Mathematical Proofs as Learning Programs

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$$\mathsf{PA} + \mathsf{SK} \vdash \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \mathcal{P}(x, y) \implies \mathsf{PA} \vdash \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \mathcal{P}(x, y)$$

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Theorem (Kleene-Gödel-Kreisel)

$\mathrm{HA} \vdash \forall x^{\mathrm{N}} \exists y^{\mathrm{N}}\, \mathcal{P}(x, y) \implies$ there is a program A $\forall x^{\mathrm{N}}\, \mathcal{P}(x, A(x))$

Avigad's Forcing

- Condition $s : \mathbb{N} \rightharpoonup \mathbb{N}$:
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- $s \Vdash A \wedge B$ if and only if $s \Vdash A$ and $s \Vdash B$
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- $s \Vdash A \to B$ if and only if $\forall s' \geq s$, if $s' \Vdash A$, then $s' \Vdash B$
- $s \Vdash \forall x^{\mathrm{N}} A$ if and only if for all n, $s \Vdash A[n/x]$
- $s \Vdash \exists x^{\mathrm{N}} A$ if and only if $\forall s' \geq s\ \exists s'' \geq s' \exists n \quad s'' \Vdash A[n/x]$

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$$\mathrm{PA} + \mathrm{SK} \vdash A \implies \mathrm{PA} \vdash (s \Vdash A)$$

$$\mathrm{PA} \vdash (s \Vdash A) \leftrightarrow A$$

A computational semantics for Peano Arithmetic with Skolem axioms:

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A way of making oracle computations effective, through the use of approximations and learning by counterexamples

Learning Based Realizability (2)

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2. CLASSICAL AXIOMS $\Longrightarrow$ LEARNING

   - An efficient method to approximate the ideal program

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- A classical version $\mathcal{T}_{\text{Class}}$ of Gödel's system $\mathcal{T}$

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- For every formula A, add to $\mathcal{T}$ a Skolem constant $\Phi_A : \mathtt{N} \to \mathtt{N}$ such that:

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- We assume to have an enumeration $\Phi_0, \Phi_1, \Phi_2, \ldots$ of all Skolem constants.

Approximations: States of Knowledge

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2. *Approximation at state s* of a term t of $\mathcal{T}_{\text{Class}}$: $t[s]$ results from t by replacing each Skolem function Φ_n with s_n.

Involutive Negation

For each arithmetical formula F, its involutive negation $F^\perp$ is defined by induction on F.

$$(\neg_{\texttt{Bool}} P)^\perp = P \text{ (if } P \text{ positive)} \qquad P^\perp = \neg_{\texttt{Bool}} P \text{ (if } P \text{ positive)}$$

$$(A \wedge B)^\perp = A^\perp \vee B^\perp \qquad (A \vee B)^\perp = A^\perp \wedge B^\perp$$

$$(A \rightarrow B)^\perp = A \setminus B \qquad (A \setminus B)^\perp = A \rightarrow B$$

$$(\forall x^{\mathrm{N}} A)^\perp = \exists x^{\mathrm{N}} A^\perp \qquad (\exists x^{\mathrm{N}} A)^\perp = \forall x^{\mathrm{N}} A^\perp$$

Truth Value of a Formula in a State

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We define $F^{s} := [\![F]\!][s]$ and call it *the truth value of F in the state s*.

Interactive Realizability for PA + SK

Definition (The new parts will appear in red)

Let t be a term of $\mathcal{T}_{\text{Class}}$. We define $t \Vdash_s F$ for any $s \in \mathbb{S}$.

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  2) $t[s] = \varnothing$ implies $P(t_1, \ldots, t_n)[s] = \mathit{True}$
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- $t \Vdash_s A \vee B$ iff either $\pi_0 t[s] = \mathit{True}$ and $\pi_1 t \Vdash_s A$ or $\pi_0 t[s] = \mathit{False}$ and $\pi_2 t \Vdash_s B$
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- $t \Vdash_s \forall x A$ iff for all numerals n, $tn \Vdash_s A[n/x]$
- $t \Vdash_s \exists x A$ iff $\pi_0 t[s] = n$ and $\pi_1 t \Vdash_s A[n/x]$

Definition (The new parts will appear in red)

Let t be a term of $\mathcal{T}_{\text{Class}}$. We define $t \Vdash_s F$ for any $s \in \mathbb{S}$.

- $t \Vdash_s P(t_1, \ldots, t_n)$ iff $t[s] \in \mathbb{P}_{Fin}(\mathrm{N} \times \mathrm{N} \times \mathrm{N})$ and

  1) $(n, m, l) \in t[s]$ and $\Phi_n = \Phi_A$, then:
  $A^s(m, l) = \mathit{True} \wedge A^s(m, s_n(m)) = \mathit{False}$

  2) $t[s] = \varnothing$ implies $P(t_1, \ldots, t_n)[s] = \mathit{True}$
- $t \Vdash_s A \wedge B$ iff $\pi_0 t \Vdash_s A$ and $\pi_1 t \Vdash_s B$
- $t \Vdash_s A \vee B$ iff either $\pi_0 t[s] = \mathit{True}$ and $\pi_1 t \Vdash_s A$ or $\pi_0 t[s] = \mathit{False}$ and $\pi_2 t \Vdash_s B$
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- $t \Vdash_s \exists x A$ iff $\pi_0 t[s] = n$ and $\pi_1 t \Vdash_s A[n/x]$
- $t \Vdash F$ iff $\forall s \in \mathbb{S}\ t \Vdash_s F$

Skolem Axiom

$$\lambda x \lambda y.\ \text{if}\ (\mathscr{T} xy \wedge \neg \mathscr{T} x \Phi_k(x))\ \text{then}\ \{(k, x, y)\}\ \text{else}\ \varnothing$$

$$\Vdash_{s}$$

$$\forall x, y.\ \mathscr{T}(x, y) \rightarrow \mathscr{T}(x, \Phi_k(x))$$

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$$\Vdash_{s}$$

$$\mathscr{T}(n, m) \rightarrow \mathscr{T}(n, s_k(n))$$

Example: Excluded Middle for Simply existential formulas

$$E := \lambda x^{\mathbb{N}} \langle P(x, \Phi_a x),\ \langle \Phi_a x,\ \varnothing \rangle, \lambda y^{\mathbb{N}} \text{ if } P(x,y) \text{ then } (a, x, y) \text{ else } \varnothing \rangle$$

$$E \Vdash \forall x^{\mathbb{N}}.\ \exists y^{\mathbb{N}}\ P(x,y) \vee \forall y^{\mathbb{N}} \neg\ P(x,y)$$

Conservativity of PA + SK over PA for arithmetical formulas

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For every arithmetical formula A

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therefore

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$$t \Vdash \exists x P(x)$$

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$$r_0 := f$$

$$\text{Choose } (n, m, l) \in \pi_1 t[r_n]$$

$$r_{n+1} := (a, b) \mapsto \begin{cases} l & \text{if } a = n, b = m \\ r_n(a, b) & \text{if complexity}(\Phi_a) \leq \text{complexity}(\Phi_n) \\ 0 & \text{otherwise} \end{cases}$$

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$$\exists n.\ \pi_1 t[r_n] = \varnothing$$

Computability for terms $t : \mathtt{N}$ of $\mathcal{T}_{\text{Class}}$, i.e. functions $\mathtt{S} \rightarrow \mathtt{N}$

Computability: Classical VS Constructive Forcing (2)

- *Classical Forcing*

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  2. Constructively true:

     $$M \Vdash t \in \mathrm{N} \equiv \forall k^{\mathrm{S} \to \mathrm{S}}\, \forall s.\ \mathcal{M}ks = s' \to t \downarrow [s', k(s')]$$

Constructive Forcing

$\mathcal{M}$ of $\mathcal{T}$ forces t of $\mathcal{T}_{\text{Class}}$ to be a computable functional of type σ:

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SECPS Translation

Negative Translation $\Leftrightarrow$ CPS translation:

$$t : \sigma \implies t^* : \sigma^{\neg\neg}$$

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Constructive Forcing $\Leftrightarrow$ SECPS translation

$$t : \sigma \implies t^* \Vdash t : \sigma$$

Model Theoretic Interpretation

Define a non-standard model M of Gödel's system $\mathcal{T}_{\text{Class}}$:

$$[\![\mathtt{N}]\!] := \{(f, \mathcal{N}) \mid f : \mathtt{S} \to \mathtt{N} \wedge \mathcal{N} \Vdash f \in \mathtt{N}\}$$

$$[\![\sigma \to \tau]\!] := [\![\sigma]\!] \to [\![\tau]\!]$$

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Then the SECPS t^* defines in Gödel's $\mathcal{T}$ the interpretation of t in the model M.