First Exam
Computation with Bounded Resources, LVA 703961

October 2, 2015

Name: **Studentnumber**:

The exam consists of 5 exercises with a total of 100 points. Please fill out your name and credentials *before* you start the exam.

1	2	3	4	5	Sum

1. Let $\mathcal{R}$ be a finite TRS over a finite signature, compatible with a polynomial interpretation $\mathcal{A}$.

   a) Prove the existence of a strictly positive constant $c \in \mathbb{R}$, such that for all terms t, $\mathsf{dh}(t, \to_{\mathcal{R}}) \leqslant 2^{2^{c \cdot |t|}}$. (15 pts)

   b) What happens for *infinite* TRSs (over an infinite signature)? (5 pts)

2. Consider TRS $\mathcal{R}$
$$(x \circ y) \circ z \to x \circ (y \circ z)$$

   a) Find a family of terms t_m such that $\mathsf{dh}(t_m, \to_{\mathcal{R}}) = \Omega(m^2)$. (15 pts)

   b) Using the above, show that there exists a quadratic lowerbound on the derivation height of t in its size $|t|$. (5 pts)

3. Let $\mathcal{A}$ be a strongly linear algebra and set $M := \max\{f_{\mathcal{A}}(0, \dots, 0) \mid f \in \mathcal{F}\}$, where $\mathcal{F}$ denotes a signature. Let α_0 denote the assignment mapping any variable to 0, i.e. $\alpha_0(x) = 0$ for all $x \in \mathcal{V}$, and let t be a term. We write $[t]$ as an abbreviation for $[\alpha_0]_{\mathcal{A}}(t)$. Prove that $[t] \leqslant M \cdot |t|$ holds. (15 pts)

4. Let $\mathcal{A}$ be as above and let $\mathcal{R}$ and $\mathcal{S}$ be TRSs over the signature $\mathcal{F}$ such that $\mathcal{S}$ is compatible with $\mathcal{A}$. We set
$$\Delta := \max\{[r] \dot{-} [l] \mid l \to r \in \mathcal{R}\} ,$$
where $\dot{-}$ is defined as usual. Consider the following claim and its "proof".

   Claim. We have $\mathsf{dh}(t, \to_{\mathcal{R} \cup \mathcal{S}}) \leqslant (1 + \Delta) \cdot \mathsf{dh}(t, \to_{\mathcal{R}/\mathcal{S}}) + M \cdot |t|$, whenever t is terminating on $\mathcal{R} \cup \mathcal{S}$.

   Proof. Let $m = \mathsf{dh}(t, \to_{\mathcal{R}/\mathcal{S}})$, let $n = |t|$. Any derivation of $\to_{\mathcal{R} \cup \mathcal{S}}$ is representable as follows
$$s_0 \to_{\mathcal{S}}^{k_0} t_0 \to_{\mathcal{R}} s_1 \to_{\mathcal{S}}^{k_1} t_1 \to_{\mathcal{R}} \cdots \to_{\mathcal{S}}^{k_m} t_m ,$$
and without loss of generality we may assume that the derivation is maximal. We observe the next two facts.

   (a) $k_i \leqslant [s_i] - [t_i]$ holds for all $0 \leqslant i \leqslant m$. This is because $[s] \geqslant [t] + 1$ whenever $s \to_{\mathcal{S}} t$ by the assumption $\mathcal{S} \subseteq >_{\mathcal{A}}$, and we have $s_i \to_{\mathcal{S}}{}^{k_i} t_i$.

   (b) $[s_{i+1}] - [t_i] \leqslant \Delta$ holds for all $0 \leqslant i < m$ as a simple inductive argument yields $[t_i] + \Delta \geqslant [s_{i+1}]$.

   We obtain the following inequalities:
$$\begin{aligned}
\mathsf{dh}(s_0, \to_{\mathcal{R} \cup \mathcal{S}}) &= m + k_0 + \cdots + k_m \\
&\leqslant m + ([s_0] - [t_0]) + \cdots + ([s_m] - [t_m]) \\
&= m + [s_0] + ([s_1] - [t_0]) + \cdots + ([s_m] - [t_{m-1}]) - [t_m] \\
&\leqslant m + [s_0] + m\Delta - [t_m] \\
&\leqslant m + [s_0] + m\Delta \\
&\leqslant m + M \cdot n + m\Delta = (1 + \Delta)m + M \cdot n .
\end{aligned}$$

Here we used (a) m-times in the second line, (b) $m-1$-times in the fourth line, and the fact that for any term t, $[t] \leqslant M \cdot |t|$ in the last line. (See exercise above.) $\square$

a) The claim and its proof are wrong. Pinpoint the mistake. (5 pts)

b) Provide a counter-example. (15 pts)

c) By whom and where has this proof been published? (5 pts)

5. Determine whether the following statements are true or false. Every correct answer is worth 2 points, every wrong answer -1 points. (The worst that can happen is that you get zero points for this exercise.) (20 pts)

Statement	**yes**	**no**
Multiset path orders induce primitive recursive runtime complexity	☐	☐
The function which maps to each n the length of the Hydra battle starting from $(\omega^{\omega^{\omega}}, n)$ is primitive recursive.	☐	☐
Weak dependency pairs are a variant of dependency pairs, where only weakly oriented rules are considered.	☐	☐
Simple terminating rewrite systems admit at most multiple recursive derivational complexity.	☐	☐
Let A be a well-founded domain (ordered with $\succ$) and let $a \in A$. The order type of a is the maximal number of descends wrt. $\succ$.	☐	☐
Cichon's Principle holds for KBO, MPO, and LPO, but not for polynomial interpretations.	☐	☐
A function is computable in polynomial time by a TRS $\mathcal{R}$ iff the runtime complexity of $\mathcal{R}$ is polynomial.	☐	☐
Matrix interpretations induces exponential runtime complexity.	☐	☐
Let $\mathcal{A}$ be a matrix interpretation and let M denote the maximum matrix wrt. $\mathcal{A}$, that is, the matrix built from the point-wise maximum of all matrices employed in $\mathcal{A}$. Then all entries of M are non-negative.	☐	☐
All methods for complexity analysis via matrix interpretations take the maximum matrix M into account.	☐	☐