. Please note that a manual setting of inference/reduction rules can result in an unsound or incomplete calculus. SPASS does not verify your manual settings.

The combination of the reduction rules gets more subtle compared to the combination of subsumption/tautology deletion presented in Section 2.2. The function *ired* serves this purpose, Table 2.6. Please recall that the terminator is integrated like an inference rule and hence does not show up. First, line 4, any newly derived clause is forward reduced with respect to the sets Wo and Us . The function *fred* is presented in detail in Table 2.5. The ordering of the tested forward reductions is determined by potential dependencies between the rules and by their respective implementation costs.

We don’t consider the lazy reduction approach introduced in Section 2.2, Table 2.3. It is a bit tricky but not too difficult to develop it out of the presented full reduction algorithms. Inside the algorithms a redundant clause is not always directly deleted, but represented by the constant \top .

In practice, tautology deletion (Table 2.5, line 2), elimination of trivial literals (line 4), condensation (line 5) and assignment equation deletion (line 6) are cheap operations, because only the derived clause has to be considered for testing their applicability. This is not completely true for the assignment equation deletion (see Definition 2.3.24), but the suggested syntactic domain size criterion can be tested once at the beginning of the search process, so no extra effort is necessary. Clauses that pass these tests, are checked for forward subsumption with respect to Wo and Us (line 7) and for forward rewriting (line 8) where *Hit* is set to true, if a rewriting step actually took place. If a rewriting step is performed, the rules tautology deletion,


```

1 fred(Given, Wo, Us)
2 Given := taut(Given);
3 If (Given =  $\top$ ) then return( $\top$ );
4 Given := obv(Given);
5 Given := cond(Given);
6 Given := aed(Given, Wo, Us);
7 If (fsub(Given, Wo, Us)) then return( $\top$ );
8 (Hit, Given) := frew(Given, Wo, Us);
9 If (Hit) then {
10   Given := taut(Given)
11   If (Given =  $\top$ ) then return( $\top$ );
12   Given := obv(Given);
13   Given := cond(Given);
14   If (fsub(Given, Wo, Us)) then return( $\top$ );
15 }
16 Given := ssi(Given, Wo, Us);
17 Given := fmrr(Given, Wo, Us);
18 Given := unc(Given, Wo, Us);
19 Given := sst(Given, Wo, Us);
20 return(Given);

```

Table 2.5: Forward Reduction

1	$ired(New, Wo, Us)$
2	While $(New \neq \emptyset)$ {
3	$(Given, New) := choose(New);$
4	$Given := fred(Given, Wo, Us);$
5	If $(Given \neq \top)$ then {
6	$(New, Wo, Us) := bsub(Given, New, Wo, Us);$
7	$(New, Wo, Us) := bmrr(Given, New, Wo, Us);$
8	$(New, Wo, Us) := brew(Given, New, Wo, Us);$
9	$Us := Us \cup \{Given\};$
10	}
11	}
12	return $(\emptyset, Wo, Us);$

Table 2.6: Interreduction

elimination of duplicate/trivial literals, condensation and forward subsumption are checked a second time (lines 10–14). Below is a simple example that demonstrates dependencies between the different reduction rules.

1:	$\rightarrow f(x) \approx x$
2:	$\rightarrow a \approx b$
3:	$P(f(x)) \rightarrow P(x)$
4:	$P(f(x)), P(b) \rightarrow$
5:	$P(a), P(c) \rightarrow$
6:	$P(g(f(x))), P(g(x)) \rightarrow$

The clauses 3–6 are completely interreduced with respect to the reductions presented in Section 2.3. Rewriting with clause 1 into clause 3 generates a syntactic tautology, rewriting with clause 1 into clause 4 enables a further condensation step on clause 4 resulting in $4': P(b) \rightarrow$ and rewriting clause 5 with clause 2 produces $5': P(b), P(c) \rightarrow$ (assuming $a \succ b$) that is forward subsumed by clause $4'$. After rewriting with clause 1, duplicate literal elimination can be applied to clause 6.

Finally the reductions sort simplification (line 16), forward clause reduction (line 17), unit conflict (line 18) and static soft typing are tested. These reduction rules don't enable further applications of other rules, because they either strictly reduce the number of literals or reduce the clause to \top (static soft typing). In order to test the applicability of these rules the overall clause stores Wo and Us must be considered.

All clauses that pass forward reduction (Table 2.6, line 4) are used for back reduction (lines 6–8) and are finally added to the Us set (line 9). Backward subsumption (line 6) deletes all clauses from Wo and Us that are subsumed by $Given$. Backward matching replacement resolution (line 7) tests all clauses in Wo and Us for matching replacement resolution with $Given$. Reduced clauses are always deleted from their respective source set and added to New . Back rewriting (line 8) behaves the same, but tests rewriting. The clauses in New are not directly tested for all these reduction rules. They are tested after having entered the Us set. This is motivated by efficiency issues which we will discuss below.

The function *choose* (Table 2.6, line 3) selects a clause with the smallest number of symbols. Small clauses have a higher probability to reduce other clauses. For example, a subsuming clause must have fewer symbols (consider the discussion after Definition 2.3.16) than the clause it subsumes. For many other reductions like rewriting, selecting small clauses is still a good heuristic because the size ordering of terms is often included in the reduction ordering.

Note that a clause can be selected several times as *Given* clause in the while-loop of the interreduction

1	PROVER(N)
2	$Wo := \emptyset$;
3	$Us := ired(N, N)$;
4	$Stack := emptystack()$;
5	While ($Us \neq \emptyset$ and ($\square \notin Us$ or not $stackempty(Stack)$)) {
6	If ($\square \in Us$) then
7	$(Stack, Wo, Us) := backtrack(Stack, Wo, Us)$;
8	else {
9	$(Given, Us) := choose(Us)$;
10	If ($splittable(Given)$) then {
11	$New := firstsplitcase(Given)$;
12	$Stack := push(Stack, secondsplitcase(Given))$;
13	}
14	else {
15	$Wo := Wo \cup \{Given\}$;
16	$New := inf(Given, Wo)$;
17	}
18	$(New, Wo, Us) := ired(New, Wo, Us)$;
19	}
20	}
21	If ($Us = \emptyset$) then print "Completion Found";
22	If ($\square \in Us$) then print "Proof Found";

Table 2.7: The Overall Loop with Splitting

algorithm, if it is successfully reduced several times. Selecting small *Given* clauses tries to minimize the number of such situations.

The main-loop presented in Table 2.7 extends the already discussed main-loop of Table 2.4 with splitting. Whether SPASS applies splitting or not depends on the input problem⁸ and can be controlled by the splitting option (see Appendix 2.5.2). In case splitting is not applied, executing the main-loop in Table 2.7 or Table 2.4 results in exactly the same behavior.

If splitting is possible (Table 2.7, line 10) it is preferred over all other inferences. The rationale behind this decision is that a splitting application results in a strictly smaller clause store collection. The exploration of the binary tree generated by the splitting rule is performed by a standard depth-first, backtracking search (lines 7, 11, 12). Compared to the SPASS Versions 1.0.x, Version 2.0 has a more sophisticated splitting clause selection. It selects the clause with the highest unit reduction potential after the split.

All functions implementing inference/reduction operations have to be refined and must also consider the *split level* of a clause. Initially, all clauses have split level zero and the current split level is zero. Then any clause generated by a splitting inference gets the current split level plus one as its split level and the current split level is incremented. Clauses generated by all other inference/reduction rules get the maximal split level of their parent clauses as their new split level. Backtracking resets the current split level to the split level of the activated branch. But what happens if a clause *C* is now subsumed by a clause *D* with a

⁸SPASS only splits non-Horn clauses, see Definition 2.3.25.

greater split level? We must not delete C , but only remove it from the current clause store, store it at D 's split level on the split stack and reinsert it if backtracking considers that level. Clauses that are rewritten or reduced by clauses with a higher split level must be copied and also kept appropriately on the split stack.

Basically that is all to integrate splitting into a saturation based prover. Nevertheless, some refinements are possible. First, since all clauses have a split level, also the empty clause has a split level. This level indicates where backtracking should start to consider open branches and all branches at a higher split level can be discarded. For example, if we derive the empty clause at split level zero, then we can immediately stop and don't have to consider any further possibly open branches. Second, if we don't only store the split level with each clause but also a bit array with length of the split level, the following improvement is possible. The bit array is updated together with the split level and indicates every level that contributed to the clause. If we now derive an empty clause at some split level and detect that it does not depend on some earlier levels above the previous backtracking level that have open branches left, we can erase these levels, their split clauses and all clauses depending on these. We call this operation *branch condensation* and it is indispensable to make splitting feasible in practice. In the AI literature branch condensation it often referred to as dependency directed backtracking.

In Section 2.2 we also introduced a main-loop with lazy reduction, Table 2.3. Although we did not present it here, lazy reduction is also possible with the extended inference rule set and it is not too difficult to think of lazy extensions of the main-loops according to Table 2.3. Therefore, we omitted an extra presentation here.

2.4.2 Proof Documentation/Checking

By default, SPASS does not output a proof in case it derives the (final) empty clause nor does SPASS provide a final saturated set of clauses, in case all possible inferences have been performed without finding an empty clause. This can be changed by activating the proof documentation option (see Appendix 2.5.1).

Proof documentation is possible by implicitly or explicitly storing all clauses during the overall search process that might contribute to a proof. As a consequence, a run with proof documentation has a higher memory consumption and thus causes the prover to slow down. This effect is further supported by splitting applications, where all clauses from all significant branches must be kept as well. Therefore, in favor of execution speed, by default SPASS does not output proofs (saturated clause stores). Nevertheless, SPASS can handle proofs of several hundred thousand steps in reasonable time.

Automated proof checking is a very important topic in any theorem proving project. The inference/reduction rules are non-trivial to (efficiently) implement, so there is a high potential for bugs. This is also shown by several a posteriori disqualifications at the CASC theorem proving competitions happened so far [49]. Proofs of automated theorem provers cannot be checked by hand in practice. So there is a need for automated proof checking. Our solution is a separately implemented proof checker. The checker takes a proof and starts with an analysis of the splitting rule applications. The checker generates the binary tree resulting from subsequent splitting inference applications and tests whether all branches contain an empty clause, whether the split level assignments are done correctly and whether the splitting inference rule is applied in a sound way. Then the checker generates for every inference/reduction rule application the corresponding first-order theorem proving problem and provides it for a separate prover. The single step proof problems can typically be easily solved if they are correct. This way, it is possible to validate proofs up to several hundred thousand steps in reasonable time and that completely automatically. As such a proof checker solely relies on logical implication, it supports most of today's saturation-based inference systems and is robust against modifications to inference/reduction rules.

2.4.3 Data Structures and Algorithms

In Section 2.2 we discussed a simple prover based on resolution. In Section 2.4.1 we extended this prover to cope with the inference/reduction rules of SPASS. For our simple resolution-based prover we already argued that

- the Us set grows very fast,

- reductions are indispensable to reduce the number of clauses in the Us set
- most of the time is spent with reductions.

The situation is getting even more dramatic if we consider the inference rules for equality introduced in Section 2.3 and the suggested main-loops (Table 2.4, Table 2.7). For example, if the selected *Given* clause C in the main-loop contains a positive equation, then any non-variable subterm of any literal of a clause in Wo that unifies with the left or right hand side of the equation generates a new clause by paramodulation. If the considered left hand side is a variable then any subterm of any different clause unifies with the variable and produces a new clause via paramodulation. An example for such a clause is one that forces a finite, two element domain: $\rightarrow x \approx a, x \approx b$. Ordering restrictions improve the situation (not for the finite domain clause), but we nevertheless have to find reasonable ways to store large Us sets and to efficiently find reduction/inference partners.

There are several solutions to these problems. We now focus on one solution and discuss alternatives at the end of this section. The first design decision in SPASS is to store all atoms in Wo , Us in a shared way, respectively. That means every occurrence of any subterm of an atom in Us (Wo) exists exactly once and is shared by all superterms containing this subterm. The idea is to save space and to keep indexing structures small. As all terms are shared, any subterm is only submitted once to the indexing structure that provides retrieval for inferences/reductions. This works fine for ground terms, but in general clauses contain variables that are considered different if they occur in different clauses. Therefore, almost nothing can be shared between two different non-ground clauses. The solution to this problem is our second design decision that is to *normalize* variables of clauses. For any clause, if the clause is considered from left to right as a sequence of its literals, the variables are named with respect to their occurrence according to a fixed variable sequence. After normalization there is a high probability that clauses with variables share non-ground subterms. This is confirmed by experiments.

A consequence of this decision is that algorithms for unification/matching/generalization have to keep track of this fact. For example, we have variants of the unification algorithm that use two substitutions (which are called *contexts* inside the unification algorithm) in order to store bindings between variables for two terms stemming from two different clauses in an appropriate way.⁹

Putting newly generated clauses into a sharing/indexing data structure is extra effort. Since newly generated clauses have a high probability of being subsumed or reduced by already existing clauses, the *New* set (Table 2.4, 2.7, 2.6) is kept unshared. Furthermore, reductions are (by definition) destructive, so they can only be efficiently applied to unshared clauses. Shared clauses have to be extracted/copied from the sharing structure before modification, because destructive manipulation of shared terms may also effect other clauses where the considered reduction is not permitted. So any forward reduction to the *Given* clause inside the interreduction algorithm (Table 2.6) is done destructively, but before a back reduction operation can actually be performed (lines 6–8) the clause has to be extracted/copied from the Wo , Us sharing structure first. As it is unshared afterwards, it can be moved to the *New* set. At line 9 of the *ired* function (Table 2.6), the *Given* clause is inserted into the sharing/indexing structures of the Us set.

An alternative solution not implemented in SPASS is to completely abstract from variable positions, e.g., by introducing one dummy variable for all occurring variables. We build one or several atom/term trees that represent all atoms/terms without considering variables to be different. These trees are then linked to the real atoms/terms and efficient algorithms can be devised to search for candidate atoms/terms out of such skeleton trees. Atoms/terms found this way are still candidates, because of the variable abstraction. They have then to be verified by the appropriate matching/unification test. The so called discrimination trees support such an approach.

If we resign from complete interreduction and focus on lazy reduction (see Table 2.3), no reduction rule needs to be tested with respect to the Us set. So inserting the Us clauses into an indexing structure for access is not needed. The Us clauses are only needed to provide a pool from which the next *Given* clause is selected. To this end, we only need the necessary information for the *choose* function and the clause itself. If the *choose* function relies on size, the necessary information is simply a number. Since all Us clauses are children of two Wo clauses, instead of storing the clause, we store the numbers of the parents

⁹See the discussion in Section 2.2.

and the way it was generated represented by, say, one extra number. So, every clause in the Us set can be represented by four numbers, in practice in constant space. This results in a huge reduction of memory consumption and hence in an increase of execution speed. The Waldmeister system (see Appendix 2.7) can treat the Us set this way. Note that the extra time needed to generate the *Given* clause can be neglected. In case a parent of a selected *Given* clause is no longer in the Wo set, it must have become redundant, hence the *Given* clause is redundant as well and needs not to be considered.

A further possibility to restrict the number of clauses in the Us set is to simply throw away clauses. This may cause incompleteness of the theorem prover. Such techniques are available in Fiesta, Otter, SPASS and Vampire, see Appendix 2.7 and the discussion in Section 2.2.

2.5 SPASS Version 2.0 Options

From a (unix) shell, SPASS is called

```
SPASS [options] [<input-file>]
```

where `options` influence the behavior of the prover. Here, we only introduce SPASS options that relate to the content of this chapter. Further options, e.g., options controlling the output, are documented in the SPASS man-pages. The options discussed here apply to SPASS Versions 2.0. Options can be set to integer values. For boolean options 0 means falsity and 1 means truth. For example, the option `-IMPm=1` enables the inference rule merging paramodulation which can be abbreviated by `-IMPm` whereas `-IMPm=0` disables the inference rule.

2.5.1 Control

<code>Auto</code>	Automatic Mode, after a problem analysis, all options are set automatically.
<code>FullRed</code>	Full Reduction, Section 2.2. If full reduction is enabled, the overall SPASS loop corresponds to the loop presented in Table 2.1, if the option is disabled, it corresponds to the lazy reduction loop presented in Table 2.3.
<code>BoundMode</code>	Bound Mode selects the mode for resource controlled generation of the search space, Section 2.2. If set to 1 clauses are weight restricted, if set to 2 clauses are depth restricted.
<code>BoundStart</code>	Bound Start determines the start value for resource restriction, Section 2.2.
<code>BoundLoops</code>	Bound Loops determines the number of resource restricted main-loop iterations.
<code>DocProof</code>	activates proof documentation.

2.5.2 Inference Rules

<code>ISoR</code>	Sort Constraint Resolution, Definition 2.3.3.
<code>IEmS</code>	Empty Sort, Definition 2.3.4.
<code>IEqR</code>	Equality Resolution, Definition 2.3.7.
<code>IERR</code>	Reflexivity Resolution, Definition 2.3.7.
<code>ISpL</code>	Superposition Left, Definition 2.3.8.
<code>IOPm</code>	Ordered Paramodulation, Definition 2.3.8 and Definition 2.3.9.
<code>ISpM</code>	(Standard) Paramodulation, Definition 2.3.8 and Definition 2.3.9.
<code>ISpR</code>	Superposition Right, Definition 2.3.9.
<code>IOFc</code>	Ordered Factoring, Definition 2.3.10.
<code>ISFc</code>	(Standard) Factoring, Definition 2.3.10.
<code>IEqF</code>	Equality Factoring, Definition 2.3.11.

IMPm	Merging Paramodulation, Definition 2.3.12.
IORe	Ordered Resolution, Definition 2.3.13.
ISRe	(Standard) Resolution, Definition 2.3.13.
IOHy	Ordered Hyper Resolution, Definition 2.3.14.
ISHy	(Standard) Hyper Resolution, Definition 2.3.14.
Splits	Splitting, Definition 2.3.25. The option determines the number of splitting applications where any negative number means that splitting is not restricted.

2.5.3 Reduction Rules

RSSi	Sort Simplification, Definition 2.3.5.
RSST	Static Soft Typing, Definition 2.3.6.
RObv	Trivial Literal Elimination, Definition 2.3.15.
RFSub	Forward Subsumption Deletion, Definition 2.3.16, Table 2.5.
RBSub	Backward Subsumption Deletion, Definition 2.3.16, Table 2.6.
RCon	Condensation, Definition 2.3.17.
RTaut	Tautology Deletion, Definition 2.3.18. If the option is set to 1 only syntactic tautologies are eliminated. If it is set to 2, semantic tautologies are deleted as well.
RUnC	Unit Conflict, Definition 2.3.19.
RTer	Terminator, Definition 2.3.19, where the value of the option determines the number of non-unit clause occurrences in the searched refutation.
RFMMR	Forward Matching Replacement Resolution, Definition 2.3.20, Table 2.5.
RBMMR	Backward Matching Replacement Resolution, Definition 2.3.20, Table 2.6.
RFrew	Forward Rewriting, Definition 2.3.21 and Definition 2.3.22, Table 2.5.
RBrew	Backward Rewriting, Definition 2.3.21, Table 2.6.
RAED	Assignment Equation Deletion, Definition 2.3.24. If set to 2, it is assumed that any model has a non-trivial domain and the corresponding eliminations are performed.

2.6 Pointers into the SPASS Source Code

The below tabular relates algorithms presented in this chapter to the actual source code. It is meant to provide a starting point to explore further details or to adapt the code to personal desires. For every topic, we point to the SPASS source file and the name of the corresponding function.

Main-Loop, Table 2.7	→ top.c
	→ top_ProofSearch
fred, Table 2.5	→ rules-red.c
	→ red_CompleteReductionOnDerivedClause
ired, Table 2.6	→ rules-red.c
	→ red_CompleteReductionOnDerivedClauses
inf, Table 2.7	→ rules-inf.c
	→ inf_DerivableClauses

2.7 Links to Saturation Based Provers

Bliksem by Hans de Nivelle
<http://www.mpi-sb.mpg.de/~nivelle/>

Discount	by Jörg Denzinger http://agent.informatik.uni-kl.de/denzinge/denzinger.html
E	by Stephan Schulz [44] http://www.jessen.informatik.tu-muenchen.de/personen/schulz.html
Fiesta	by Robert Nieuwenhuis, Pilar Nivela and Guillem Godoy http://www.lsi.upc.es/~roberto/
Gandalf	by Tanel Tammet http://www.cs.chalmers.se/~tammet/gandalf/
Otter	by William McCune [32] http://www-unix.mcs.anl.gov/AR/otter/
Saturate	by Harald Ganzinger, Robert Nieuwenhuis and Pilar Nivela [36] http://www.mpi-sb.mpg.de/SATURATE/
SPASS	by Christoph Weidenbach, Bijan Afshordel, Enno Keen, Christian Theobalt, Dalinor Topić [54] http://spass.mpi-sb.mpg.de/
Vampire	by Alexandre Riazanov and Andrei Voronkov [40] http://www.cs.man.ac.uk/fmethods/vampire/
Waldmeister	by Arnim Buch, Thomas Hillenbrand, Roland Vogt, Bernd Löchner and Andreas Jaeger [25] http://agent.informatik.uni-kl.de/waldmeister/

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As mentioned in the introduction the development of a competitive theorem prover is a challenging software project, exemplified in the following for the SPASS theorem prover: Although there existed some preliminary versions of SPASS before 1994, the first version called SPASS was started in that year and was finished in 1995 by Bernd Gaede in the context of his diploma thesis. This version already relied on a library of data structures we called EARL.¹⁰ The library already contained indexing support and was developed by Peter Graf and Christoph Meyer and myself. Clause normal form translation was added to SPASS by Georg Rock as a diploma project. Further development of SPASS took place in paid student projects that typically lasted for several months each. Christian Cohrs introduced splitting to SPASS, Enno Keen was responsible for inference rules and parsing support, Thorsten Engel wrote our proof checker, Dalibor Topić significantly improved our memory management module and contributed to the implementation of reduction rules and Christian Theobalt wrote a whole bunch of documentation support scripts and mastered the challenge to port SPASS to the Windows world. As a prerequisite he developed a neat graphical user interface. Bijan Afshordel contributed to reductions on the formula level and programmed the atom definition module, Uwe Brahm was indispensable for putting SPASS on the Web and Christof Brinker added the most recent development, the detection and deletion of non-syntactic tautologies. Thanks to all of them.

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¹⁰Efficient Automated Reasoning Library

Bibliography

- [1] Grigorios Antoniou and Hans Jürgen Ohlbach. Terminator. In Alan Bundy, editor, *Proceedings of 8th International Joint Conference on Artificial Intelligence, IJCAI-83*, pages 916–919, 1983.
- [2] Franz Baader and Tobias Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998.
- [3] Leo Bachmair and Harald Ganzinger. Rewrite-based equational theorem proving with selection and simplification. *Journal of Logic and Computation*, 4(3):217–247, 1994. Revised version of Max-Planck-Institut für Informatik technical report, MPI-I-91-208, 1991.
- [4] Leo Bachmair and Harald Ganzinger. Resolution theorem proving. In Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning*, volume I, chapter 2, pages 19–99. Elsevier, 2001.
- [5] Leo Bachmair, Harald Ganzinger, and Uwe Waldmann. Superposition with simplification as a decision procedure for the monadic class with equality. In Georg Gottlob, Alexander Leitsch, and Daniele Mundici, editors, *Computational Logic and Proof Theory, Third Kurt Gödel Colloquium*, volume 713 of *LNCS*, pages 83–96. Springer, August 1993.
- [6] Dan Benanav. Simultaneous paramodulation. In M. E. Stickel, editor, *Proceedings of the 10th International Conference on Automated Deduction*, volume 449 of *LNAI*, pages 442–455. Springer, 1990.
- [7] Chin-Liang Chang and Richard Char-Tung Lee. *Symbolic Logic and Mechanical Theorem Proving*. Computer Science and Applied Mathematics. Academic Press, 1973.
- [8] Witold Charatonik, David McAllester, Damian Niwinski, Andreas Podelski, and Igor Walukiewicz. The horn mu-calculus. In *Proceedings 13th IEEE Symposium on Logic in Computer Science, LICS'98*, pages 58–69. IEEE Computer Society Press, 1998.
- [9] Marcello D'Agostino. Are tableaux an improvement on truth-tables? *Journal of Logic, Language and Information*, 1:235–252, 1992.
- [10] Martin Davis and Hilary Putnam. A computing procedure for quantification theory. *Journal of the ACM*, 7:201–215, 1960.
- [11] Nachum Dershowitz. Orderings for term-rewriting systems. *Theoretical Computer Science*, 17:279–301, 1982.
- [12] Nachum Dershowitz. Termination of rewriting. *Journal of Symbolic Computation*, 3(1):69–115, July 1987.
- [13] William F. Dowling and Jean H. Gallier. Linear-time algorithms for testing the satisfiability of propositional horn formulae. *Journal of Logic Programming*, 1(3):267–284, 1984.
- [14] Peter J. Downey, Ravi Sethi, and Robert Endre Tarjan. Variations on the common subexpression problem. *Journal of the ACM*, 27(4):758–771, 1980.

- [15] Norbert Eisinger. *Completeness, Confluence, and Related Properties of Clause Graph Resolution*. Research Notes in Artificial Intelligence. Pitman Ltd., London, 1991.
- [16] Bernd Fischer, Johann Schumann, and Gregor Snelting. Deduction-based software component retrieval. In Wolfgang Bibel and Peter H. Schmitt, editors, *Automated Deduction - A Basis for Applications*, volume 3 of *Applied Logic*, chapter 11, pages 265–292. Kluwer, 1998.
- [17] Thom Frühwirth, Ehud Shapiro, Moshe Y. Vardi, and Eyal Yardeni. Logic programs as types for logic programs. In Albert R. Meyer, editor, *Proceedings of the 6th Annual IEEE Symposium on Logic in Computer Science, LICS'91*, pages 300–309. IEEE Computer Society Press, July 1991.
- [18] Harald Ganzinger, Christoph Meyer, and Christoph Weidenbach. Soft typing for ordered resolution. In *Proceedings of the 14th International Conference on Automated Deduction, CADE-14*, volume 1249 of *LNAI*, pages 321–335, Townsville, Australia, 1997. Springer.
- [19] Michael R. Garey and David S. Johnson. *Computers and intractability : A guide to the theory of NP-completeness*. Mathematical Sciences Series. Freeman, New York, 1979.
- [20] Georg Gottlob and Christian G. Fermüller. Removing redundancy from a clause. *Artificial Intelligence*, 61:263–289, 1993.
- [21] Georg Gottlob and Alexander Leitsch. On the efficiency of subsumption algorithms. *Journal of the ACM*, 32(2):280–295, 1985.
- [22] Peter Graf. *Term Indexing*, volume 1053 of *LNAI*. Springer, 1996.
- [23] Reiner Hähnle, Manfred Kerber, and Christoph Weidenbach. Common syntax of the dfg-schwerpunktprogramm “deduktion”. Interner Bericht 10/96, Universität Karlsruhe, Fakultät für Informatik, Germany, 1996. Current version available from <http://spass.mpi-sb.mpg.de/>.
- [24] Nevin Heintze and Edmund Clarke, editors. *Workshop on Formal Methods and Security Protocols*, Trento, Italy, July 1999. Self Publishing.
- [25] Thomas Hillenbrand, Andreas Jaeger, and Bernd Löchner. Waldmeister – improvements in performance and ease of use. In Harald Ganzinger, editor, *16th International Conference on Automated Deduction, CADE-16*, *LNAI*, pages 232–236. Springer, 1999.
- [26] Ullrich Hustadt and Renate A. Schmidt. On evaluating decision procedures for modal logics. In *Proceedings of 15th International Joint Conference on Artificial Intelligence, IJCAI-97*, pages 202–207, 1997.
- [27] Florent Jacquemard, Christoph Meyer, and Christoph Weidenbach. Unification in extensions of shallow equational theories. In Tobias Nipkow, editor, *Rewriting Techniques and Applications, 9th International Conference, RTA-98*, volume 1379 of *LNCS*, pages 76–90. Springer, 1998.
- [28] William H. Joyner Jr. Resolution strategies as decision procedures. *Journal of the ACM*, 23(3):398–417, July 1976.
- [29] Henry Kautz and Bart Selman. Pushing the envelope: Planning, propositional logic and stochastic search. In *Proceedings of the 13th National Conference on AI, AAAI'96*, volume 2, pages 1194–1201. AAAI Press / MIT Press, 1996.
- [30] Donald E. Knuth and Peter B. Bendix. Simple word problems in universal algebras. In I. Leech, editor, *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon Press, 1970.
- [31] William McCune and Larry Vos. Experiments in automated deduction with condensed detachment. In *11th International Conference on Automated Deduction, CADE-11*, volume 607 of *LNCS*, pages 209–223. Springer, 1992.
- [32] William McCune and Larry Vos. Otter. *Journal of Automated Reasoning*, 18(2):211–220, 1997.

- [33] Christoph Meyer. *Soft Typing for Clausal Inference Systems*. Dissertation, Technische Fakultät der Universität des Saarlandes, Saarbrücken, Germany, 1999.
- [34] Robert Nieuwenhuis. Basic paramodulation and decidable theories (extended abstract). In *Proceedings 11th IEEE Symposium on Logic in Computer Science, LICS'96*, pages 473–482. IEEE Computer Society Press, 1996.
- [35] Robert Nieuwenhuis and Albert Rubio. Paramodulation-based theorem proving. In Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning*, volume I, chapter 7, pages 371–443. Elsevier, 2001.
- [36] Pilar Nivela and Robert Nieuwenhuis. Saturation of first-order (constrained) clauses with the *Saturate* system. In Claude Kirchner, editor, *Rewriting Techniques and Applications, 5th International Conference, RTA-93*, volume 690 of *Lecture Notes in Computer Science, LNCS*, pages 436–440, Montreal, Canada, June 16–18, 1993. Springer.
- [37] Andreas Nonnengart, Georg Rock, and Christoph Weidenbach. On generating small clause normal forms. In Claude Kirchner and Hélène Kirchner, editors, *15th International Conference on Automated Deduction, CADE-15*, volume 1421 of *LNAI*, pages 397–411. Springer, 1998.
- [38] Andreas Nonnengart and Christoph Weidenbach. Computing small clause normal forms. In Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning*, volume 1, chapter 6, pages 335–367. Elsevier, 2001.
- [39] Gerald E. Peterson. A technique for establishing completeness results in theorem proving with equality. *SIAM Journal of Computation*, 12(1):82–100, February 1983.
- [40] Alexandre Riazanov and Andrei Voronkov. Vampire. In Harald Ganzinger, editor, *16th International Conference on Automated Deduction, CADE-16*, volume 1632 of *LNAI*, pages 292–296. Springer, 1999.
- [41] G. Robinson and L. Wos. Paramodulation and theorem-proving in first-order theories with equality. In B. Meltzer and D. Michie, editors, *Machine Intelligence 4*, pages 135–150, 1969.
- [42] John Alan Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12(1):23–41, January 1965.
- [43] Manfred Schmidt-Schauß. Implication of clauses is undecidable. *Theoretical Computer Science*, 59:287–296, 1988.
- [44] Stephan Schulz. System abstract: E 0.3. In Harald Ganzinger, editor, *16th International Conference on Automated Deduction, CADE-16*, volume 1632 of *LNAI*, pages 297–301. Springer, 1999.
- [45] Wayne Snyder. On the complexity of recursive path orderings. *Information Processing Letters*, 46:257–262, 1993.
- [46] Rolf Socher. A subsumption algorithm based on characteristic matrices. In E. Lusk and R. Overbeek, editors, *9th International Conference on Automated Deduction, CADE-9*, volume 310 of *LNCS*, pages 573–581. Springer, 1988.
- [47] Rona B. Stillman. The concept of weak substitution in theorem-proving. *Journal of the ACM*, 20(4):648–667, 1973.
- [48] Geoff Sutcliffe and Christian B. Suttner. The tptp problem library – cnf release v1.2.1. *Journal of Automated Reasoning*, 21(2):177–203, 1998.
- [49] Geoff Sutcliffe and Christian B. Suttner. The cade-15 atp system competition. *Journal of Automated Reasoning*, 23(1):1–23, 1999.

- [50] Tanel Tammet. Towards efficient subsumption. In Claude Kirchner and H el ene Kirchner, editors, *15th International Conference on Automated Deduction, CADE-15*, volume 1421 of *LNAI*, pages 427–441. Springer, 1998.
- [51] Christoph Weidenbach. *Computational Aspects of a First-Order Logic with Sorts*. Dissertation, Technische Fakult at der Universit at des Saarlandes, Saarbr ucken, Germany, 1996.
- [52] Christoph Weidenbach. Sorted unification and tree automata. In Wolfgang Bibel and Peter H. Schmitt, editors, *Automated Deduction - A Basis for Applications*, volume 1 of *Applied Logic*, chapter 9, pages 291–320. Kluwer, 1998.
- [53] Christoph Weidenbach. Towards an automatic analysis of security protocols in first-order logic. In Harald Ganzinger, editor, *16th International Conference on Automated Deduction, CADE-16*, volume 1632 of *LNAI*, pages 378–382. Springer, 1999.
- [54] Christoph Weidenbach, Bijan Afshordel, Uwe Brahm, Christian Cohrs, Thorsten Engel, Enno Keen, Christian Theobalt, and Dalibor Topic. System description: Spass version 1.0.0. In Harald Ganzinger, editor, *16th International Conference on Automated Deduction, CADE-16*, volume 1632 of *LNAI*, pages 314–318. Springer, 1999.
- [55] L. Wos, G.A. Robinson, and D.F. Carson. Efficiency and completeness of the set of support strategy in theorem proving. *Journal of the ACM*, 12(4):536–541, 1965.

Chapter 3

dfg2dfg

3.1 Introduction

The tool `dfg2dfg` allows the user to calculate some approximation of a clause set. It is named `dfg2dfg` because it first reads clauses from an input file in DFG syntax [1]. It then calculates some approximation of the clause set depending on command line options. Finally it writes the approximated clause set in DFG syntax to a file.

3.2 Synopsis

```
dfg2dfg [-horn] [-monadic] [-linear] [-shallow] infile [outfile]
```

If neither `infile` nor `outfile` are given, `dfg2dfg` reads from standard input and writes to standard output. If one file name is given, it reads from that file and writes the output to standard output. If more than one file name is given, `dfg2dfg` reads from the first file and writes to the second.

The following sections describe the effects of the command line options. We are using a notation similar to the notation of reduction rules [2]:

$$\frac{\Theta \parallel \Gamma \rightarrow \Delta}{\Psi_1 \parallel \Pi_1 \rightarrow \Lambda_1}$$
$$\vdots$$
$$\Psi_n \parallel \Pi_n \rightarrow \Lambda_n$$

Such a rule is applied to a clause set P by selecting a clause $\Theta \parallel \Gamma \rightarrow \Delta$ from P and replacing it by the clauses $\Psi_i \parallel \Pi_i \rightarrow \Lambda_i$. A transformation is calculated by recursively applying the corresponding rule. The calculation stops when the rule isn't applicable to any clause from the clause set.

3.3 Transforming a clause to a Horn clause

This transformation is enabled with the `-horn` command line option. The rule is

$$\frac{\Theta \parallel \Gamma \rightarrow E_1, \dots, E_n}{\Theta \parallel \Gamma \rightarrow E_1}$$
$$\vdots$$
$$\Theta \parallel \Gamma \rightarrow E_n$$

where $n \geq 2$ and E_1, \dots, E_n are equality or non-equality literals.

3.4 Transformation to monadic literals

The following two transformations are enabled with the `-monadic` command line option. They transform non-monadic non-equality literals into monadic literals. Note that equality literals are not transformed.

3.4.1 Transformation by term encoding

This transformation is enabled with the option `-monadic`, which is equivalent to `-monadic=1`. It is described by the rule

$$\frac{\Theta \parallel \Gamma \rightarrow P(t_1, \dots, t_n), \Delta}{\Theta \parallel \Gamma \rightarrow T(p(t_1, \dots, t_n)), \Delta}$$

where p is a new function corresponding to the predicate P and T is a special predicate. All occurrences of P in the clause set are transformed into the same function p . Constraint and antecedent literals are transformed in a similar way. This approximation is equivalence preserving.

3.4.2 Transformation by projection

This transformation is enabled with the option `-monadic=2`. It is described by the two rules

$$\frac{\Theta \parallel P(t_1, \dots, t_n), \Gamma \rightarrow \Delta}{\Theta \parallel P_1(t_1), \dots, P_n(t_n), \Gamma \rightarrow \Delta} \quad \frac{\Theta \parallel \Gamma \rightarrow P(t_1, \dots, t_n), \Delta}{\Theta \parallel \Gamma \rightarrow P_1(t_1), \Delta}$$

$$\vdots$$

$$\Theta \parallel \Gamma \rightarrow P_n(t_n), \Delta$$

where P_1, \dots, P_n are some new predicates. All occurrences of P in the clause set are transformed into the same predicates P_1, \dots, P_n . Constraint literals are transformed similar to antecedent literals.

3.5 The linear approximation of a clause

A term is called *linear*, if it contains no repeated variables. This transformation generates the *linear approximation* of a clause with monadic literals by replacing a variable x repeated within the succedent by some new variable x' . Note that the transformation isn't applicable to clauses containing equality or non-monadic literals. This transformation is enabled with the `-linear` option. It is described by the rule

$$\frac{\Theta \parallel \Gamma \rightarrow A[x]_p, B[x]_q, \Delta}{\Theta' \sigma, \Theta \parallel \Gamma' \sigma, \Gamma \rightarrow A, B[q/x'], \Delta}$$

where (i) all literals are monadic, (ii) $A \neq B$ or $p \neq q$, (iii) x' is a new variable, (iv) $\sigma = \{x \rightarrow x'\}$, (v) $\Theta' = \{L \in \Theta \mid x \in \text{vars}(L)\}$ and (vi) $\Gamma' = \{L \in \Gamma \mid x \in \text{vars}(L)\}$.

3.6 The shallow approximation of a clause

We implemented three kinds of transformations with different requirements for the input clause and different output clauses. The transformations are enabled with the `-shallow` option. Note that the rules aren't applicable to non-horn clauses.

3.6.1 The strict version

This transformation is enabled with the `-shallow` option, which is equivalent to `-shallow=1`. It is described by the rule

$$\frac{\Theta \parallel \Gamma \rightarrow P(t[s]_{p_1})}{S(x), \Theta_1 \parallel \Gamma_1 \rightarrow P(t[p_1, \dots, p_n/x]), \Theta_2 \parallel \Gamma_2 \rightarrow S(s)}$$

where (i) all literals are monadic and all terms in Θ, Γ are variables, (ii) s is a complex term at non-top position p in t , (iii) $\text{vars}(s) \cap \text{vars}(P(t[p_1, \dots, p_n/c])) = \emptyset$, where c is an arbitrary constant and p_1, \dots, p_n are all positions of s in t , (iv) x is a new variable and S is a new predicate, (v) Θ_1 and Θ_2 are a partition of Θ , with $\Theta_2 = \{L \in \Theta \mid \text{vars}(L) \subseteq \text{vars}(s)\}$ and (vi) Γ_1 and Γ_2 are a partition of Γ , with $\Gamma_2 = \{L \in \Gamma \mid \text{vars}(L) \subseteq \text{vars}(s)\}$. Note that all occurrences of s in t are replaced simultaneously. This transformation is equivalence preserving with respect to the extension of P in the minimal model.

3.6.2 A more relaxed version

This version uses the same rule as the strict version, but condition (iii) is omitted. This results in an upper approximation of P . This transformation is enabled with the `-shallow=2` option.

3.6.3 The least restricted version

This transformation is enabled with the `-shallow=3` option. It uses the rule

$$\frac{\Theta \parallel \Gamma \rightarrow P(t[s]_{p_1})}{\frac{S(x), \Theta \parallel \Gamma \rightarrow P(t[p_1, \dots, p_n/x])}{\Theta \parallel \Gamma \rightarrow S(s)}}$$

where (i) all literals are monadic, (ii) s is a complex term at non-top position p_1 in t and p_1, \dots, p_n refer to all positions of s in t and (iii) x is a new variable and S is a new predicate. Note that Θ and Γ may contain non-variable terms and that s and $t[p_1, \dots, p_n/c]$ (where c is an arbitrary constant) may share variables. In contrast to the other two versions all negative literals are copied into all resulting clauses. This transformation is an upper approximation of P .

3.7 Combining several transformations

It is possible to combine several of the transformations described above. However, transformations from the sections 3.4 or 3.6 aren't combinable with other transformations from the same section, because they have the same goal. But for the other transformations the order of their application becomes important if several transformations are combined. For example a transformation to monadic literals should be applied *before* the linear transformation because the latter requires monadic literals. If the transformations are applied in this order it might be possible to apply the second transformation more often. `dfg2dfg` therefore applies the transformations in the same order as the order of their description in this paper. That means transformations are applied in the following order:

1. transformation to horn clauses
2. transformation to monadic literals
3. linear transformation
4. shallow transformation.

The shallow transformation is applied last because it has the most preconditions.

Bibliography

- [1] Reiner Hähnle, Manfred Kerber, and Christoph Weidenbach. Common syntax of the dfg-schwerpunktprogramm “deduktion”. Interner Bericht 10/96, Universität Karlsruhe, Fakultät für Informatik, Germany, 1996. Current version available from <http://spass.mpi-sb.mpg.de/>.
- [2] Christoph Weidenbach. Combining superposition, sorts and splitting. In Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning*, volume 2, chapter 27, pages 1965–2012. Elsevier, 2001.

Chapter 4

EML Logic Features

4.1 Introduction

This chapter describes the features and facilities for supporting automated reasoning in a large class of related logics which we refer to as *EML* logics (short for extended modal logics). Support for *EML* logics was implemented from 1998 onwards in the MSPASS theorem prover [11, 13, 20] as an extension of SPASS 1.0. This code has been integrated and upgraded with respect to SPASS 3.0 (and FLOTTER 3.0) so that support for modal, relational and description logic reasoning is now immediately available to SPASS users (as had been planned from the outset) and the latest SPASS technology is immediately available to MSPASS users.

EML logics include (traditional) propositional modal logics such as $K_{(m)}$, $KD_{(m)}$, $KT^4_{(m)}$ etc., which play an important role e.g. in the specification of multi-agent systems. *EML* logics also include dynamic modal logics [22] which are *PDL*-like modal logics in which the modal operators are parameterized by relational formulas. These can be used to formalize dynamic notions such as actions or programs and are useful in linguistic or AI applications. Examples of dynamic modal logics are Boolean modal logic, tense logic, information logics, logics expressing inaccessibility and sufficiency, Peirce logic, as well as a large class of description logics. The *EML* class further includes relational logics, i.e. logical versions of Tarski's relation algebras. The strongest dynamic modal logic that the prover supports is Peirce logic [22, 23] or equivalent logics. SPASS handles these logics by translation to first-order logic.

With the *EML* facilities SPASS supports the following kind of reasoning for *EML* logics.

1. For dynamic modal logics and relational logics: non-logical axioms, modal operators characterized by any first-order frame correspondence properties, relational symbols satisfying any first-order properties, specification of concrete worlds as constants, (first-order) relationships between concrete worlds, specifications of frames and models.
2. For description logics: the corresponding features, including terminological axioms, TBox and RBox statements, and ABox statements for concept and role expressions.

Because SPASS is a first-order resolution prover its capabilities as a modal, relational or description logic prover are very different and more varied than those of other provers for these logics. It is possible to use SPASS as a decision procedure for a large class of *EML* logics. For instance, it decides extensions of Boolean modal logic with converse, domain/range restriction, and positive occurrences of composition, and the corresponding description logics, i.e. extensions of \mathcal{ALB} with positive occurrences of composition. SPASS is a decision procedure for many solvable first-order fragments including the guarded fragment, Maslov's class K, first-order logic in two variables, the clausal class DL^* , and many decidable quantifier prefix classes. No other (special-purpose) prover currently decides these logics. Using the features of SPASS it is possible to approximate the behaviour of modal and description logic tableau provers with SPASS. Additionally, SPASS can be used as a model finder. For details and references please consult the survey papers [12, 22] (see also [4, 8, 21]).

In the following we assume terminology and notation of dynamic modal logic and Peirce logic as described in [22]. We do not discuss description logics and relational logics separately because

1. description logics and dynamic modal logics can be viewed as syntactic variants of each other, and
2. dynamic modal logics can be viewed as two-sorted combinations of relational logics and propositional logic and therefore cater also for relational logics.

4.2 Dynamic modal logics

This section defines the class of dynamic modal logics [22] in which Peirce logic is the strongest logic and the basic multi-modal logic $K_{(m)}$ is the weakest logic. Peirce logic is a logical formalisation of representable Peirce algebras [14, 23] and is closely related to de Rijke's dynamic modal logic [5].

Formally, Peirce logic is the modal logic defined over relations which form a relation algebra. The language of Peirce logic consists of two syntactic types: dynamic modal formulas and relational formulas. The logical connectives are

1. the connectives of the modal logic $K_{(m)}$, with the difference that the modal operators are indexed with relational formulas, instead of just numbers,
2. the standard connectives of relational logics, namely: $;$ (composition), \smile (converse), id (identity), and
3. a left cylindrification operator c .

Instead of the left cylindrification operator one could have chosen the test operator of *PDL*, domain restriction, range restriction, or cross product, cf. [3]; the symbol id is a logical constant which is interpreted as the identity relation. Given countably many propositional variables denoted by p_j , and countably many relational variables, denoted by r_i , *dynamic modal formulas* and *relational formulas* are generated according to the following production rules.

$$\begin{array}{ll} \text{Dynamic modal formulas:} & \phi \longrightarrow p_j \mid \neg\phi \mid \phi \wedge \phi \mid [\alpha]\phi \\ \text{Relational formulas:} & \alpha \longrightarrow r_j \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha ; \alpha \mid \alpha^\smile \mid id \mid \phi^c \end{array}$$

are defined inductively as follows.

We define the set of *formulas* of Peirce logic to be the set of dynamic modal formulas. The definition in [23] allows also relational formulas as first-class citizens, but these can be expressed in terms of dynamic modal formulas (see below). This variation in the definition is thus inconsequential.

The semantics of Peirce logic is defined in terms of frames, where a frame is a tuple (W, R) of a non-empty set W (of worlds) and a mapping R from relational formulas to binary relations over W satisfying:

$$\begin{aligned} R_{\neg\alpha} &= W^2 \setminus R_\alpha \\ R_{\alpha \wedge \beta} &= R_\alpha \cap R_\beta \\ R_{\alpha ; \beta} &= R_\alpha ; R_\beta \\ R_{\alpha^\smile} &= R_\alpha^\smile \\ R_{id} &= Id_W \\ R_{\phi^c} &= \{(x, y) \in W^2 \mid x \in v(\phi)\}. \end{aligned}$$

Here and in the rest of the paper we prefer to use the notation R_α instead of $R(\alpha)$. Id_W denotes the identity relation on the set W , while R^\smile denotes the converse (or inverse) of a relation R . A model is now given by a triple $\mathcal{M} = (W, R, v)$, where (W, R) is a frame and v is a mapping from propositional variables to

subsets of W satisfying:

$$\begin{aligned} \mathcal{M}, x \models p &\text{ iff } x \in v(p) \\ \mathcal{M}, x \models \neg\phi &\text{ iff } \mathcal{M}, x \not\models \phi \\ \mathcal{M}, x \models \phi \wedge \psi &\text{ iff both } \mathcal{M}, x \models \phi \text{ and } \mathcal{M}, x \models \psi \\ \mathcal{M}, x \models [\alpha]\phi &\text{ iff } (x, y) \in R_\alpha \text{ implies } \mathcal{M}, y \models \phi, \text{ for any } y \in W \end{aligned}$$

If $\mathcal{M}, x \models \varphi$ holds then we say φ is true at x in \mathcal{M} and that \mathcal{M} *satisfies* φ . A modal formula φ is *satisfiable* iff there exists a model \mathcal{M} and a world x in \mathcal{M} such that $\mathcal{M}, x \models \varphi$. A modal formula is *valid in a frame* iff it is valid in all models based on the frame.

We can define numerous other connectives in Peirce logic. For example:

$$\begin{aligned} \text{Relational sum:} & \quad \alpha \dagger \beta =^{\text{def}} \neg((\neg\alpha); (\neg\beta)) \\ \text{Domain restriction:} & \quad \alpha \upharpoonright \phi =^{\text{def}} \alpha \wedge \phi^c \\ \text{Range restriction:} & \quad \alpha \downharpoonright \phi =^{\text{def}} (\alpha^\smile \upharpoonright \phi)^\smile \\ \text{Test:} & \quad \phi? =^{\text{def}} id \wedge \phi^c \\ \text{Universal modality:} & \quad [\top]\phi =^{\text{def}} [r \vee \neg r]\phi \quad \text{where } r \text{ denotes some relational variable} \end{aligned}$$

Relational formulas as ‘independent’ formulas are implicit in all dynamic modal logics with relational negation and relational conjunction or disjunction. For example, implication between relational formulas can be defined by

$$(\alpha \supset \beta) =^{\text{def}} \top \supset [\alpha \wedge \neg\beta] \perp.$$

Then it is also possible to specify properties of the underlying accessibility relations. Figure 4.1 gives some properties of the accessibility relation associated with r which can be specified by relational formulas in Peirce logic. In the figure the relational operators are assumed to have higher priority than \supset .

Property of R_r	Relational formula	Property of R_r	Relational formula
reflexivity	$id \supset r$	transitivity	$r; r \supset r$
symmetry	$r \supset r^\smile$	functionality	$r^\smile; r \supset id$
seriality	$\top \supset r; \top$ or $\top \supset \langle r \rangle \top$	Euclideaness	$r; r^\smile \supset r$

Figure 4.1: Relational properties expressed as relational formulas.

The basic multi-modal logic $K_{(m)}$ is the reduct of Peirce logic in which relational formulas are limited to m relational variables r_j . *Dynamic modal logics* are defined over the language of $K_{(m)}$ but include in addition to the operators of $K_{(m)}$ and relational variables also finitely many relational operators \star_1, \dots, \star_k . We assume that these relational operators are definable in terms of the operators of Peirce logic. A logic $K_{(m)}(\star_1, \dots, \star_k)$ is then defined to be the multi-modal logic defined over relations closed under the set-theoretic operations corresponding to the relational operators \star_1, \dots, \star_k . The class of dynamic modal logics forms therefore a lattice in which $K_{(m)}$ is the weakest logic and every other logic is obtained by enhancing the language with one or more relational operators \star_i . Clearly some of the logics in this lattice are expressively equivalent. Peirce logic is equivalent to the top element in the lattice. Tense logic coincides with the logic $K_{(m)}(\smile)$, Boolean modal logic coincides with $K_{(m)}(\neg, \wedge)$, and the description logic \mathcal{ALB} [10] coincides with $K_{(m)}(\neg, \wedge, \smile, \upharpoonright)$.

4.3 Overview of EML input specifications

The `dfg` input language of SPASS was extended to support the input of EML problems without changing the syntax for formulas in first-order logic or clause form. EML problems are specified with new, special types of formulas using

```
list_of_special_formulas(axioms, EML)
```

for axioms and

```
list_of_special_formulas (conjectures, EML)
```

for conjectures. Currently the only class of special formulas are EML formulas. EML formulas include first-order formulas, Boolean type formulas and relational type formulas. First-order formulas are specified with `formula` using the familiar `dfg` syntax. Boolean type formulas are specified with either

```
prop_formula or concept_formula,
```

and relational formulas with either

```
rel_formula or role_formula.
```

Boolean and relational type formulas can be constructed using common modal, relational and description logic operators. The pre-defined logical operators include the following.

- The standard Boolean operators (for all three types of formulas): `true`, `false`, `not`, `and`, `or`, `implies` (subsumed by), `implied` (subsumes), and `equiv`. These operators have different semantics depending on the type of formulas they combine.
- Multi-modal operators with atomic or complex relational arguments: `dia` and `box` (synonyms are `some` and `all`), as well as `domain` and `range`.
- Additional relational operators: `comp` (composition), `sum` (relational sum), `conv` (converse), `id` (the identity relation), and `div` (the diversity relation).
- `test` (`test`), `domrestr` (domain restriction) and `ranrestr` (range restriction).

Apart from their usual interpretation in propositional logic and first-order logic, the (nullary) operators `true` and `false` may also be used as Boolean or relational formulas. `true` used as a Boolean type, resp. relational type, represents the universal set, the universal (binary) relation, or truth in first-order logic. Similarly, `false` represents either, the empty set, the empty relation or `falsum`.

We give three examples of *EML* formulas, two Boolean type formulas and one relational type formula.

```
(4.1) prop_formula( implies(box(bell,p), box(know1,box(bell,p))) ).
```

```
(4.2) concept_formula( implies(expert_AR,
                             not(some(not(has_studied), proof_methods))) ).
```

```
(4.3) rel_formula( implies(comp(r,r), r) ).
```

(4.1) is an example from modal logic and says that if agent 1 believes `p` then it knows that it believes `p`, i.e. it is aware that it believes `p`. The example (4.2) is a description logic example; it says that an expert in automated reasoning is someone who has studied every proof method. This kind of example cannot be handled by current tableau-based description logic provers because it requires negation of roles. (4.3) is the way to express transitivity of a relation in relational logic (or in description logics).

Boolean- and relational-type atomic formulas must be declared as nullary predicates in the declaration section of the input file (because the supported EML logics are propositional non-classical logics defined over languages free of first-order variables).

For EML problems it is sometimes useful to specify the first-order predicate symbols to which the nullary Boolean and relational predicate symbols are mapped. This can be done with a new kind of declaration statement. For example,

```
translpairs(p, Qp)
```

causes the nullary symbol `p` to be translated to a first-order predicate symbol with the name `Qp`. `Qp` should be declared as a unary (resp. binary) predicate symbol if `p` is a Boolean (resp. relational) symbol.

The new extended `dfg` is sufficient to support the following.

- For dynamic modal logics and relational logics: non-logical axioms, modal operators characterised by any first-order frame correspondence properties, relational symbols satisfying any first-order properties, specification of concrete worlds as constants, (first-order) relationships between concrete worlds, specifications of frames and models.
- For description logics: the corresponding features, including terminological axioms, TBox and RBox statements, ABox statements for concept and role expressions,

A more detailed, formal specification of the input language with examples of input files in the extended `dfg` format can be found in [24].

4.4 Translation mappings

Boolean and relational type formulas are translated into first-order formulas using one of numerous translation methods. Table 4.1 summarizes the implemented translation methods.

Translation method	Options
relational translation	<code>-EMLTranslation=0</code> (default)
(monadic) functional translation	<code>-EMLTranslation=1</code>
polyadic functional translation	<code>-EMLTranslation=1 -EMLFuncNary=1</code>
(monadic) optimized functional translation	<code>-EMLTranslation=2</code>
polyadic optimized functional translation	<code>-EMLTranslation=2 -EMLFuncNary=1</code>
semi-functional translation	<code>-EMLTranslation=3</code>
relational-functional translation	<code>-EMLTranslation=0 -EMLFuncNary=1</code>
relational-relational translation	<code>-EML2Rel=1 [-EMLTranslation=0]</code>

Table 4.1: Available translation methods

The different translation methods are based on first-order encodings of the different ways of defining the semantics of the logics. The basis for the *relational translation*, or *standard translation*, method is the standard set-theoretic semantics of *EML* logics. It is implemented for all Boolean and relational *EML* formulas. The basis for the different *functional translations* is the functional semantics of traditional modal logics. The *optimized functional translations* are obtained from the functional translations by a non-standard quantifier exchange operation, which is implemented by replacing non-constant Skolem terms by Skolem constants. The *polyadic functional translation* methods are variations of functional translation methods and differ in the way they encode world paths (transition sequences). The polyadic translations avoid the use of an extra function symbol by using n-ary predicates of different arities. The *semi-functional translation* approach is a mixture of the relational and functional translation approaches. It translates box modalities in the standard relational way, while diamond modalities are translated functionally. The *relational-functional translation* method, or *tree-layered relational translation*, is a variation of the relational translation specialized for the basic modal logic $K_{(m)}$. The *relational-relational translation* converts Boolean *EML* formulas into the relational formulas via a cylindrification operation. All translation methods are sound and complete for the logics they are implemented for and have linear time complexity.

The transformation of non-first-order *EML* formulas into first-order logic can be specified with the `-EMLTranslation` option. Unless output is disabled, the prover outputs the translated formulas using the indicators as specified in Table 4.2.

The different transformation mappings are described in more detail in the following sections. The relational translation, the relational-functional translation and the relational-relational translation are collectively referred to as *relational translation mappings* and are discussed in Sections 4.4.1, 4.4.7, and 4.4.8. The monadic and polyadic, optimised and non-optimised functional translations and the semi-functional translation are collectively referred to as *functional translation mappings* and are discussed in Sections 4.4.2–4.4.6.

Translation method	Output indicator
relational translation	RelTr
(monadic) functional translation	FuncTr
polyadic functional translation	FuncFTr
(monadic) optimised functional translation	(FuncTr)
polyadic optimised functional translation	(FuncFTr)
semi-functional translation	SemiFuncTr
relational-functional translation	RelFuncTr
relational-relational translation	Rel

Table 4.2: Translation output indicators

4.4.1 Standard relational translation method

The standard relation translation is enabled by `-EMLTranslation=0`. This is the default setting for `-EMLTranslation`. It is implemented for all *EML* formulas. It maps formulas into fragments of first-order logic defined over unary and binary symbols [12, 22]. These are uninterpreted unless properties have been specified in the input file.

The relational translation is determined by the usual Kripke semantics. For instance, the standard relational translation of Peirce logic into first-order logic is specified by the following.

For dynamic modal formulas:

$$\begin{aligned}
\pi_r(p, x) &= Q_p(x) \\
\pi_r(\neg\phi, x) &= \neg\pi_r(\phi, x) \\
\pi_r(\phi \wedge \psi, x) &= \pi_r(\phi, x) \wedge \pi_r(\psi, x) \\
\pi_r([\alpha]\phi, x) &= \forall y (\tau_r(\alpha, x, y) \supset \pi_r(\phi, y))
\end{aligned}$$

For relational formulas:

$$\begin{aligned}
\tau_r(r, x, y) &= Q_r(x, y) \\
\tau_r(\neg\alpha, x, y) &= \neg\tau_r(\alpha, x, y) \\
\tau_r(\alpha \wedge \beta, x, y) &= \tau_r(\alpha, x, y) \wedge \tau_r(\beta, x, y) \\
\tau_r(\alpha^\sim, x, y) &= \tau_r(\alpha, y, x) \\
\tau_r(\alpha ; \beta, x, y) &= \exists z (\tau_r(\alpha, x, z) \wedge \tau_r(\beta, z, y)) \\
\tau_r(\phi^c, x, y) &= \pi_r(\phi, x) \\
\tau_r(id, x, y) &= x \approx y
\end{aligned}$$

Q_p , respectively Q_r , denote a unary, respectively binary, predicate symbol uniquely associated with the propositional symbol p , respectively the relational symbol r .

Let L be a logic in the lattice of dynamic modal logics and let Σ be a (possibly empty) set of first-order relational frame properties. Then we have that, for any formula φ ,

1. $\pi_r(\varphi, x)$ can be computed in linear time, and
2. φ is satisfiable in L with respect to Σ iff $\Sigma \wedge \exists x \pi_r(\varphi, x)$ is first-order satisfiable [22].

4.4.2 (Monadic) Functional translation method

The (monadic) functional translation method is enabled by `-EMLTranslation=1`. It is implemented for the basic multi-modal logic $K_{(m)}$ possibly with serial (total) modalities (`-EMLTheory=1`), plus frames (atom structures) and models, and non-logical axioms. The corresponding description logics are \mathcal{ALC} with concept ABox statements (concept and role assertions) and TBox statements (general inclusion and equivalence axioms), and total roles.

The functional translation transforms modal formulas into a fragment of monadic many-sorted first-order logic. The sorts are: S_W for the set of worlds W , S_i for each modality \Box_i in the logic. For each i there is a binary, left-associative function $[\cdot, \cdot]_i$ of sort $S_W \times S_i \rightarrow W$. There are special unary predicates nde_i of sort S_W representing subsets of W . Each propositional variable p is uniquely associated with a unary predicate symbols Q_p of sort S_W . The (monadic) optimised functional translation π_f is defined by:

$$\begin{aligned}\pi_f(p, s) &= Q_p(s) \\ \pi_f(\neg\varphi, s) &= \neg\pi_f(\varphi, s) \\ \pi_f(\varphi \wedge \psi, s) &= \pi_{of}(\varphi, s) \wedge \pi_f(\psi, s) \\ \pi_f(\Box_i\varphi, s) &= \forall y:S_i(\text{nde}_i(s) \supset \pi_f(\varphi, [s y:S_i]_i))\end{aligned}$$

If \Box_i is a D modality then π_f is given by the following.

$$\pi_f(\Box_i\varphi, s) = \forall y:S_i\pi_f(\varphi, [s y:S_i]_i)$$

The symbol s denotes a path and $y:S_i$ denotes a variable of sort S_i . The intuition of the term $[s y:S_i]_i$ is that it represents an i -successor world which is reached via the path s to its predecessor world followed by a y transition of type S_i . This means $[s y:S_i]_i$ represents both a world and the path via which it is reached from the initial world. The intuition of $\text{nde}_i(s)$ is that the world represented by s is not a dead-end.

The following holds: A modal formula φ is satisfiable in $K_{(m)}$ or $KD_{(m)}$ iff $\exists x\pi_{of}(\varphi, x:S_W)$ is first-order satisfiable [2, 6, 16, 19].

In the prover the $[\cdot, \cdot]_i$ function is implemented by the `appl` function and the nde_i predicates are implemented as uninterpreted special unary `nde` predicates. There is currently no support for specifying additional properties for the `nde` predicates. Similarly for the other functional translations.

4.4.3 Polyadic functional translation method

The polyadic functional translation method is enabled by `-EMLTranslation=1 -EMLFuncNary=1`. The polyadic functional translation methods are variations of functional translation methods. Instead of using the $[\cdot, \cdot]_i$ (or `appl`) functions, and unary predicates, to encode truth in world paths, k -ary predicates are used. The polyadic functional translation method is implemented for the basic multi-modal logic $K_{(m)}$ possibly with D modalities.

The k -ary predicate symbols are $Q_{p,\sigma}$ and $\text{nde}_{i,\sigma}$ where p denotes a propositional symbol, and σ is a sequence of length k of natural numbers. We use \bar{x} to denote a sequence of variables x_1, \dots, x_k , and we denote by ‘ ϵ ’ and ‘ \cdot ’ the empty sequence and the concatenation operation on sequences, respectively. Then the polyadic functional translation π'_f is given by the following.

$$\begin{aligned}\pi'_f(p, \bar{x}, k, \sigma) &= \begin{cases} Q_{p,\epsilon} & \text{if } \sigma = \epsilon \text{ and } k = 0 \\ Q_{p,\sigma}(x_1, \dots, x_k) & \text{otherwise} \end{cases} \\ \pi'_f(\neg\varphi, \bar{x}, k, \sigma) &= \neg\pi'_f(\varphi, \bar{x}, k, \sigma) \\ \pi'_f(\varphi \wedge \psi, \bar{x}, k, \sigma) &= \pi'_f(\varphi, \bar{x}, k, \sigma) \wedge \pi'_f(\psi, \bar{x}, k, \sigma) \\ \pi'_f(\Box_i\varphi, \bar{x}, k, \sigma) &= \forall x_{n+1}(\text{nde}_{i,\sigma}(\bar{x}) \supset \pi'_f(\varphi, \bar{x}.x_{k+1}, k+1, \sigma.i))\end{aligned}$$

In $KD_{(m)}$ the translation of \Box_i formulas is given by the following,

$$\pi'_f(\Box_i\varphi, \bar{x}, k, \sigma) = \forall x_{n+1}\pi'_f(\varphi, \bar{x}.x_{k+1}, k+1, \sigma.i)$$

In this definition the variable sequence in the argument position two of π'_f represents the world, and its path from the initial world, where the formula in argument position one is true. The translation of a modal formula φ is given by $\pi'_f(\varphi, \epsilon, 0, \epsilon)$.

For any multi-modal logic L with K -modalities and D -modalities only, a modal formula φ is satisfiable in L iff $\pi'_f(\varphi, \epsilon, 0, \epsilon)$ is first-order satisfiable [7]. π'_f embeds modal formulas into fluted logic [12, 18].

4.4.4 (Monadic) Optimised functional translation method

The (monadic) optimised functional translation method is enabled by `-EMLTranslation=2`. It is implemented for $K_{(m)}$, possibly with serial modalities.

It is an optimisation of the functional translation. A crucial difference is that it eliminates the dependency of existentially quantified first-order variables on universally quantified first-order variables via a non-standard quantifier exchange operator.

The (monadic) optimised functional translation maps the basic modal logic $K_{(m)}$ to basic path logic [18]. Basic path logic is a fragment of monadic many-sorted first-order logic. Basic path logic has a sort S_W for the set of worlds W and a sort S_i for each modality \Box_i in the logic. For each i there is a binary, left-associative function $[\cdot, \cdot]_i$ of sort $S_W \times S_i \rightarrow W$. There are special unary predicates nde_i of sort S_W representing subsets of W . Each propositional variable p is uniquely associated with a unary predicate symbols Q_p of sort S_W . The (monadic) optimised functional translation π_{of} is implemented as a two step process:

1. The application of the functional translation to a modal formula which translates it to basic path logic, followed by
2. the application of a quantifier exchange operation which converts the first-order formula obtained from the functional translation into prenex normal form and moves all existential quantifiers outwards as far as possible (or inwards, depending on one's point of view).

The following defines the optimised functional translation obtained as a result of both steps.

$$\begin{aligned}
\pi_{of}(p, s) &= Q_p(s) \\
\pi_{of}(\neg\varphi, s) &= \neg\pi_{of}(\varphi, s) \\
\pi_{of}(\varphi \wedge \psi, s) &= \pi_{of}(\varphi, s) \wedge \pi_{of}(\psi, s) \\
\pi_{of}(\Box_i\varphi, s) &= \forall y:S_i(\text{nde}_i(s) \supset \pi_{of}(\varphi, [s y:S_i]_i)) \\
\pi_{of}(\Diamond_i\varphi, s) &= \text{nde}_i(s) \wedge \pi_{of}(\varphi, [s y:S_i]_i)
\end{aligned}$$

The symbol s denotes a path and $y:S_i$ denotes a variable of sort S_i . The intuition of the term $[s y:S_i]_i$ is that it represents an i -successor world which is reached via the path s to its predecessor world followed by a y transition of type S_i . This means $[s y:S_i]_i$ represents both a world and the path via which it is reached from the initial world. The inclusion of a specification for diamond formulas in the above definition is intentional and so is the omission of the quantifiers. The optimised functional translation of a modal formula φ is given by $\pi_{of}(\varphi, x:S_W)$, where $x:S_W$ is an arbitrary variable of sort S_W , and $x:S_W$ as well as the $y:S_i$ from $\pi_{of}(\Diamond_i\varphi, s)$, are free variables which are implicitly existentially quantified.

In $KD_{(m)}$ the translation of \Box_i and \Diamond_i formulas is given by:

$$\begin{aligned}
\pi_{of}(\Box_i\varphi, s) &= \forall y:S_i\pi_{of}(\varphi, [s y:S_i]_i) \\
\pi_{of}(\Diamond_i\varphi, s) &= \pi_{of}(\varphi, [s y:S_i]_i).
\end{aligned}$$

The following holds: A modal formula φ is satisfiable in $K_{(m)}$ or $KD_{(m)}$ iff $\exists x\bar{y}\pi_{of}(\varphi, x:S_W)$ is first-order satisfiable [17, 18].

4.4.5 Polyadic optimised functional translation method

The setting `-EMLTranslation=2 -EMLFuncNary=1` uses the optimised version of the polyadic functional translation.

The following specifies the polyadic optimised functional translation π'_{of} .

$$\begin{aligned}\pi'_{of}(p, \bar{x}, k, \sigma) &= \begin{cases} Q_{p, \epsilon} & \text{if } \sigma = \epsilon \text{ and } k = 0 \\ Q_{p, \sigma}(x_1, \dots, x_k) & \text{otherwise} \end{cases} \\ \pi'_{of}(\neg\varphi, \bar{x}, k, \sigma) &= \neg\pi'_{of}(\varphi, \bar{x}, k, \sigma) \\ \pi'_{of}(\varphi \wedge \psi, \bar{x}, k, \sigma) &= \pi'_{of}(\varphi, \bar{x}, k, \sigma) \wedge \pi'_{of}(\psi, \bar{x}, k, \sigma) \\ \pi'_{of}(\Box_i\varphi, \bar{x}, k, \sigma) &= \forall x_{n+1} (\text{nde}_{i, \sigma}(\bar{x}) \supset \pi'_{of}(\varphi, \bar{x}.x_{k+1}, k+1, \sigma.i)) \\ \pi'_{of}(\Diamond_i\varphi, \bar{x}, k, \sigma) &= \text{nde}_{i, \sigma}(\bar{x}) \wedge \pi'_{of}(\varphi, \bar{x}.x_{k+1}, k+1, \sigma.i)\end{aligned}$$

In $KD_{(m)}$ the translation of \Box_i and \Diamond_i formulas is given by:

$$\begin{aligned}\pi'_{of}(\Box_i\varphi, \bar{x}, k, \sigma) &= \forall x_{n+1} \pi'_{of}(\varphi, \bar{x}.x_{k+1}, k+1, \sigma.i) \\ \pi'_{of}(\Diamond_i\varphi, \bar{x}, k, \sigma) &= \pi'_{of}(\varphi, \bar{x}.x_{k+1}, k+1, \sigma.i).\end{aligned}$$

The translation of a modal formula φ is given by $\pi'_{of}(\varphi, \epsilon, 0, \epsilon)$.

For any multi-modal logic L with K -modalities and D -modalities only, a modal formula φ is satisfiable in L iff $\exists \bar{x} \pi'_{of}(\varphi, \epsilon, 0, \epsilon)$ is first-order satisfiable [9, 17]. π'_{of} embeds modal formulas into the Bernays-Schönfinkel class, the class of $\exists^*\forall^*$ prefix quantifier formulas [9].

4.4.6 Semi-functional translation method

The semi-functional translation method is enabled by `-EMLTranslation=3`. It is defined for numerous traditional-style modal logics [15], but is implemented only for the basic multi-modal logic $K_{(m)}$ and $KD_{(m)}$ (`-EMLTheory=1`), plus frames (atom structures) and models, and non-logical axioms. The corresponding description logics are \mathcal{ALC} with concept ABox statements (concept and role assertions) and TBox statements (general inclusion and equivalence axioms), and total roles.

The semi-functional translation maps modal formulas to many-sorted first-order formulas. We distinguish between the sorts W and AF for worlds and accessibility functions. Unary predicate symbols have sort W , the binary predicate symbol R associated with the accessibility relation has sort $W \times W$, the constant symbol ϵ has sort W , and the binary (left-associative) function $[\cdot, \cdot]$ has sort $W \times AF \rightarrow W$. Then the semi-functional translation π_{sf} is defined as follows.

$$\begin{aligned}\pi_{sf}(p, s) &= Q_p(s) \\ \pi_{sf}(\neg\psi, s) &= \neg\pi_{sf}(\psi, s) \\ \pi_{sf}(\phi \wedge \psi, s) &= \pi_{sf}(\phi, s) \wedge \pi_{sf}(\psi, s) \\ \pi_{sf}(\Box\psi, s) &= \forall y (R(s, y) \supset \pi_{sf}(\psi, y)) \\ \pi_{sf}(\Diamond\psi, s) &= \text{nde}(s) \wedge \exists \alpha \pi_{sf}(\psi, [s\alpha])\end{aligned}$$

[15] proves the following: A modal formula φ in negation normal form is satisfiable in $K_{(m)}$ iff $\pi_{sf}(\varphi, \epsilon)$ is satisfiable. And, a modal formula φ in negation normal form is satisfiable in $KD_{(m)}$ iff $\pi_{sf}(\varphi, \epsilon) \wedge \forall x \forall \alpha (\text{nde}(x) \wedge R(x, [x\alpha]))$ is satisfiable.

4.4.7 Relational-functional translation method

The settings `-EMLTranslation=0 -EMLFuncNary=1` enable the relational-functional translation method. This translation is implemented for multi-modal $K_{(m)}$.

The relational-functional translation method was introduced as the tree-layered relational translation method in [1]. It is a variation of the relational translation specialised for the basic modal logic $K_{(m)}$ but there is also a close connection to the (optimised) functional translation [22].

The relational-functional translation π_{rf} is specified by the following.

$$\begin{aligned}\pi_{rf}(p, x, \sigma) &= Q_{p, \sigma}(x) \\ \pi_{rf}(\neg\varphi, x, \sigma) &= \neg\pi_{rf}(\varphi, x, \sigma) \\ \pi_{rf}(\varphi \wedge \psi, x, \sigma) &= \pi_{rf}(\varphi, x, \sigma) \wedge \pi_{rf}(\psi, x, \sigma) \\ \pi_{rf}(\Box_i\varphi, x, \sigma) &= \forall y (R_{\sigma, i}(x, y) \supset \pi_{rf}(\varphi, y, \sigma.i))\end{aligned}$$

σ denotes a sequence of natural numbers of length k , ϵ denotes the empty sequence, and ‘.’ is the concatenation operation on sequences.

We have that a modal formula φ is satisfiable in $K_{(m)}$ iff $\pi_{rf}(\varphi, x, \epsilon)$ is first-order satisfiable [1].

4.4.8 Relational-relational translation method

If the option `-EML2Rel=1` is used, dynamic modal formulas are embedded into the relational calculus. The option is disabled by default. It can be combined only with the relational translation to first-order logic, option `-EMLTranslation=0`, which is the default. The translation is specified by the following and the obvious extension to all other operators of the language.

For dynamic modal formulas:

$$\begin{aligned}\pi_{rr}(p) &= p; \top \\ \pi_{rr}(\neg\phi) &= \neg\pi_{rr}(\phi) \\ \pi_{rr}(\phi \wedge \psi) &= \pi_{rr}(\phi) \wedge \pi_{rr}(\psi) \\ \pi_{rr}([\alpha]\phi) &= \neg(\tau_{rr}(\alpha); \neg\pi_{rr}(\phi)) \\ \pi_{rr}(\langle\alpha\rangle\phi) &= \tau_{rr}(\alpha); \pi_{rr}(\phi)\end{aligned}$$

For relational formulas with relational operators τ_{rr} is a homomorphism, and:

$$\begin{aligned}\tau_{rr}(\phi?) &= \pi_{rr}(\phi) \wedge id \\ \tau_{rr}(\alpha|\phi) &= \tau_{rr}(\alpha) \wedge \pi_{rr}(\phi) \\ \tau_{rr}(\alpha|\phi) &= \tau_{rr}(\alpha) \wedge \pi_{rr}(\phi)^\smile\end{aligned}$$

The definition amounts to a reduction of Boolean type formulas via a cylindrification operation (which maps Boolean type formulas to right-ideal relations). There are other ways of embedding dynamic modal logic into the relational calculus, cf. [3].

4.5 Background theories

The `-EMLTheory` option can be used to add certain relational properties to the background theory, and can in particular be used to add the frame correspondence properties for all modalities.

1. `-EMLTheory=1` for $D = \Box_i p \supset \Diamond_i p$.
When the relational translation is used, it adds the seriality condition for every non-equality binary predicate symbol. For the functional, optimised functional and semi-functional translation methods, it simplifies the translation of \Box and \Diamond subformulas.
2. `-EMLTheory=2` for $T = \Box_i p \supset p$.
When the relational translation is used, it adds the reflexivity condition for every non-equality binary predicate symbol. Not implemented for any of the functional translations.
3. `-EMLTheory=3` for $B = \Diamond_i \Box_i p \supset p$.
When the relational translation is used, it adds the symmetry condition for every non-equality binary predicate symbol. Not implemented for any of the functional translations.
4. `-EMLTheory=4` for $4 = \Box_i p \supset \Box_i \Box_i p$.
When the relational translation is used, it adds the transitivity condition for every non-equality binary predicate symbol. Not implemented for any of the functional translations.

5. `-EMLTheory=5` for $S = \diamond_i \Box_i p \supset \Box_i p$.
When the relational translation is used, it adds the Euclideaness condition for every non-equality binary predicate symbol. Not implemented for any of the functional translations.
6. `-EMLTheory=6` for *S4* axioms, i.e. *T* and *4*.
When the relational translation is used, it adds reflexivity and transitivity for every non-equality binary predicate symbol. Not implemented for any of the functional translations.
7. `-EMLTheory=7` for *S5* axioms, in particular, *T*, *B* and *4*.
When the relational translation is used, it adds reflexivity, transitivity and symmetry for every non-equality binary predicate symbol. Not implemented for any of the functional translations.

4.6 Additional options

There are various additional *EML* options.

4.6.1 Path representations

The main purpose of the option `-EMLFuncNary` is to vary the way paths are encoded when the functional translations are used. With `-EMLFuncNary=1` enabled the polyadic versions of the functional translations is used. The option can be combined with the functional translation and the optimised functional translation, but also the relational translation, see Sections 4.4.3, 4.4.5 and 4.4.7 above. This option is switched off by default.

4.6.2 Dead-end predicates

When a functional translation is used, the option `-EMLFuncNdeQ=1` places the nde_i literals outside the scope of the quantifiers. For example, for the functional translation \Box_i formulas are translated according to

$$\pi_f(\Box_i \varphi, s) = \text{nde}_i(s) \supset \forall y:S_i \pi_f(\varphi, [s y:S_i]_i)$$

rather than

$$\pi_f(\Box_i \varphi, s) = \forall y:S_i (\text{nde}_i(s) \supset \pi_f(\varphi, [s y:S_i]_i)).$$

The pattern used is ‘ nde_i literal followed by quantifier’ rather than ‘quantifier followed by nde_i literal’. The translation for \diamond_i formulas and the other functional translations is defined similarly. Enabling this option is only meaningful together with a functional translation method. The default is enabled.

4.6.3 Sorts

When enabled the option `-EMLFFSorts=1` uses explicit sorts in the translation rather than encoding the information into predicate names. Enabling this option is only meaningful together with a functional translation method. By default this option is disabled.

4.6.4 Eliminating composition

When enabled the option `-EMLElimComp=1` attempts to eliminate composition using these replacement rules.

$$\frac{\langle \alpha; \beta \rangle \phi}{\langle \alpha \rangle \langle \beta \rangle \phi} \qquad \frac{[\alpha; \beta] \phi}{[\alpha] [\beta] \phi}$$

By default this option is disabled.

4.6.5 Exchanging quantifiers

When enabled, the option `-QuantExch=1` replaces non-constant Skolem terms in the clausal form of the conjecture are replaced by constants. The option is automatically set for the optimised functional translation methods (`-EMLTranslation=2`). The option is not limited to *EML* logics. It can also be used for classical formulas and clauses. By default this option is disabled.

4.7 Combinations of options

Table 4.3 summarises the combinations of options which are meaningful.

Relational translations		Functional translations	
<code>EMLTranslation</code>	0	<code>EMLTranslation</code>	1, 2, 3
<code>EMLTheory</code>	0, 1, 2, 3, 4, 5, 6, 7	<code>EMLTheory</code>	0, 1, 2, 3, 4, 5, 6, 7
<code>EMLFuncNary</code>	0, 1	<code>EMLFuncNary</code>	0, 1
		<code>EMLFuncNdeQ</code>	0, 1
		<code>EMLFFSorts</code>	0, 1
<code>EMLElimComp</code>	0, 1	<code>EMLElimComp</code>	0, 1
<code>EML2Rel</code>	0, 1		

Table 4.3: Meaningful combinations of options

Bibliography

- [1] C. Areces, R. Gennari, J. Heguiabehere, and M. de Rijke. Tree-based heuristics in modal theorem proving. In W. Horn, editor, *Proceedings of the 14th European Conference on Artificial Intelligence (ECAI-2000)*, pages 199–203. IOS Press, 2000.
- [2] Y. Auffray and P. Enjalbert. Modal theorem proving: An equational viewpoint. *Journal of Logic and Computation*, 2(3):247–297, 1992.
- [3] C. Brink, K. Britz, and R. A. Schmidt. Peirce algebras. *Formal Aspects of Computing*, 6(3):339–358, 1994.
- [4] H. De Nivelle, R. A. Schmidt, and U. Hustadt. Resolution-based methods for modal logics. *Logic Journal of the IGPL*, 8(3):265–292, 2000. Commissioned survey paper.
- [5] M. de Rijke. *Extending Modal Logic*. PhD thesis, Univ. of Amsterdam, December 1993.
- [6] L. Fariñas del Cerro and A. Herzig. Modal deduction with applications in epistemic and temporal logics. In D. M. Gabbay, C. J. Hogger, and J. A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming: Epistemic and Temporal Reasoning*, volume 4, pages 499–594. Clarendon Press, Oxford, 1995.
- [7] A. Herzig. A new decidable fragment of first order logic, June 1990. In Abstracts of the 3rd Logical Biennial, Summer School & Conference in honour of S. C. Kleene, Varna, Bulgaria.
- [8] I. Horrocks, U. Hustadt, U. Sattler, and R. A. Schmidt. Computational modal logic. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, pages 181–245. Elsevier, Amsterdam, 2007. Commissioned overview paper.
- [9] U. Hustadt and R. A. Schmidt. An empirical analysis of modal theorem provers. *Journal of Applied Non-Classical Logics*, 9(4):479–522, 1999.
- [10] U. Hustadt and R. A. Schmidt. Issues of decidability for description logics in the framework of resolution. In R. Caferra and G. Salzer, editors, *Automated Deduction in Classical and Non-Classical Logics*, volume 1761 of *Lecture Notes in Artificial Intelligence*, pages 191–205. Springer, 2000.
- [11] U. Hustadt and R. A. Schmidt. MSPASS: Modal reasoning by translation and first-order resolution. In R. Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods, International Conference (TABLEAUX 2000)*, volume 1847 of *Lecture Notes in Artificial Intelligence*, pages 67–71. Springer, 2000.
- [12] U. Hustadt, R. A. Schmidt, and L. Georgieva. A survey of decidable first-order fragments and description logics. *Journal of Relational Methods in Computer Science*, 1:251–276, 2004. Invited overview paper.
- [13] U. Hustadt, R. A. Schmidt, and C. Weidenbach. MSPASS: Subsumption testing with SPASS. In P. Lambrix, A. Borgida, M. Lenzerini, R. Möller, and P. Patel-Schneider, editors, *Proc. of Intern. Workshop on Description Logics '99*, pages 136–137. Linköping University, 1999.

- [14] K. Nellas. Reasoning about sets and relations: A tableaux-based automated theorem prover for Peirce logic. Master's thesis, University of Manchester, UK, 2001.
- [15] A. Nonnengart. *A Resolution-Based Calculus for Temporal Logics*. PhD thesis, Universität des Saarlandes, Saarbrücken, 1995.
- [16] H. J. Ohlbach. Semantics based translation methods for modal logics. *Journal of Logic and Computation*, 1(5):691–746, 1991.
- [17] H. J. Ohlbach and R. A. Schmidt. Functional translation and second-order frame properties of modal logics. *Journal of Logic and Computation*, 7(5):581–603, 1997.
- [18] R. A. Schmidt. *Optimised Modal Translation and Resolution*. PhD thesis, Universität des Saarlandes, Saarbrücken, Germany, 1997.
- [19] R. A. Schmidt. Decidability by resolution for propositional modal logics. *Journal of Automated Reasoning*, 22(4):379–396, 1999.
- [20] R. A. Schmidt. MSPASS, 1999. <http://www.cs.man.ac.uk/~schmidt/mypass/>.
- [21] R. A. Schmidt and U. Hustadt. Mechanised reasoning and model generation for extended modal logics. In H. C. M. de Swart, E. Orłowska, G. Schmidt, and M. Roubens, editors, *Theory and Applications of Relational Structures as Knowledge Instruments*, volume 2929 of *Lecture Notes in Computer Science*, pages 38–67. Springer, 2003. Survey paper commissioned for the Kickoff Volume of EU COST Action 274.
- [22] R. A. Schmidt and U. Hustadt. First-order resolution methods for modal logics. In A. Podelski, A. Voronkov, and R. Wilhelm, editors, *Volume in memoriam of Harald Ganzinger*, Lecture Notes in Computer Science. Springer, 2006. Invited overview paper, to appear.
- [23] R. A. Schmidt, E. Orłowska, and U. Hustadt. Two proof systems for Peirce algebras. In R. Berghammer, B. Möller, and G. Struth, editors, *Relational and Kleene-Algebraic Methods in Computer Science (RelMiCS 7)*, volume 3051 of *Lecture Notes in Computer Science*, pages 238–251. Springer, 2004.
- [24] Christoph Weidenbach, Renate A. Schmidt, and Dali Topic. Spass input syntax version 3.0. Contained in the documentation of SPASS Version 3.0, 2007.

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