Weighted Automata Theory for Complexity Analysis of Rewrite Systems

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Overview

- Motivation
  - Quiz
  - Termination and Polynomial Interpretations
- Matrix Interpretation
- Weighted Automata
- Growth Rate of Automata
- Runtime Complexity Analysis
- Conclusion

Example ("A common implementation of purely functional queues")

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empty x = (nil, nil);

checkF (f,r) = match f with
                 | nil -> (rev(r),nil)
                 | (x::xs) -> (f,r);

snoc (queue,x) = match queue with
                 | (f,r) -> checkF(f,x::r);

enq n = match n with
                 | 0 -> empty()
                 | S n' -> snoc(enq(n'),n');

main = enq 3;
main = ([0],[3,2,1])
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Comment

queues are implemented as pairs of list f and r; checkF guarantees the invariant that f is only empty, if the queue is empty

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linear? ☐ quadratic? ☐ exponential? ☐

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- the potential function Φ is a mapping from values to $\mathbb{R}^+$
- suppose d_i is the output of the i^{th} operation and input to the $i+1^{th}$ operation; t_i the actual cost of the i^{th} operation
- the amortised cost of operation i is defined as follows:

$$a_i = t_i + \Phi(d_i) - \Phi(d_{i-1})$$

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Bounding the Total Cost

$$\sum_{i=1}^{j} t_i = \sum_{i=1}^{j} (a_i + \Phi(d_{i-1}) - \Phi(d_i)) = \sum_{i=1}^{j} a_i + \Phi(d_0) - \Phi(d_j) \leqslant \sum_{i=1}^{j} a_i$$

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[1] J. Hoffmann, K. Aehlig, and M. Hofmann. Multivariate amortized resource analysis. In *Proc. 38th POPL*, pages 357–370. ACM, 2011.

[2] M. Hofmann and GM. Amortised resource analysis and typed polynomial interpretations. In *Proc. of 25th RTA & 12th TLCA*, 2014. Tuesday.

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\mathsf{chk}(\mathsf{q}(\mathsf{nil}, r)) &\to \mathsf{q}(\mathsf{rev}(r), \mathsf{nil}) & \mathsf{enq}(0) &\to \mathsf{q}(\mathsf{nil}, \mathsf{nil}) \\
\mathsf{chk}(\mathsf{q}(x :: xs, r)) &\to \mathsf{q}(x :: xs, r) & \mathsf{rev}'(\mathsf{nil}, ys) &\to ys \\
\mathsf{tl}(\mathsf{q}(x :: f, r)) &\to \mathsf{chk}(\mathsf{q}(f, r)) & \mathsf{rev}(xs) &\to \mathsf{rev}'(xs, \mathsf{nil}) \\
\mathsf{snoc}(\mathsf{q}(f, r), x) &\to \mathsf{chk}(\mathsf{q}(f, x :: r)) & \mathsf{hd}(\mathsf{q}(x :: f, r)) &\to x \\
\mathsf{rev}'(x :: xs, ys) &\to \mathsf{rev}'(xs, x :: ys) & \mathsf{hd}(\mathsf{q}(\mathsf{nil}, r)) &\to \mathsf{err\_head} \\
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- $\mathrm{dh}(t, \rightarrow) = \max\{n \mid \exists u\ t \rightarrow^{n} u\}$
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Example (continued)

- the following well-founded monotone algebra $\mathcal{A} = (\mathbb{N}, >)$ is a polynomial interpretation

$$\mathsf{s}_{\mathcal{A}}(n) = 28n + 24 \qquad \mathsf{snoc}_{\mathcal{A}}(q, x) = 16q + 16x + 24$$
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Lemma

a matrix interpretation $(\mathbb{N}^n, >)$ *is a well-founded monotone algebra*

Example

consider the TRS

$$(x \circ y) \circ z \to x \circ (y \circ z)$$

together with the matrix interpretation $\mathcal{M}$

$$\circ_{\mathcal{M}}(\vec{x}, \vec{y}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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$$t = t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} t_4 \to_{\mathcal{R}} \cdots$$

implies

$$[\alpha]_{\mathcal{M}}(t) > [\alpha]_{\mathcal{M}}(t_1) > [\alpha]_{\mathcal{M}}(t_2) > [\alpha]_{\mathcal{M}}(t_3) > [\alpha]_{\mathcal{M}}(t_4) > \cdots$$

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Observation

if $\mathcal{M}$ is compatible with $\mathcal{R}$ then $\mathsf{dc}_{\mathcal{R}}(k) \leqslant \mathsf{c}_{\mathcal{M}}(k)$; thus it suffices to bound the complexity induced by matrix interpretations suitably

Weighted Automata

Definitions

- a weighted automaton is a quintuple $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$
  1. $Q = \{1, \dots, n\}$ is a set of states
  2. Σ a finite alphabet
  3. λ, γ are weight functions (in $\mathbb{N}$) for entering and leaving a state
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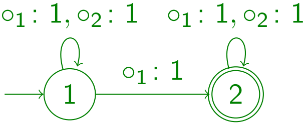
- the growth function of $\mathcal{A}$ is defined as

$$\mathrm{growth}_{\mathcal{A}}(k) = \max \{\mathrm{weight}_{\mathcal{A}}(x) \mid x \in \Sigma^k\}$$

Example

consider $\mathcal{A} = (\{1,2\}, \Sigma, \lambda, \mu, \gamma)$:



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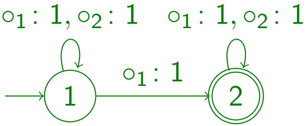

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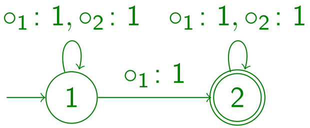

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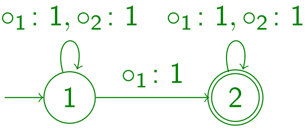

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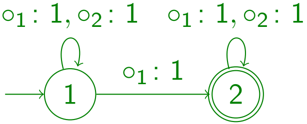

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$C^{\mathcal{M}}$ is the component-wise maximum of all absolute vectors

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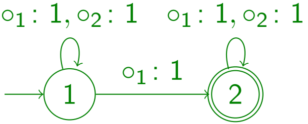

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From the Complexity of Interpretations to the Growth Rate of Automata

Lemma

let $\mathcal{M}$ be a matrix interpretation with corresponding automaton $\mathcal{A}$, if $\mathsf{growth}_{\mathcal{A}}(k) \in O(k^d)$ *then* $\mathsf{c}_{\mathcal{M}}(k) \in O(k^{d+1})$

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- lemma follows from the claim

∎

Polynomially Bounded Growth

Definition

consider the following criterion:

$$\exists\, q \in Q\, \exists\, x \in \Sigma^* \text{ such that weight}(q, x, q) \geqslant 2 \text{ and } q \text{ useful} \qquad \text{(EDA)}$$

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[1] R. Jungers. *The Joint Spectral Radius: Theory and Applications*. Springer Verlag, 2009.

[2] W. Kuich. Finite automata and ambiguity. Technical Report 253, Institut für Informationsverarbeitung, Technische Universität Graz und ÖCG, 1988.

[3] A. Weber and H. Seidl. On the degree of ambiguity of finite automata. *TCS*, 88(2):325–349, 1991.

Precise Growth Rate

Definition

consider the following criterion:

$$\exists\, p_1, q_1, \ldots, p_d, q_d \in Q\ \exists\, v_1, u_2, v_2, \ldots, u_d, v_d \in \Sigma^* \text{ so that}$$
$$\textit{forall } i \geqslant 1\ \forall j \geqslant 2 p_i \neq q_i,\ p_i \xrightarrow{v_i} p_i,\ p_i \xrightarrow{v_i} q_i,\ q_i \xrightarrow{v_i} q_i,\ q_{j-1} \xrightarrow{u_j} p_j \qquad (\mathrm{IDA}_d)$$

which is visualised as follows:

Theorem

$\mathcal{A}$ be an automaton that does not comply with EDA*;*
$\text{growth}_{\mathcal{A}}(k) = \Omega(k^d)$ *iff $\mathcal{A}$ complies with* IDA_d

Proof.

- by assumption the growth rate is polynomially bounded
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Theorem

$\mathrm{growth}_{\mathcal{A}}(k) = \Theta(k^d)$ *iff $\mathcal{A}$ does not comply with* EDA *nor with* IDA_{d+1}*, but complies with* IDA_d*; furthermore thes properties are polynomially decidable*

Summing Up

Corollary

1. *let $\mathcal{R}$ be a TRS and let $\mathcal{M}$ be a compatible matrix interpretation of dimension n*
2. *let $\mathcal{A}$ be the corresponding weighted automaton such that $\mathcal{A}$ does not comply with* EDA *nor with* IDA_{d+1}

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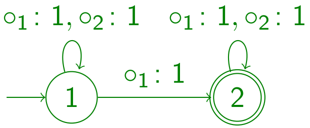
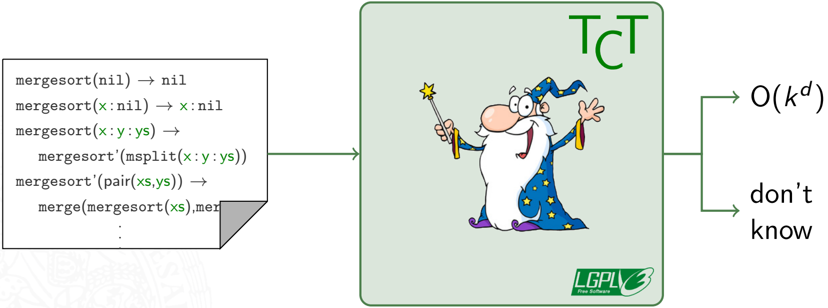


$$\left.\begin{array}{l}\mathcal{A} \text{ does not comply with EDA nor with } \mathsf{IDA}_2 \\ \mathcal{A} \text{ complies with } \mathsf{IDA}_1\end{array}\right\} \Rightarrow \mathsf{dc}_{\mathcal{R}}(k) \in \mathsf{O}(k^2)$$

Automation

(runtime) complexity analyser for rewrite systems

http://cl-informatik.uibk.ac.at/software/tct



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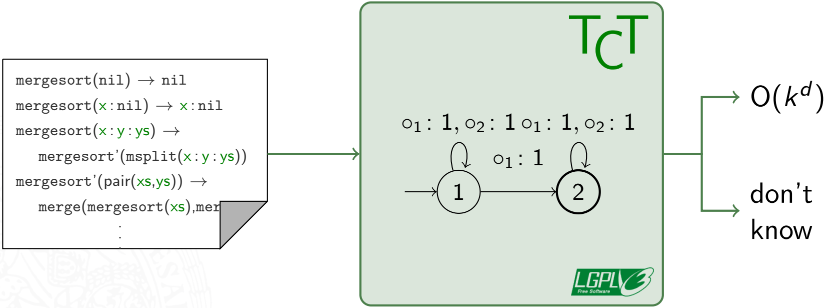

Back to the Beginning

Example

$$\begin{aligned}
\mathsf{chk}(\mathsf{q}(\mathsf{nil}, r)) &\to \mathsf{q}(\mathsf{rev}(r), \mathsf{nil}) & \mathsf{enq}(0) &\to \mathsf{q}(\mathsf{nil}, \mathsf{nil}) \\
\mathsf{chk}(\mathsf{q}(x :: xs, r)) &\to \mathsf{q}(x :: xs, r) & \mathsf{rev}'(\mathsf{nil}, ys) &\to ys \\
\mathsf{tl}(\mathsf{q}(x :: f, r)) &\to \mathsf{chk}(\mathsf{q}(f, r)) & \mathsf{rev}(xs) &\to \mathsf{rev}'(xs, \mathsf{nil}) \\
\mathsf{snoc}(\mathsf{q}(f, r), x) &\to \mathsf{chk}(\mathsf{q}(f, x :: r)) & \mathsf{hd}(\mathsf{q}(x :: f, r)) &\to x \\
\mathsf{rev}'(x :: xs, ys) &\to \mathsf{rev}'(xs, x :: ys) & \mathsf{hd}(\mathsf{q}(\mathsf{nil}, r)) &\to \mathsf{err_head} \\
\mathsf{enq}(\mathsf{s}(n)) &\to \mathsf{snoc}(\mathsf{enq}(n), n) & \mathsf{tl}(\mathsf{q}(\mathsf{nil}, r)) &\to \mathsf{err_tail}
\end{aligned}$$

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TCT says

```
TcT> apply $ matrix `withDimension` 4 `withDegree` Just 1
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  No Progress :(
```

Runtime Complexity Analysis

Definitions

- a term $f(t_1, \dots, t_n)$ is basic if f is defined and $\forall\ i$: $t_i \in \mathcal{T}(\mathcal{C}, \mathcal{V})$

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Runtime Complexity Analysis via Weighted Automata

Corollary

1. *let* $\mathcal{R}$ *be a TRS and let* $\mathcal{M}$ *be a compatible matrix interpretation of dimension* n
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  Hurray, the problem was solved with certicficate
  YES(?,O(n^1)). Use 'proof' to show the complete proof.
```

Example (continued)

- consider the compatible matrix interpretation $\mathcal{M}$

$$\vec{x} ::_{\mathcal{M}} \vec{xs} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \vec{xs} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathsf{q}_{\mathcal{M}}(\vec{f}, \vec{r}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{f} + \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \vec{r} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

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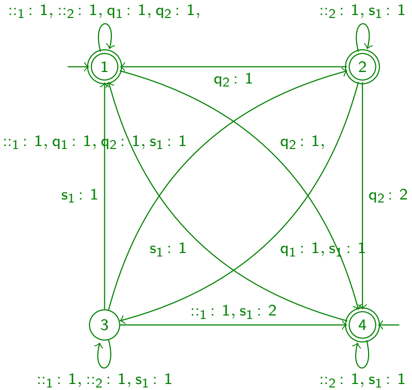

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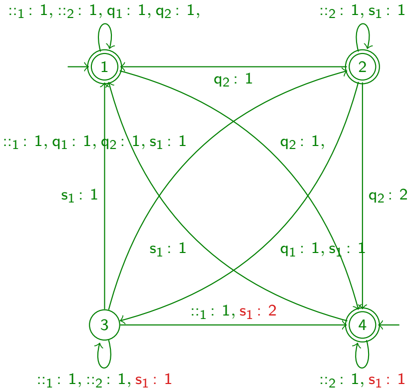

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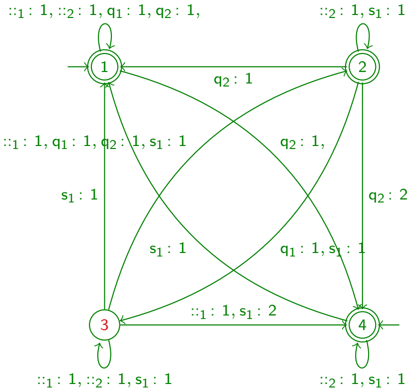

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Conclusion

Summary

- termination techniques like interpretation methods yield bounds on the complexity of rewrite systems
- complexity induced by matrix interpretations best estimated via growth rates of automata (or joint spectral radius)
- (very) expensive
- surprisingly versatile

Thank You for Your Attention!