Complexity Analysis of Term Rewriting Based on Matrix and Context Dependent Interpretations

Georg Moser Andreas Schnabl Johannes Waldmann

Institute of Computer Science
University of Innsbruck

FSTTCS'08, Bengaluru, December 9, 2008

Overview

- Introduction
- Automation
- Context Dependent Interpretations
- Triangular Matrix Interpretations
- Correspondence Result
- Picture

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

$$\mathsf{app} \qquad :: \qquad \mathsf{reverse} \qquad \mathsf{shuffle} \qquad \mathsf{s} \qquad \mathsf{nil} \qquad 0$$

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

$$\mathsf{app} \qquad :: \qquad \mathsf{reverse} \qquad \mathsf{shuffle} \qquad \mathsf{s} \qquad \mathsf{nil} \qquad 0$$

Rules

$$\begin{aligned} \mathsf{app}(\mathsf{nil}, y) &\to y & \mathsf{app}(n :: x, y) &\to n :: \mathsf{app}(x, y) \\ \mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{reverse}(n :: x) &\to \mathsf{app}(\mathsf{reverse}(x), n :: \mathsf{nil}) \\ \mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{shuffle}(n :: x) &\to n :: \mathsf{shuffle}(\mathsf{reverse}(x)) \end{aligned}$$

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

$$\mathsf{app} \qquad :: \qquad \mathsf{reverse} \qquad \mathsf{shuffle} \qquad \mathsf{s} \qquad \mathsf{nil} \qquad \mathsf{0}$$

Rules

$$\begin{aligned} \mathsf{app}(\mathsf{nil}, y) &\to y & \mathsf{app}(n :: x, y) &\to n :: \mathsf{app}(x, y) \\ \mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{reverse}(n :: x) &\to \mathsf{app}(\mathsf{reverse}(x), n :: \mathsf{nil}) \\ \mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{shuffle}(n :: x) &\to n :: \mathsf{shuffle}(\mathsf{reverse}(x)) \end{aligned}$$

$$\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{reverse}(2 :: 3 :: 4 :: 5 :: \mathsf{nil}))$$

(we abbreviate $\mathsf{s}^m(0)$ as m)

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

$$\mathsf{app} \qquad :: \qquad \mathsf{reverse} \qquad \mathsf{shuffle} \qquad \mathsf{s} \qquad \mathsf{nil} \qquad 0$$

Rules

$$\mathsf{app}(\mathsf{nil}, y) \to y \qquad \mathsf{app}(n :: x, y) \to n :: \mathsf{app}(x, y)$$

$$\mathsf{reverse}(\mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{reverse}(n :: x) \to \mathsf{app}(\mathsf{reverse}(x), n :: \mathsf{nil})$$

$$\mathsf{shuffle}(\mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{shuffle}(n :: x) \to n :: \mathsf{shuffle}(\mathsf{reverse}(x))$$

$$\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{reverse}(2 :: 3 :: 4 :: 5 :: \mathsf{nil}))$$

$$\to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{reverse}(3 :: 4 :: 5 :: \mathsf{nil}), 2 :: \mathsf{nil}))$$

(we abbreviate $\mathsf{s}^m(0)$ as m)

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

$$\mathsf{app} \quad :: \quad \mathsf{reverse} \quad \mathsf{shuffle} \quad \mathsf{s} \quad \mathsf{nil} \quad 0$$

Rules

$$\begin{aligned} \mathsf{app}(\mathsf{nil}, y) &\to y & \mathsf{app}(n :: x, y) &\to n :: \mathsf{app}(x, y) \\ \mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{reverse}(n :: x) &\to \mathsf{app}(\mathsf{reverse}(x), n :: \mathsf{nil}) \\ \mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{shuffle}(n :: x) &\to n :: \mathsf{shuffle}(\mathsf{reverse}(x)) \end{aligned}$$

$$\begin{aligned} &\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{reverse}(2 :: 3 :: 4 :: 5 :: \mathsf{nil})) \\ &\to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{reverse}(3 :: 4 :: 5 :: \mathsf{nil}), 2 :: \mathsf{nil})) \\ &\to^{*}_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{nil}, 5 :: \mathsf{nil}), 4 :: \mathsf{nil}), 3 :: \mathsf{nil}), 2 :: \mathsf{nil})) \end{aligned}$$

(we abbreviate $\mathsf{s}^m(0)$ as m)

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

$$\mathsf{app} \qquad :: \qquad \mathsf{reverse} \qquad \mathsf{shuffle} \qquad \mathsf{s} \qquad \mathsf{nil} \qquad 0$$

Rules

$$\begin{aligned}
\mathsf{app}(\mathsf{nil}, y) &\to y & \mathsf{app}(n :: x, y) &\to n :: \mathsf{app}(x, y) \\
\mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{reverse}(n :: x) &\to \mathsf{app}(\mathsf{reverse}(x), n :: \mathsf{nil}) \\
\mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{shuffle}(n :: x) &\to n :: \mathsf{shuffle}(\mathsf{reverse}(x))
\end{aligned}$$

$$\begin{aligned}
&\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{reverse}(2 :: 3 :: 4 :: 5 :: \mathsf{nil})) \\
&\to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{reverse}(3 :: 4 :: 5 :: \mathsf{nil}), 2 :: \mathsf{nil})) \\
&\to_{\mathcal{R}}^{*} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{nil}, 5 :: \mathsf{nil}), 4 :: \mathsf{nil}), 3 :: \mathsf{nil}), 2 :: \mathsf{nil})) \\
&\to_{\mathcal{R}}^{*} 1 :: \mathsf{shuffle}(5 :: 4 :: 3 :: 2 :: \mathsf{nil})
\end{aligned}$$

(we abbreviate $\mathsf{s}^m(0)$ as m)

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

$$\mathsf{app} \qquad :: \qquad \mathsf{reverse} \qquad \mathsf{shuffle} \qquad \mathsf{s} \qquad \mathsf{nil} \qquad \mathsf{0}$$

Rules

$$\begin{aligned} \mathsf{app}(\mathsf{nil}, y) &\to y & \mathsf{app}(n :: x, y) &\to n :: \mathsf{app}(x, y) \\ \mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{reverse}(n :: x) &\to \mathsf{app}(\mathsf{reverse}(x), n :: \mathsf{nil}) \\ \mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{shuffle}(n :: x) &\to n :: \mathsf{shuffle}(\mathsf{reverse}(x)) \end{aligned}$$

$$\begin{aligned} &\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{reverse}(2 :: 3 :: 4 :: 5 :: \mathsf{nil})) \\ &\to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{reverse}(3 :: 4 :: 5 :: \mathsf{nil}), 2 :: \mathsf{nil})) \\ &\to_{\mathcal{R}}^{*} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{nil}, 5 :: \mathsf{nil}), 4 :: \mathsf{nil}), 3 :: \mathsf{nil}), 2 :: \mathsf{nil})) \\ &\to_{\mathcal{R}}^{*} 1 :: \mathsf{shuffle}(5 :: 4 :: 3 :: 2 :: \mathsf{nil}) \to_{\mathcal{R}}^{*} 1 :: 5 :: 2 :: 4 :: 3 :: \mathsf{nil} \end{aligned}$$

(we abbreviate $\mathsf{s}^m(0)$ as m)

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

$$\mathsf{app} \qquad :: \qquad \mathsf{reverse} \qquad \mathsf{shuffle} \qquad \mathsf{s} \qquad \mathsf{nil} \qquad 0$$

Rules

$$\begin{aligned}
\mathsf{app}(\mathsf{nil}, y) &\to y & \mathsf{app}(n :: x, y) &\to n :: \mathsf{app}(x, y) \\
\mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{reverse}(n :: x) &\to \mathsf{app}(\mathsf{reverse}(x), n :: \mathsf{nil}) \\
\mathsf{shuffle}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{shuffle}(n :: x) &\to n :: \mathsf{shuffle}(\mathsf{reverse}(x))
\end{aligned}$$

$$\begin{aligned}
&\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{reverse}(2 :: 3 :: 4 :: 5 :: \mathsf{nil})) \\
&\to_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{reverse}(3 :: 4 :: 5 :: \mathsf{nil}), 2 :: \mathsf{nil})) \\
&\to_{\mathcal{R}}^{*} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{nil}, 5 :: \mathsf{nil}), 4 :: \mathsf{nil}), 3 :: \mathsf{nil}), 2 :: \mathsf{nil})) \\
&\to_{\mathcal{R}}^{*} 1 :: \mathsf{shuffle}(5 :: 4 :: 3 :: 2 :: \mathsf{nil}) \to_{\mathcal{R}}^{*} 1 :: 5 :: 2 :: 4 :: 3 :: \mathsf{nil}
\end{aligned}$$

(we abbreviate $\mathsf{s}^m(0)$ as m)

Term Rewriting

Signature

let signature $\mathcal{F}$ consists of

app :: reverse shuffle s nil 0

Rules

$$\mathsf{app}(\mathsf{nil}, y) \to y \qquad \mathsf{app}(n :: x, y) \to n :: \mathsf{app}(x, y)$$

$$\mathsf{reverse}(\mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{reverse}(n :: x) \to \mathsf{app}(\mathsf{reverse}(x), n :: \mathsf{nil})$$

$$\mathsf{shuffle}(\mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{shuffle}(n :: x) \to n :: \mathsf{shuffle}(\mathsf{reverse}(x))$$

$$\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to^*_{\mathcal{R}} 1 :: 5 :: 2 :: 4 :: 3 :: \mathsf{nil}$$

(we abbreviate $\mathsf{s}^m(0)$ as m)

Complexity of Term Rewriting Systems

Definition

the derivation length of a term t with respect to a TRS $\mathcal{R}$ and rewrite relation $\rightarrow_{\mathcal{R}}$ is defined as:

$$\mathsf{dl}_{\mathcal{R}}(t) = \max\{n \mid \exists u \; t \rightarrow^{n} u\}$$

Complexity of Term Rewriting Systems

Definition

the derivation length of a term t with respect to a TRS $\mathcal{R}$ and rewrite relation $\to_{\mathcal{R}}$ is defined as:

$$\mathsf{dl}_{\mathcal{R}}(t) = \max\{n \mid \exists u\ t \to^n u\}$$

consider $t = \mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil})$, then $\mathsf{dl}_{\mathcal{R}}(t) = 41$:

$$\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to^{1}_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{reverse}(2 :: 3 :: 4 :: 5 :: \mathsf{nil}))$$
$$\to^{5}_{\mathcal{R}} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{nil}, 5 :: \mathsf{nil}), 4 :: \mathsf{nil}), 3 :: \mathsf{nil}), 2 :: \mathsf{nil}))$$
$$\to^{10}_{\mathcal{R}} 1 :: \mathsf{shuffle}(5 :: 4 :: 3 :: 2 :: \mathsf{nil}) \to^{25}_{\mathcal{R}} 1 :: 5 :: 2 :: 3 :: 4 :: \mathsf{nil}$$

Complexity of Term Rewriting Systems

Definition

the derivation length of a term t with respect to a TRS $\mathcal{R}$ and rewrite relation $\to_{\mathcal{R}}$ is defined as:

$$\mathsf{dl}_{\mathcal{R}}(t) = \max\{n \mid \exists u \; t \to^n u\}$$

consider $t = \mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil})$, then $\mathsf{dl}_{\mathcal{R}}(t) = 41$:

$$\begin{aligned}
&\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: 5 :: \mathsf{nil}) \to_{\mathcal{R}}^{1} 1 :: \mathsf{shuffle}(\mathsf{reverse}(2 :: 3 :: 4 :: 5 :: \mathsf{nil})) \\
&\to_{\mathcal{R}}^{5} 1 :: \mathsf{shuffle}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{app}(\mathsf{nil}, 5 :: \mathsf{nil}), 4 :: \mathsf{nil}), 3 :: \mathsf{nil}), 2 :: \mathsf{nil})) \\
&\to_{\mathcal{R}}^{10} 1 :: \mathsf{shuffle}(5 :: 4 :: 3 :: 2 :: \mathsf{nil}) \to_{\mathcal{R}}^{25} 1 :: 5 :: 2 :: 3 :: 4 :: \mathsf{nil}
\end{aligned}$$

in general we obtain a cubic bound on

$$\mathsf{dl}_{\mathcal{R}}(\mathsf{shuffle}(1 :: 2 :: 3 :: 4 :: \cdots :: n :: \mathsf{nil}))$$

in n

Derivational Complexity Analysis

Definition

the derivational complexity (with respect to $\mathcal{R}$) is defined as:

$$\mathsf{dc}_{\mathcal{R}}(n) = \max\{\mathsf{dl}_{\mathcal{R}}(t) \mid \underbrace{|t|}_{\text{size of } t} \leqslant n\}$$

Derivational Complexity Analysis

Definition

the derivational complexity (with respect to $\mathcal{R}$) is defined as:

$$\mathsf{dc}_{\mathcal{R}}(n) = \max\{\mathsf{dl}_{\mathcal{R}}(t) \mid \underbrace{|t|}_{\text{size of } t} \leqslant n\}$$

Known Results

- polynomial interpretations: double-exponential — Hofbauer et al, 1991
- MPO: primitive recursive — Hofbauer, 1992
- LPO: multiple recursive — Weiermann, 1995
- match-bounds: linear — Geser et al, 2007
- matrix interpretations: exponential — Endrullis et al, 2008

Derivational Complexity Analysis

Definition

the derivational complexity (with respect to $\mathcal{R}$) is defined as:

$$\mathsf{dc}_{\mathcal{R}}(n) = \max\{\mathsf{dl}_{\mathcal{R}}(t) \mid \underbrace{|t|}_{\text{size of } t} \leqslant n\}$$

Known Results

- polynomial interpretations: double-exponential — Hofbauer et al, 1991
- MPO: primitive recursive — Hofbauer, 1992
- LPO: multiple recursive — Weiermann, 1995
- match-bounds: linear — Geser et al, 2007
- matrix interpretations: exponential — Endrullis et al, 2008

in this talk we consider linear and quadratic case

Automated Derivational Complexity Analysis

consider the TRS $\mathcal{R}$

$$(x \circ y) \circ z \to x \circ (y \circ z)$$

and polynomial interpretation $\mathcal{A}$ on $\mathbb{N} - \{0\}$:

$$\circ_{\mathcal{A}}(n, m) = 2n + m \qquad\qquad c_{\mathcal{A}} = 1$$

Automated Derivational Complexity Analysis

consider the TRS $\mathcal{R}$

$$(x \circ y) \circ z \to x \circ (y \circ z)$$

and polynomial interpretation $\mathcal{A}$ on $\mathbb{N} - \{0\}$:

$$\circ_{\mathcal{A}}(n, m) = 2n + m \qquad\qquad \mathsf{c}_{\mathcal{A}} = 1$$

Observation

TRS $\mathcal{R}$ is compatible with $\mathcal{A}$ and $\mathsf{dl}_{\mathcal{R}}(t) \leqslant \underbrace{[\alpha]_{\mathcal{A}}(t)}_{\text{evaluation function}}$

Automated Derivational Complexity Analysis

consider the TRS $\mathcal{R}$

$$(x \circ y) \circ z \to x \circ (y \circ z)$$

and polynomial interpretation $\mathcal{A}$ on $\mathbb{N} - \{0\}$:

$$\circ_{\mathcal{A}}(n, m) = 2n + m \qquad\qquad \mathsf{c}_{\mathcal{A}} = 1$$

Observation

TRS $\mathcal{R}$ is compatible with $\mathcal{A}$ and $\mathsf{dl}_{\mathcal{R}}(t) \leqslant \underbrace{[\alpha]_{\mathcal{A}}(t)}_{\text{evaluation function}}$

Theorem

polynomial interpretations induce double-exponential derivational complexity and this bound is optimal in general

Automated Derivational Complexity Analysis

consider the TRS $\mathcal{R}$

$$(x \circ y) \circ z \to x \circ (y \circ z)$$

and polynomial interpretation $\mathcal{A}$ on $\mathbb{N} - \{0\}$:

$$\circ_{\mathcal{A}}(n, m) = 2n + m \qquad \mathsf{c}_{\mathcal{A}} = 1$$

Observation

TRS $\mathcal{R}$ is compatible with $\mathcal{A}$ and $\mathsf{dl}_{\mathcal{R}}(t) \leqslant \underbrace{[\alpha]_{\mathcal{A}}(t)}_{\text{evaluation function}}$

Theorem

polynomial interpretations induce double-exponential derivational complexity and this bound is optimal in general

new ideas are necessary

First Idea

Observation

recall TRS $(x \circ y) \circ z \to x \circ (y \circ z)$ and consider terms: $(t_n)_{n \in \mathbb{N}}$

$$t_0 = \mathsf{c} \qquad\qquad t_{n+1} = t_n \circ \mathsf{c}$$

it is not difficult to see that

$$\mathsf{dl}_{\mathcal{R}}(t_n) \leqslant \frac{n \cdot (n-1)}{2}$$

however $2^n \leqslant [\alpha]_{\mathcal{A}}(t_n)$

First Idea

Observation

recall TRS $(x \circ y) \circ z \to x \circ (y \circ z)$ and consider terms: $(t_n)_{n \in \mathbb{N}}$

$$t_0 = \mathsf{c} \qquad\qquad t_{n+1} = t_n \circ \mathsf{c}$$

it is not difficult to see that

$$\mathsf{dl}_{\mathcal{R}}(t_n) \leqslant \frac{n \cdot (n-1)}{2}$$

however $2^n \leqslant [\alpha]_{\mathcal{A}}(t_n)$

Context Dependent Interpretations (CDI)

First Idea

Observation

recall TRS $(x \circ y) \circ z \to x \circ (y \circ z)$ and consider terms: $(t_n)_{n \in \mathbb{N}}$

$$t_0 = \mathsf{c} \qquad\qquad t_{n+1} = t_n \circ \mathsf{c}$$

it is not difficult to see that

$$\mathrm{dl}_{\mathcal{R}}(t_n) \leqslant \frac{n \cdot (n-1)}{2}$$

however $2^n \leqslant [\alpha]_{\mathcal{A}}(t_n)$

Context Dependent Interpretations (CDI)

- extend polynomial interpretations by an additional parameter that changes in the course of evaluation
- influence of the context in the evaluation of a term is limited

Definition

context dependent $\mathcal{F}$-algebra (CDA) $\mathcal{C}$ assigns to each n-ary f:

1. $f_{\mathcal{C}} \colon \mathbb{R}^+ \times (\mathbb{R}_0^+)^n \to \mathbb{R}_0^+$ — interpretation function
2. $f_{\mathcal{C}}^i \colon \mathbb{R}^+ \to \mathbb{R}^+$ for all $1 \leqslant i \leqslant n$ — parameter function

Definition

context dependent $\mathcal{F}$-algebra (CDA) $\mathcal{C}$ assigns to each n-ary f:

1. $f_{\mathcal{C}} \colon \mathbb{R}^+ \times (\mathbb{R}_0^+)^n \to \mathbb{R}_0^+$ — interpretation function
2. $f_{\mathcal{C}}^i \colon \mathbb{R}^+ \to \mathbb{R}^+$ for all $1 \leqslant i \leqslant n$ — parameter function

and

3. consider set $\{>_\Delta \mid \Delta \in \mathbb{R}^+\}$, where $z >_\Delta z'$ if and only if $z - z' \geqslant \Delta$

Definition

context dependent $\mathcal{F}$-algebra (CDA) $\mathcal{C}$ assigns to each n-ary f:

1. $f_{\mathcal{C}} \colon \mathbb{R}^+ \times (\mathbb{R}_0^+)^n \to \mathbb{R}_0^+$ — interpretation function
2. $f_{\mathcal{C}}^i \colon \mathbb{R}^+ \to \mathbb{R}^+$ for all $1 \leqslant i \leqslant n$ — parameter function

and

3. consider set $\{>_\Delta \mid \Delta \in \mathbb{R}^+\}$, where $z >_\Delta z'$ if and only if $z - z' \geqslant \Delta$

Definition

- a Δ-assignment is a mapping: $\alpha \colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$;

Definition

context dependent $\mathcal{F}$-algebra (CDA) $\mathcal{C}$ assigns to each n-ary f:

1. $f_{\mathcal{C}} \colon \mathbb{R}^+ \times (\mathbb{R}_0^+)^n \to \mathbb{R}_0^+$ — interpretation function
2. $f_{\mathcal{C}}^i \colon \mathbb{R}^+ \to \mathbb{R}^+$ for all $1 \leqslant i \leqslant n$ — parameter function

and

3. consider set $\{>_\Delta \mid \Delta \in \mathbb{R}^+\}$, where $z >_\Delta z'$ if and only if $z - z' \geqslant \Delta$

Definition

- a Δ-assignment is a mapping: $\alpha : \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$; extend to terms:

$$[\alpha, \Delta]_{\mathcal{C}}(t) = \begin{cases} \alpha(\Delta, t) & \text{if } t \in \mathcal{V} \\ f_{\mathcal{C}}(\Delta, [\alpha, f_{\mathcal{C}}^1(\Delta)]_{\mathcal{C}}(t_1), \dots, [\alpha, f_{\mathcal{C}}^n(\Delta)]_{\mathcal{C}}(t_n)) & \text{otherwise} \end{cases}$$

Definition

context dependent $\mathcal{F}$-algebra (CDA) $\mathcal{C}$ assigns to each n-ary f:

1. $f_\mathcal{C} \colon \mathbb{R}^+ \times (\mathbb{R}_0^+)^n \to \mathbb{R}_0^+$ interpretation function
2. $f_\mathcal{C}^i \colon \mathbb{R}^+ \to \mathbb{R}^+$ for all $1 \leqslant i \leqslant n$ parameter function

and

3. consider set $\{>_\Delta \mid \Delta \in \mathbb{R}^+\}$, where $z >_\Delta z'$ if and only if $z - z' \geqslant \Delta$

Definition

- a Δ-assignment is a mapping: $\alpha \colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$; extend to terms:

$$[\alpha, \Delta]_\mathcal{C}(t) = \begin{cases} \alpha(\Delta, t) & \text{if } t \in \mathcal{V} \\ f_\mathcal{C}(\Delta, [\alpha, f_\mathcal{C}^1(\Delta)]_\mathcal{C}(t_1), \ldots, [\alpha, f_\mathcal{C}^n(\Delta)]_\mathcal{C}(t_n)) & \text{otherwise} \end{cases}$$

- a Δ-assignment is linear, if $\forall\ x \in \mathcal{V}$, $\exists\ a, b \in \mathbb{N}$
  $\alpha(\Delta, x) = a + b\Delta$

Definition

a CDA $\mathcal{C}$ is

- Δ-monotone if
  $\forall\ \Delta \in \mathbb{R}^+$, $\forall\ a_1, \ldots, a_n, b \in \mathbb{R}_0^+$ with $a_i >_{f_{\mathcal{C}}^i(\Delta)} b$:

  $$f_{\mathcal{C}}(\Delta, a_1, \ldots, a_i, \ldots, a_n) >_\Delta f_{\mathcal{C}}(\Delta, a_1, \ldots, b, \ldots, a_n)$$

Definition

a CDA $\mathcal{C}$ is

- Δ-monotone if
  $\forall\ \Delta \in \mathbb{R}^+$, $\forall\ a_1, \ldots, a_n, b \in \mathbb{R}_0^+$ with $a_i >_{f_{\mathcal{C}}^i(\Delta)} b$:

  $$f_{\mathcal{C}}(\Delta, a_1, \ldots, a_i, \ldots, a_n) >_{\Delta} f_{\mathcal{C}}(\Delta, a_1, \ldots, b, \ldots, a_n)$$

- compatible with a TRS $\mathcal{R}$ if
  $\forall\ l \to r \in \mathcal{R}$, $\forall\ \Delta \in \mathbb{R}^+$, $\forall\ \Delta$-assignment α:

  $$[\alpha, \Delta]_{\mathcal{C}}(l) >_{\Delta} [\alpha, \Delta]_{\mathcal{C}}(r)$$

Definition

a CDA $\mathcal{C}$ is

- Δ-monotone if
  $\forall\ \Delta \in \mathbb{R}^+,\ \forall\ a_1, \ldots, a_n, b \in \mathbb{R}_0^+$ with $a_i >_{f_\mathcal{C}^i(\Delta)} b$:

  $f_\mathcal{C}(\Delta, a_1, \ldots, a_i, \ldots, a_n) >_\Delta f_\mathcal{C}(\Delta, a_1, \ldots, b, \ldots, a_n)$

- compatible with a TRS $\mathcal{R}$ if
  $\forall\ l \to r \in \mathcal{R},\ \forall\ \Delta \in \mathbb{R}^+,\ \forall\ \Delta$-assignment α:

  $[\alpha, \Delta]_\mathcal{C}(l) >_\Delta [\alpha, \Delta]_\mathcal{C}(r)$

Theorem

Hofbauer, 2001

let $\mathcal{R}$ be a TRS and let $\mathcal{C}$ denote a Δ-monotone and compatible CDA;
then $\mathcal{R}$ is terminating and $\forall$ terms t

$$\mathsf{dl}_\mathcal{R}(t) \leqslant \liminf_{\Delta \in \mathbb{R}^+} \frac{[\alpha, \Delta]_\mathcal{C}(t)}{\Delta}$$

Second Idea

Observation

a TRS $\mathcal{R}$ is terminating if

- $\exists$ algebra $\mathcal{A}$
- $\exists$ well-founded order $>_{\mathcal{A}}$ on $\mathcal{A}$ such that

$$[\alpha]_{\mathcal{A}}(l) >_{\mathcal{A}} [\alpha]_{\mathcal{A}}(r) \qquad \forall\ l \to r \in \mathcal{R} \quad \forall \text{ assignments } \alpha$$

Second Idea

Observation

a TRS $\mathcal{R}$ is terminating if

- $\exists$ algebra $\mathcal{A}$
- $\exists$ well-founded order $>_{\mathcal{A}}$ on $\mathcal{A}$ such that

$$[\alpha]_{\mathcal{A}}(l) >_{\mathcal{A}} [\alpha]_{\mathcal{A}}(r) \qquad \forall\ l \to r \in \mathcal{R} \quad \forall \text{ assignments } \alpha$$

consider the matrix interpretation $\mathcal{A}$ with domain $\mathbb{N}^2$:

$$\circ_{\mathcal{A}}(\vec{x}, \vec{y}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Second Idea

Observation

a TRS $\mathcal{R}$ is terminating if

- $\exists$ algebra $\mathcal{A}$
- $\exists$ well-founded order $>_{\mathcal{A}}$ on $\mathcal{A}$ such that

$$[\alpha]_{\mathcal{A}}(l) >_{\mathcal{A}} [\alpha]_{\mathcal{A}}(r) \qquad \forall\, l \to r \in \mathcal{R} \quad \forall \text{ assignments } \alpha$$

consider the matrix interpretation $\mathcal{A}$ with domain $\mathbb{N}^2$:

$$\circ_{\mathcal{A}}(\vec{x}, \vec{y}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we obtain

- $\forall$ assignments α: $[\alpha]_{\mathcal{A}}((x \circ y) \circ z) >_{\mathcal{A}} [\alpha]_{\mathcal{A}}(x \circ (y \circ z))$
- $\forall$ matrices M considered in $[\alpha]_{\mathcal{A}}(t)$:
  coefficients of M are bounded polynomially in $|t|$

Definition

a matrix interpretation $\mathcal{A}$ for dimension d assigns to each n-ary f:

1. $f_{\mathcal{A}} : \mathbb{N}^d \times \cdots \times \mathbb{N}^d \to \mathbb{N}^d$
   $(\vec{v}_1, \ldots, \vec{v}_n) \mapsto F_1 \vec{v}_1 + \ldots + F_n \vec{v}_n + \vec{f}$

Definition

a matrix interpretation $\mathcal{A}$ for dimension d assigns to each n-ary f:

1. $f_{\mathcal{A}} : \mathbb{N}^d \times \cdots \times \mathbb{N}^d \to \mathbb{N}^d$
   $(\vec{v}_1, \ldots, \vec{v}_n) \mapsto F_1\vec{v}_1 + \ldots + F_n\vec{v}_n + \vec{f}$

and

2. $(x_1, x_2, \ldots, x_d) >_{\mathcal{A}} (y_1, y_2, \ldots, y_d)$ is defined as
   $x_1 > y_1$ and $\forall\ i \geqslant 2$: $x_i \geqslant y_i$

Definition

a matrix interpretation $\mathcal{A}$ for dimension d assigns to each n-ary f:

1. $f_{\mathcal{A}} : \mathbb{N}^d \times \cdots \times \mathbb{N}^d \to \mathbb{N}^d$
   $(\vec{v}_1, \ldots, \vec{v}_n) \mapsto F_1\vec{v}_1 + \ldots + F_n\vec{v}_n + \vec{f}$

and

2. $(x_1, x_2, \ldots, x_d) >_{\mathcal{A}} (y_1, y_2, \ldots, y_d)$ is defined as
   $x_1 > y_1$ and $\forall\ i \geqslant 2\colon\ x_i \geqslant y_i$

Definition

- an upper triangular matrix is a matrix M such that
  $\forall\ i > j\colon\ M_{i,j} = 0$ and $\forall\ i\colon\ M_{i,i} \leqslant 1$
- a triangular matrix interpretation (TMI) is a monotone matrix interpretation using only upper triangular matrices

Definition

a matrix interpretation $\mathcal{A}$ for dimension d assigns to each n-ary f:

1. $f_{\mathcal{A}} : \mathbb{N}^d \times \cdots \times \mathbb{N}^d \to \mathbb{N}^d$
   $(\vec{v}_1, \ldots, \vec{v}_n) \mapsto F_1 \vec{v}_1 + \ldots + F_n \vec{v}_n + \vec{f}$

and

2. $(x_1, x_2, \ldots, x_d) >_{\mathcal{A}} (y_1, y_2, \ldots, y_d)$ is defined as
   $x_1 > y_1$ and $\forall\ i \geqslant 2\colon\ x_i \geqslant y_i$

Definition

- an upper triangular matrix is a matrix M such that
  $\forall\ i > j\colon\ M_{i,j} = 0$ and $\forall\ i\colon\ M_{i,i} \leqslant 1$
- a triangular matrix interpretation (TMI) is a monotone matrix interpretation using only upper triangular matrices

Theorem

if TRS $\mathcal{R}$ admits a TMI of dimension d: $\mathrm{dc}_{\mathcal{R}}$ bounded by a polynomial of degree d

Definition

a linear Δ assignment $\alpha \colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$ and an assignment $\alpha' \colon \mathcal{V} \to \mathbb{N}^2$ are corresponding if $\alpha(\Delta, x) = a + b\Delta \iff \alpha'(x) = \begin{pmatrix} b \\ a \end{pmatrix}$

Definition

a linear Δ assignment $\alpha\colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$ and an assignment $\alpha'\colon \mathcal{V} \to \mathbb{N}^2$ are corresponding if $\alpha(\Delta, x) = a + b\Delta \iff \alpha'(x) = \begin{pmatrix} b \\ a \end{pmatrix}$

Lemma

let $\mathcal{C}$ a CDA:

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

Definition

a linear Δ assignment $\alpha\colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$ and an assignment $\alpha'\colon \mathcal{V} \to \mathbb{N}^2$ are corresponding if $\alpha(\Delta, x) = a + b\Delta \iff \alpha'(x) = \begin{pmatrix} b \\ a \end{pmatrix}$

Lemma

let $\mathcal{C}$ a CDA:

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

and $\mathcal{A}$ a matrix interpretation

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} \begin{pmatrix} d_i & b_i \\ c_i & a_i \end{pmatrix} \cdot x_i\right) + \begin{pmatrix} h \\ g \end{pmatrix}$$

Definition

a linear Δ assignment $\alpha \colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$ and an assignment $\alpha' \colon \mathcal{V} \to \mathbb{N}^2$ are corresponding if $\alpha(\Delta, x) = a + b\Delta \iff \alpha'(x) = \begin{pmatrix} b \\ a \end{pmatrix}$

Lemma

let $\mathcal{C}$ a CDA:

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

and $\mathcal{A}$ a matrix interpretation

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = (\sum_{i=1}^{n} \begin{pmatrix} d_i & b_i \\ c_i & a_i \end{pmatrix} \cdot x_i) + \begin{pmatrix} h \\ g \end{pmatrix}$$

Definition

a linear Δ assignment $\alpha\colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$ and an assignment $\alpha'\colon \mathcal{V} \to \mathbb{N}^2$ are corresponding if $\alpha(\Delta, x) = a + b\Delta \iff \alpha'(x) = \begin{pmatrix} b \\ a \end{pmatrix}$

Lemma

let $\mathcal{C}$ a CDA:

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^i(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

and $\mathcal{A}$ a matrix interpretation ($\forall\ i$: either $a_i > 0$ or $b_i > 0$)

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = (\sum_{i=1}^{n} \begin{pmatrix} d_i & b_i \\ c_i & a_i \end{pmatrix} \cdot x_i) + \begin{pmatrix} h \\ g \end{pmatrix}$$

Definition

a linear Δ assignment $\alpha\colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$ and an assignment $\alpha'\colon \mathcal{V} \to \mathbb{N}^2$ are corresponding if $\alpha(\Delta, x) = a + b\Delta \iff \alpha'(x) = \begin{pmatrix} b \\ a \end{pmatrix}$

Lemma

let $\mathcal{C}$ a CDA:

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

and $\mathcal{A}$ a matrix interpretation

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = (\sum_{i=1}^{n} \begin{pmatrix} d_i & b_i \\ c_i & a_i \end{pmatrix} \cdot x_i) + \begin{pmatrix} h \\ g \end{pmatrix}$$

then, if α and α' are corresponding:

$$[\alpha, \Delta]_{\mathcal{C}}(t) = s_1 + s_2\Delta \quad \iff \quad [\alpha']_{\mathcal{A}}(t) = \begin{pmatrix} s_2 \\ s_1 \end{pmatrix}$$

Theorem

1. let $\mathcal{R}$ a TRS compatible with CDA $\mathcal{C}$ of form

$$f_{\mathcal{C}}(\Delta, z_1, \dots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i \Delta}{a_i + b_i \Delta}$$

then $\exists$ matrix interpretation $\mathcal{A}$ (of dimension 2) compatible with $\mathcal{R}$

Theorem

1. let $\mathcal{R}$ a TRS compatible with CDA $\mathcal{C}$ of form

$$f_{\mathcal{C}}(\Delta, z_1, \dots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i \Delta}{a_i + b_i \Delta}$$

   then $\exists$ matrix interpretation $\mathcal{A}$ (of dimension 2) compatible with $\mathcal{R}$

2. if matrix interpretation $\mathcal{A}$ is triangular, then $\mathrm{dc}_{\mathcal{R}}(n) = \mathrm{O}(n^2)$

Theorem

1. let $\mathcal{R}$ a TRS compatible with CDA $\mathcal{C}$ of form

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i \Delta}{a_i + b_i \Delta}$$

   then $\exists$ matrix interpretation $\mathcal{A}$ (of dimension 2) compatible with $\mathcal{R}$

2. if matrix interpretation $\mathcal{A}$ is triangular, then $\mathrm{dc}_{\mathcal{R}}(n) = \mathrm{O}(n^2)$

3. let $\mathcal{R}$ be compatible with a matrix interpretation $\mathcal{A}$ of dimension 2, then

Theorem

1. let $\mathcal{R}$ a TRS compatible with CDA $\mathcal{C}$ of form

$$f_{\mathcal{C}}(\Delta, z_1, \dots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

   then $\exists$ matrix interpretation $\mathcal{A}$ (of dimension 2) compatible with $\mathcal{R}$

2. if matrix interpretation $\mathcal{A}$ is triangular, then $\mathrm{dc}_{\mathcal{R}}(n) = \mathrm{O}(n^2)$

3. let $\mathcal{R}$ be compatible with a matrix interpretation $\mathcal{A}$ of dimension 2, then
   - $\exists$ CDA of above form such that
   - $\forall$ (linear) Δ-assignments α, $\forall\ \Delta \in \mathbb{R}^+$, $\forall\ l \to r \in \mathcal{R}$:

Theorem

1. let $\mathcal{R}$ a TRS compatible with CDA $\mathcal{C}$ of form

$$f_{\mathcal{C}}(\Delta, z_1, \dots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

   then $\exists$ matrix interpretation $\mathcal{A}$ (of dimension 2) compatible with $\mathcal{R}$

2. if matrix interpretation $\mathcal{A}$ is triangular, then $\mathrm{dc}_{\mathcal{R}}(n) = \mathrm{O}(n^2)$

3. let $\mathcal{R}$ be compatible with a matrix interpretation $\mathcal{A}$ of dimension 2, then
   - $\exists$ CDA of above form such that
   - $\forall$ (linear) Δ-assignments α, $\forall$ $\Delta \in \mathbb{R}^+$, $\forall$ $l \to r \in \mathcal{R}$:
   - $[\alpha, \Delta]_{\mathcal{C}}(l) >_{\Delta} [\alpha, \Delta]_{\mathcal{C}}(r)$

Theorem

1. let $\mathcal{R}$ a TRS compatible with CDA $\mathcal{C}$ of form

$$f_{\mathcal{C}}(\Delta, z_1, \dots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

   then $\exists$ matrix interpretation $\mathcal{A}$ (of dimension 2) compatible with $\mathcal{R}$

2. if matrix interpretation $\mathcal{A}$ is triangular, then $\mathrm{dc}_{\mathcal{R}}(n) = \mathrm{O}(n^2)$

3. let $\mathcal{R}$ be compatible with a matrix interpretation $\mathcal{A}$ of dimension 2, then
   - $\exists$ CDA of above form such that
   - $\forall$ (linear) Δ-assignments α, $\forall\ \Delta \in \mathbb{R}^+$, $\forall\ l \to r \in \mathcal{R}$:
   - $[\alpha, \Delta]_{\mathcal{C}}(l) >_{\Delta} [\alpha, \Delta]_{\mathcal{C}}(r)$

   no zero column must occur in any matrix

Theorem

1. let $\mathcal{R}$ a TRS compatible with CDA $\mathcal{C}$ of form

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

   then $\exists$ matrix interpretation $\mathcal{A}$ (of dimension 2) compatible with $\mathcal{R}$

2. if matrix interpretation $\mathcal{A}$ is triangular, then $\mathrm{dc}_{\mathcal{R}}(n) = \mathrm{O}(n^2)$

3. let $\mathcal{R}$ be compatible with a matrix interpretation $\mathcal{A}$ of dimension 2, then
   - $\exists$ CDA of above form such that
   - $\forall$ (linear) Δ-assignments α, $\forall$ $\Delta \in \mathbb{R}^+$, $\forall$ $l \to r \in \mathcal{R}$:
   - $[\alpha, \Delta]_{\mathcal{C}}(l) >_{\Delta} [\alpha, \Delta]_{\mathcal{C}}(r)$

Picture

exponential

polynomial

quadratic

Picture

matrix — exponential

---

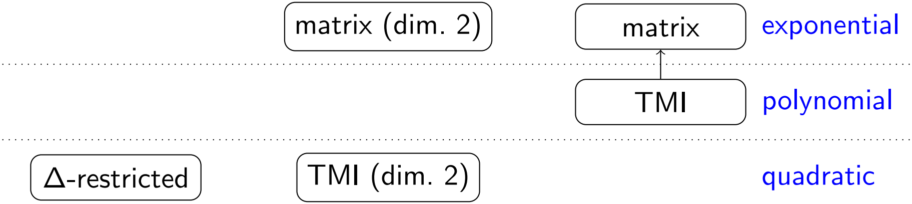
polynomial

Δ-restricted — quadratic

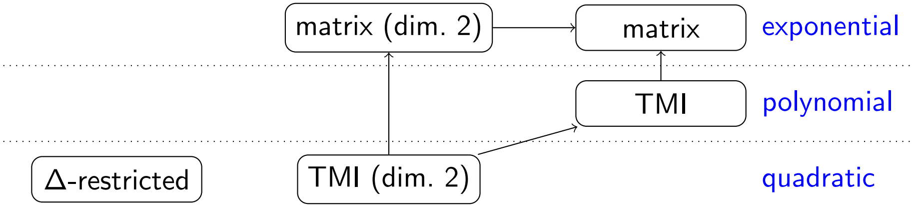
Picture


matrix
exponential
TMI
polynomial
Δ-restricted
quadratic

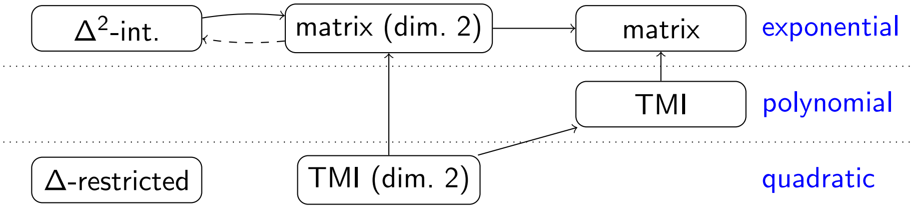
Picture


matrix (dim. 2)
matrix
exponential
TMI
polynomial
Δ-restricted
TMI (dim. 2)
quadratic

Picture


matrix (dim. 2)
matrix
exponential
TMI
polynomial
Δ-restricted
TMI (dim. 2)
quadratic

Picture



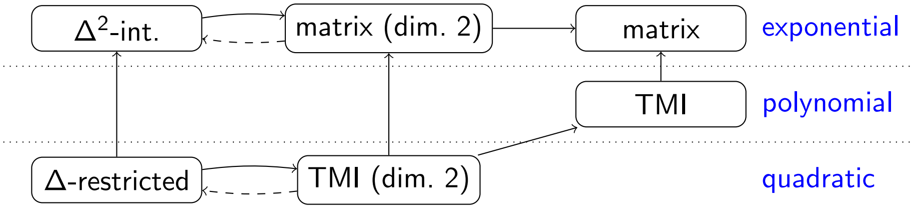
NB

- Δ^2-interpretations refers to CDAs $\mathcal{C}$ of form:

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

Picture



NB

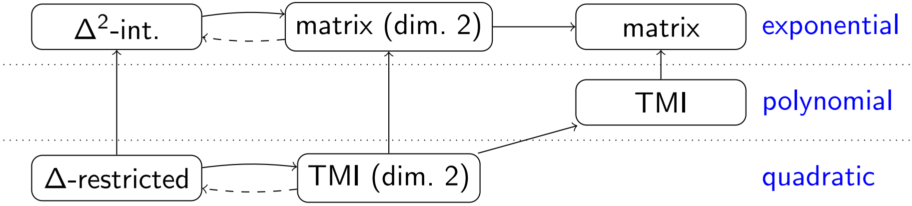
- Δ^2-interpretations refers to CDAs $\mathcal{C}$ of form:

$$f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i\Delta}{a_i + b_i\Delta}$$

- Δ-restricted is as above, but $a_i \leqslant 1$, $c_i = 0$, and $d_i = 1$

Automated Derivational Complexity Analysis



NB

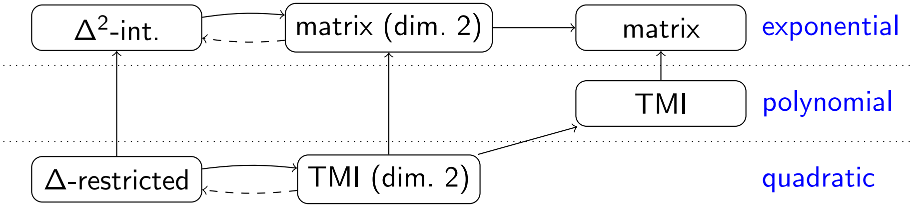
- Δ^2-interpretations refers to CDAs $\mathcal{C}$ of form:

$$f_{\mathcal{C}}(\Delta, z_1, \dots, z_n) = \sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} b_i z_i \Delta + g + h\Delta$$

$$f_{\mathcal{C}}^{i}(\Delta) = \frac{c_i + d_i \Delta}{a_i + b_i \Delta}$$

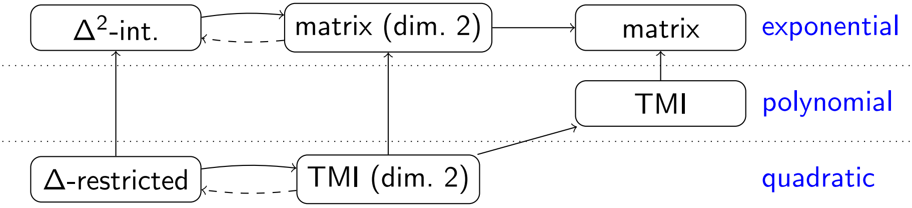
- Δ-restricted is as above, but $a_i \leqslant 1$, $c_i = 0$, and $d_i = 1$

Automated Derivational Complexity Analysis



NB

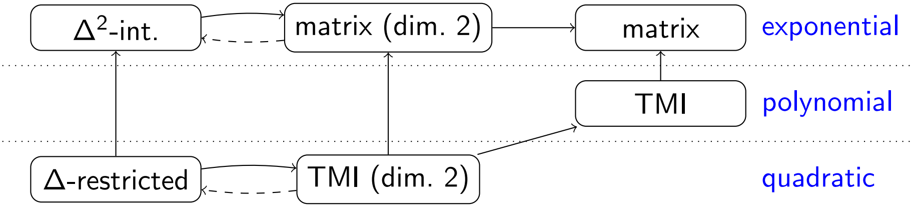
Automated Derivational Complexity Analysis



NB

	CDI	TMI				TMI $\cup$ BOUNDS
• dimension		2	3	4	5	3
# successes						

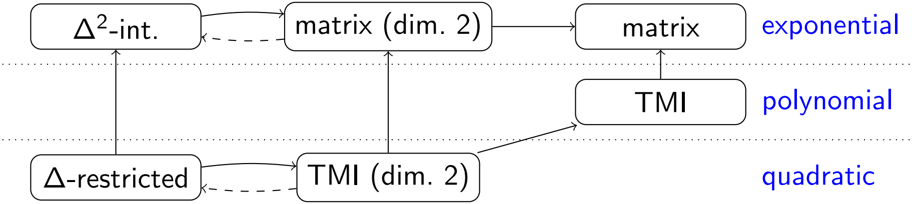
Automated Derivational Complexity Analysis



NB

	CDI	TMI				TMI ∪ BOUNDS
• dimension		2	3	4	5	3
# successes	85					

Automated Derivational Complexity Analysis

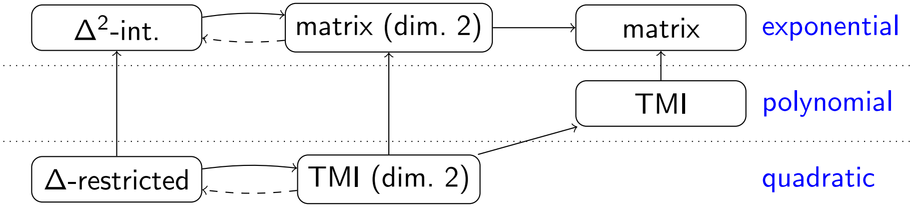


NB

-

	CDI	TMI				TMI ∪ BOUNDS
dimension		2	3	4	5	3
# successes	85	143				

Automated Derivational Complexity Analysis

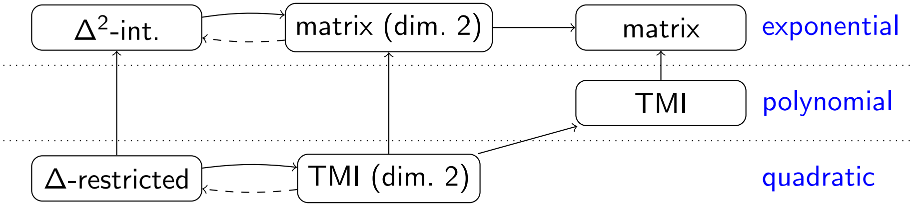


NB

-

dimension	CDI	TMI 2	TMI 3	TMI 4	TMI 5	TMI ∪ BOUNDS 3
# successes	85	143	158			

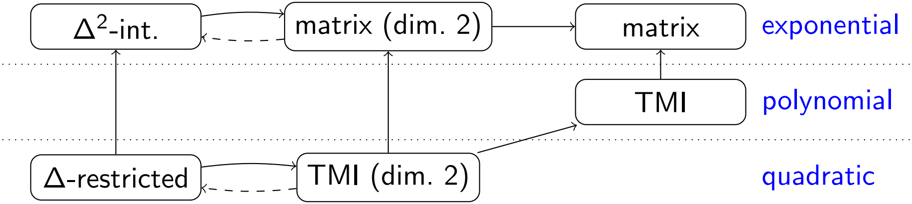
Automated Derivational Complexity Analysis



NB

	CDI	TMI				TMI ∪ BOUNDS
dimension		2	3	4	5	3
# successes	85	143	158	154		

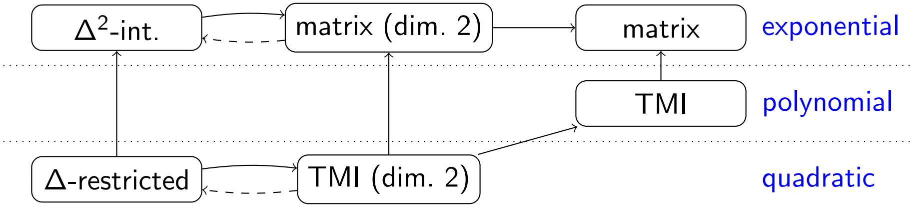
Automated Derivational Complexity Analysis



NB

• dimension	CDI	TMI				TMI ∪ BOUNDS
		2	3	4	5	3
# successes	85	143	158	154	156	

Automated Derivational Complexity Analysis



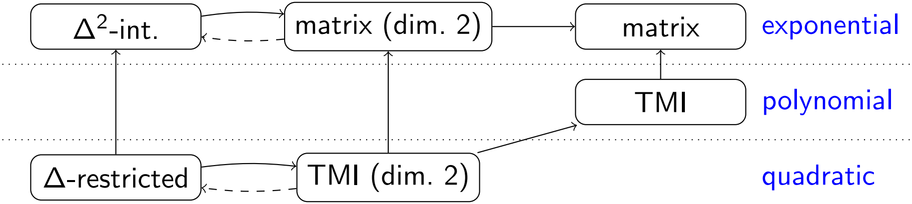
NB

-

dimension	CDI	TMI 2	TMI 3	TMI 4	TMI 5	TMI ∪ BOUNDS 3
# successes	85	143	158	154	156	216

- TPDB contains 445 non-duplicating, known to be automatically terminating TRSs

Automated Derivational Complexity Analysis



NB

dimension	CDI	TMI				TMI ∪ BOUNDS
		2	3	4	5	3
# successes	85	143	158	154	156	216

- TPDB contains 445 non-duplicating, known to be automatically terminating TRSs