

Interactive Theorem Proving

Week 2

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Summary

So far

Proof Assistants

HOL Light

- HOL Systems
- OCaml and LCF style
- Types and Terms
- Rules
- Introducing Connectives

Today

- Typed λ -calculus, STT
- Type Assignment
- Curry-Howard Isomorphism and example derivations

Different Foundations

Set Theory

- sets and membership
- semantic information
- “collections of things”
- membership is undecidable
- extensional; talk about things that exist

Type Theory

- typing judgement
- syntactic information
- what objects can be constructed
- intentional
- type checking (and sometimes inference) is decidable

Typed λ -calculus

Basis for a Proof Assistant

- Terms: Programs and Proofs
- Types: Specifications and Formulas

Brings together

- Programming
- Proving

Simple Type Theory (STT) or λ_{\rightarrow}

Types

- Atomic types $\alpha \ \beta \ \gamma \ \dots$,
- Function types $\alpha \rightarrow \beta$.

For example: $(\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$

Terms

- Variables with explicit types: $x_1^\sigma, x_2^\sigma, \dots$
 - Countably many for each σ
- Applications: if $M : \sigma \rightarrow \tau$ and $N : \sigma$ then $(MN) : \tau$
- Abstractions: if $P : \tau$ then $(\lambda x^\sigma.P) : \sigma \rightarrow \tau$

Examples

$$\lambda x^\sigma . \lambda y^\tau . x : \sigma \rightarrow \tau \rightarrow \sigma$$

$$\lambda x^{\alpha \rightarrow \beta \rightarrow \gamma} . \lambda y^{\alpha \rightarrow \beta} . \lambda z^\alpha . xz : \beta \rightarrow \gamma$$

Conventions

Parentheses

- Types associate to the right
- Applications associate to the left

α -convertibility

$$\lambda x^\sigma \dots x \dots x \dots \approx_\alpha \lambda y^\sigma \dots y \dots y \dots$$

Capture avoiding substitution

$$M[x := N]$$

β -reduction

$$(\lambda x^\sigma . M)N \longrightarrow_\beta M[x := N]$$

Terms in STT (λ_{\rightarrow})

- Can we find a term for every type?

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$$x^\alpha : \alpha$$

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$$x^\alpha : \alpha$$

- Can we find a closed term for every type?

$$(\alpha \rightarrow \alpha) \rightarrow \alpha$$

- No! Not every type is inhabited.

Type assignment

Typing à la Church

- All terms have the type information in the λ -abstractions
- Unique term types can be computed from the variable types

Typing à la Curry

- Given an untyped λ -term assign types
- Types are no longer unique
- Unification gives principal types

Example: Type $\lambda x.\lambda y.x(\lambda z.y)$

Type assignment

Typing à la Church

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Example: Type $\lambda x.\lambda y.x(\lambda z.y)$

- $((\beta \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$
- $((\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$
- $((\beta \rightarrow \alpha \rightarrow \alpha) \rightarrow \gamma) \rightarrow (\alpha \rightarrow \alpha) \rightarrow \gamma$

Type assignment

Typing à la Church

- All terms have the type information in the λ -abstractions
- Unique term types can be computed from the variable types
- Useful in proving

Typing à la Curry

- Given an untyped λ -term assign types
- Types are no longer unique
- Unification gives principal types
- Useful in programming

Example: Type $\lambda x.\lambda y.x(\lambda z.y)$

- $((\beta \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$
- $((\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$
- $((\beta \rightarrow \alpha \rightarrow \alpha) \rightarrow \gamma) \rightarrow (\alpha \rightarrow \alpha) \rightarrow \gamma$

Connection between STT à la Church and à la Curry

Erasure map: $|\cdot|$

$$|x^\alpha| = x$$

$$|MN| = |M||N|$$

$$|\lambda x^\alpha.M| = \lambda x.|M|$$

Theorem

If $M : \sigma$ in STT à la Church, then $|M| : \sigma$ in STT à la Curry

Theorem

If $N : \sigma$ in STT à la Curry, then $\exists M. |M| = N \wedge M : \sigma$ in STT à la Church

Inductive definition of terms

Rule form

$$\frac{\cdot}{x^\sigma : \sigma}$$

$$\frac{M : \sigma \rightarrow \tau \quad N : \sigma}{MN : \tau}$$

$$\frac{P : \tau}{\lambda x^\sigma. P : \sigma \rightarrow \tau}$$

With a context

- Declare the free variables

$$x_1 : \sigma_1 \dots, x_n : \sigma_n \vdash t : \tau$$

- Usually denoted Γ
- Derivation tree

The three typing rules with a context

Γ treated as a set: not possible for a variable to appear twice

variable rule

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma}$$

abstraction rule

$$\frac{\Gamma, x : \sigma \vdash P : \tau}{\Gamma \vdash (\lambda x : \sigma. P) : (\sigma \rightarrow \tau)}$$

application rule

$$\frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

Provability in λ_{\rightarrow}

$$\Gamma \vdash_{\lambda_{\rightarrow}} M : \sigma$$

iff there exists a derivation using the rules with the conclusion $\Gamma \vdash M : \sigma$

Formulas as Types (Curry-Howard isomorphism)

A typing judgement $M : \sigma$ can be read in two ways:

M is a function with the type σ

- term is an algorithm (program)
- type is its specification

M is a proof of the proposition σ

- type is a proposition
- term is its proof

One to one correspondence between

- Terms in λ_{\rightarrow} (typable)
- Derivations in minimal propositional logic

Blackboard

Minimal Proposition Logic

Subset of Intuitionistic Propositional Logic

Only one connective: \rightarrow

Definition *cut*

$$\frac{\frac{[\sigma^1]}{\mathbb{D}_1} \tau}{\sigma \rightarrow \tau} 1 \quad \mathbb{D}_2 \quad \sigma}{\tau}$$

Minimal Proposition Logic

Subset of Intuitionistic Propositional Logic

Only one connective: \rightarrow

Definition *cut*-elimination

$$\begin{array}{ccc} \frac{[\sigma^1]}{\mathbb{D}_1} & & \mathbb{D}_2 \\ \tau & & \sigma \\ \frac{\tau}{\sigma \rightarrow \tau} \mathbb{1} & \frac{\mathbb{D}_2}{\sigma} & \mathbb{D}_1 \\ \hline \tau & & \tau \end{array}$$

Cut Elimination vs λ_{\rightarrow}

Lemma

Cut-elimination in minimal proposition logic corresponds to β -reduction in λ_{\rightarrow} .

if $\mathbb{D}_1 \longrightarrow_{cut} \mathbb{D}_2$ then $\mathbb{D}_1 \longrightarrow_{\beta} \mathbb{D}_2$

Summary

Today

- λ_{\rightarrow} , Curry, Church styles
- Type Assignment, Erasure map
- Inductive definition of Terms
- Curry-Howard, Example derivations
- Cut-elimination
- Principal Types

Next time

- Untyped lambda calculus
- Type Checking Problem, Synthesis, Inhabitation
- HOL Light tactics