Relating derivation lengths with the slow-growing hierarchy directly

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joint work with Andreas Weiermann

[1] Supported by a Marie Curie fellowship, grant number HPMF-CT-2002-015777.

Why Ordinals are Good for You

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Complexity of Rewrite Systems

Let R be a finite rewrite system. Let $\succ$ denote a termination order such that $\rightarrow_R$ is contained in $\succ$. I.e. R has already been shown to be terminating.
Assume

$$s \rightarrow_R t_1 \rightarrow_R \cdots \rightarrow_R t_n$$

holds.

Goal
Calculate, depending on R and the term depth of s ($\leq m$) an upper bound for n.

Some results ...

MPO yields primitive recursive upper bounds on the derivation length (or complexity) ...(Hofbauer,1992 [2])
LPO...multiple recursive upper bounds....(Weiermann,1995,[5]).
The KBO.....derivation length functions lies within $Ack(2^{O(m)}, 0)$ ...(Lepper, 2001 [3]).

Same proof idea:

Idea

Define an interpretation function $\mathcal{I}\colon \mathcal{T}(\Sigma, \mathcal{V}) \to \mathbb{N}$ *such that*

$$\mathcal{I}(s) > \mathcal{I}(t_1) > \cdots > \mathcal{I}(t_n)$$

holds.

Ordinals and Order Types

Idea

We blatantly claim that a uniform measure for the <u>strength</u> of (termination) orders is their order type, i.e. an <u>ordinal</u>.

Idea

Separate $\mathcal{I}$ *into* $\pi\colon \mathcal{T}(\Sigma) \to \mathrm{ON}$ *and a function* $g\colon \mathrm{ON} \to \mathbb{N}$*. Prove for* $l, r \in \mathcal{T}(\Sigma)$

$$l \succ_{\mathrm{LPO}} r \Rightarrow \pi(l) > \pi(r) \Rightarrow g(\pi(l)) > g(\pi(r)) \quad .$$

Fix some notations: Let $\Sigma = \{f_1, \dots, f_N\}$ denote a finite signature; let K denote the maximal arity in Σ. Suppose R is a (simply) terminating TRS, compatible with some lexicographic path order $\succ_{\mathrm{LPO}}$.

How to represent ordinals

Definition

Definition of a set T of ordinal terms, and a subset $P \subset T$, built from 0, $+$, and the $(K+1)$-ary function symbol ψ.

Definition

Recursive definition of a partial order $>$ on T. Some cases
Let $\alpha = \psi(\alpha_1, \ldots, \alpha_{K+1})$, $\beta = \psi(\beta_1, \ldots, \beta_{K+1})$. Then $\alpha > \beta$ iff
(a) there exists k with $\alpha_k \geq \beta$, or
(b) $\alpha > \beta_l$ for all l and there exists an i s.t.
$\alpha_1 = \beta_1, \ldots \alpha_{i-1} = \beta_{i-1}$ and $\alpha_i > \beta_i$.

Links to "usual" denotations for ordinals: Set $1 := \psi(\overline{0})$;
$\omega := \psi(\overline{0}, 1)$; $\epsilon_0 := \psi(\overline{0}, 1, 0)$; $\Gamma_0 := \psi(\overline{0}, 1, 0, 0)$.

LPO and Ordinal (Terms)

Fact

Let $(T, <)$ be defined as above. Then $(T, <)$ is a linear, strict, well-founded partial order.

Theorem

- *For any number k, there exists an embedding from $(\mathcal{T}(\Sigma(k)), \prec_{\mathrm{LPO}})$ into $(T(k+1), <)$.*
- *For any number $k > 2$, there exists an embedding from $(T(k), <)$ into $(\mathcal{T}(\Sigma(k)), \prec_{\mathrm{LPO}})$.*
- $\sup_{k<\omega}(\mathrm{otyp}(T(k), <)) = \sup_{k<\omega}(\mathrm{otyp}(\mathcal{T}(\Sigma(k)), \prec_{\mathrm{LPO}})) = \Lambda$.

A Collapsing Function

Definition

(The slow-growing hierarchy). $\mathrm{G}_\alpha : \mathbb{N} \to \mathbb{N}$ for $\alpha \in T$.

$$\begin{array}{lcl} \mathrm{G}_0(x) & := & 0 \\ \mathrm{G}_{\alpha+1}(x) & := & \mathrm{G}_\alpha(x) + 1 \\ \mathrm{G}_\lambda(x) & := & \mathrm{G}_{\lambda[x]}(x) \end{array} .$$

Fact

For all $\alpha, \beta \in T$.
(1) G_α *is increasing in x (strictly if* α *is infinite).*
(2) Assume $\alpha < \beta$. *Then* G_α *eventually dominates* G_β. *That is* *for almost all x,* $\mathrm{G}_\alpha(x) > \mathrm{G}_\beta(x)$.

What is the meaning of $\lambda[x]$?

Definition

(some easy (!) cases).

$$
\begin{aligned}
0[x] &:= 0 \\
(\alpha_1 + \cdots + \alpha_m)[x] &:= \alpha_1 + \cdots + \alpha_m[x] \\
\psi(\overline{0})[x] &:= 0 \\
\psi(\overline{0}, \beta + 1)[x] &:= \psi(\overline{0}, \beta) \cdot (x + 1) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, 0)[x] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \overline{0})^{x+1}(0) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \beta + 1)[x] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \overline{0})^{x+1}(\psi(\overline{\alpha}, \overline{0}, \beta)) \\
\psi(\alpha_1, \ldots, \alpha_i, \overline{0}, \beta + 1)[x] &:= \psi(\alpha_1, \ldots, \alpha_i[x], \overline{0}, \psi(\overline{\alpha}, \overline{0}, \beta)) \quad \alpha_i \in \mathrm{L{\small I}}
\\
&\vdots
\end{aligned}
$$

All Cases

$$
\begin{aligned}
0[x] &:= 0 \\
(\alpha_1 + \cdots + \alpha_m)[x] &:= \alpha_1 + \cdots + \alpha_m[x] \\
\psi(\overline{0})[x] &:= 0 \\
\psi(\overline{0}, \beta + 1)[x] &:= \psi(\overline{0}, \beta) \cdot (x + 1) \\
\psi(\overline{0}, \lambda)[x] &:= \psi(\overline{0}, \lambda[x]) \qquad \lambda \notin \textsc{Fix}(\overline{0}) \\
\psi(\overline{0}, \lambda)[x] &:= \lambda \cdot (x + 1) \qquad \lambda \in \textsc{Fix}(\overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, 0)[x] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \overline{0})^{x+1}(0) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \beta + 1)[x] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \overline{0})^{x+1}(\psi(\overline{\alpha}, \overline{0}, \beta)) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \lambda)[x] &:= \psi(\overline{\alpha}, \overline{0}, \lambda[x]) \qquad \lambda \notin \textsc{Fix}(\overline{\alpha}, \overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \lambda)[x] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \overline{0})^{x+1}(\lambda) \\
\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, 0)[x] &:= \psi(\alpha_1, \ldots, \lambda_i[x], \overline{0}, \mathrm{MS}_{\overline{\alpha}, \lambda_i, \overline{0}}(\overline{\alpha}, \lambda_i)) \\
\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \beta + 1)[x] &:= \psi(\alpha_1, \ldots, \lambda_i[x], \overline{0}, \psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \beta)) \\
\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \lambda)[x] &:= \psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \lambda[x]) \qquad \lambda \notin \textsc{Fix}(\overline{\alpha}, \overline{0}) \\
\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \lambda)[x] &:= \psi(\alpha_1, \ldots, \lambda_i[x], \overline{0}, \lambda)
\end{aligned}
$$

Relation between the G's and $\textsc{Mrec}$

The G_α functions are really slow growing. Note

$$\mathrm{G}_x(\omega) = \mathrm{G}_x(\psi(\bar{0}) \cdot (x+1)) = \mathrm{G}_x(x+1) = \mathrm{G}_x(x) + 1 = x + 1.$$

Recall the class of multiple recursive functions $\textsc{Mrec}$. It can be characterized as the elementarz closure of the union of all *k*-ary Ackermann functions.

Fact (Robbin,1995,Girard,1981 [4, 1])

The multiple recursive functions ($\textsc{Mrec}$) can be classified as

$$\bigcup_{\alpha \in T} \mathrm{G}_\alpha.$$

Relation between the G's and $\textsc{Mrec}$ (cont.)

Proof.

(*First attempt*). Working in the ground term algebra, we start with

$$s \to_R t_1 \to_R \cdots \to_R t_n \quad .$$

As $\to_R \subset \succ_{\mathrm{LPO}}$, this yields

$$s \succ_{\mathrm{LPO}} t_1 \succ_{\mathrm{LPO}} \cdots \succ_{\mathrm{LPO}} t_n \quad .$$

The first part of Idea 3 should establish

$$\alpha > \pi(s) > \pi(t_1) > \cdots > \pi(t_n)$$

for some $\alpha \in T$. Now what we want is

$$\mathrm{G}_\alpha(h) > \mathrm{G}_{\pi(s)}(h) > \mathrm{G}_{\pi(t_1)} > \cdots > \mathrm{G}_{\pi(t_n)} \quad .$$

for some (effectively given) natural number h depending on R and $\mathrm{dp}(s)$ only. □

How-to compute h?

Definition

Let $<_{(x)}$ denote the transitive closure of $.[x]$. I.e. the closure of $.[x][x][x]\ldots$.

Fact

Then $\alpha >_{(x)} \beta$ implies $\mathrm{G}_m(\alpha) > \mathrm{G}_m(\beta)$ for all $m > x$.

Idea

Find a number h such that $>_{(h)}$ serves as an approximation of $\succ_{\mathrm{LPO}}$. That is

$$l \succ_{\mathrm{LPO}} r \Rightarrow \pi(l) >_{(h)} \pi(r)$$

holds.

Doesn't work! ... Problematic are the Subterm Property and Monotonicty Property of the lexicographic path order!

Solution: Extend the definitions!

Definition (System of fundamental sequences of $(T, <)$.)

- Assume $\alpha = 0$, then $(\alpha)^x := \emptyset$.
- Assume $\alpha = \psi(\overline{\alpha})$. Then $\beta \in (\alpha)^x$ if
  (a) $\beta = \psi(\alpha_1, \ldots, \alpha_i^*, \ldots, \alpha_{N+1})$, and $\alpha_i^* \in (\alpha_i)^x$, or
  (b) $\beta = \alpha_i + x$, where $\alpha_i > 0$, or
  (c) $\beta = \psi(\overline{\alpha})[x]$.

Now (re)define $\alpha >_{(x)} \beta$ recursively as
$\exists \gamma \in (\alpha)^x \ (\gamma = \beta \vee \gamma >_{(x)} \beta)$.

Almost done

Definition

Definition of the function $\widetilde{\mathrm{G}}_\alpha \colon \mathbb{N} \to \mathbb{N}$ for $x \in \mathbb{N}$.

$$\begin{array}{lcl} \widetilde{\mathrm{G}}_0(x) & := & 0 \\ \widetilde{\mathrm{G}}_\alpha(x) & := & \max\{\widetilde{\mathrm{G}}_\beta(x) \colon \beta \in (\alpha)^x\} + 1 \quad . \end{array}$$

Theorem

Let $\alpha \in T$, $\alpha > 0$ be given. Assume $x \in \mathbb{N}$ is arbitrary.
(1) $\widetilde{\mathrm{G}}_\alpha(x)$ is increasing in x. (Even strictly if $\alpha > \omega$.)
(2) If $\alpha >_{(x)} \beta$, then $\widetilde{\mathrm{G}}_\alpha(x) > \widetilde{\mathrm{G}}_\beta(x)$ for all x.

Theorem

$\bigcup_{\alpha \in T} \mathrm{G}_\alpha \approx \bigcup_{\alpha \in T} \widetilde{\mathrm{G}}_\alpha$.

LPO implies Multiple Recursive Upper Bounds

Definition

(Interpretation function $\pi\colon \mathcal{T}(\Sigma) \to T$.) If $s = f_j \in \Sigma$, then set $\pi(s) := \psi(j, \overline{0})$. Otherwise, let $s = f_j(s_1, \ldots, s_m)$ and set $\pi(s) := \psi(j, \pi(s_1), \ldots, \pi(s_m) + 1, \overline{0})$.

Theorem

Then we can show the existence of a number k, such that for all $l, r \in \mathcal{T}(\Sigma, \mathcal{V})$, and any ground substitution ρ

$$l \to_R r \Rightarrow \pi(l\rho) >_{(k)} \pi(r\rho) \quad .$$

Recall $\mathrm{dp}(s) \leq m$. Let $h := O(\max\{k, m\})$. Then finally we obtain

$$\widetilde{\mathrm{G}}_{\psi(K+1,\overline{0})}(h) \geq n \quad .$$

Conclusion

This approach is uniformly applicable. In addition to LPO, it works smoothly when the rewrite system R is compatible with

1. a multiset path order, or
2. a Knuth Bendix order.

The necessary changes for MPO are almost trivial. The changes necessary for KBO are more severe. As KBO takes account of the weight of functions, the ordinal interpretation becomes less canonical.

Conclusion (cont.)

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Conclusion (cont.)

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