The Hydra Battle revisted

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Motivating question

Question

- In 1991, Adam Cichon asked: *Must any termination ordering used for proving termination of the Battle of Hydra and Hercules-system have the Howard[-Bachmann] ordinal as its order type?*

Open Problem

- What is the solution to Problem 23, in the RTA list of open problems?

Consider the TRS $\mathcal{H}$—the Battle of Hydra and Hercules-system:

$$
\begin{aligned}
h(e(x), y) &\rightarrow h(d(x, y), S(y)) \\
d(g(0,0), y) &\rightarrow e(0) \\
d(g(x, y), z) &\rightarrow g(e(x), d(y, z)) \\
d(g(g(x, y), 0), S(z)) &\rightarrow g(d(g(x, y), S(z)), d(g(x, y), z)) \\
g(e(x), e(y)) &\rightarrow e(g(x, y))
\end{aligned}
$$

Question Is the order type of any reduction order compatible with $\mathcal{H}$ the Howard-Bachmann ordinal?[1]

Answer No

[1] The proof-theoretic ordinal of the arithmetical theory of one inductive definition ID_1.

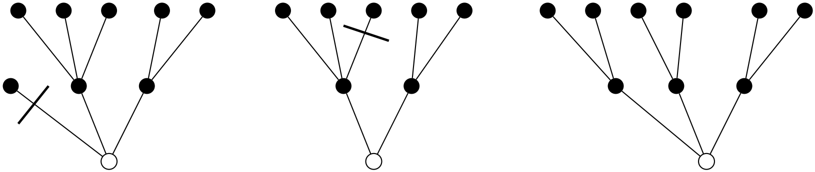
Outline

The Hydra Battle by Kirby and Paris

Definition

- The beast is a finite tree, each leaf corresponds to a head; Hercules chops off heads of the Hydra, but the Hydra regrows:
  - If the cut head has a pre-predecessor, then the remaining subtree issued from this node is multiplied by the stage of the game.
  - Otherwise the Hydra ignores the loss.
- Hercules wins, when the beast is reduced to the empty tree.

Example



$(H_1, 1)$ $(H_2, 2)$ $(H_3, 3)$

(Well-known) Facts

Definition A strategy is a mapping determining which head Hercules chops off at each stage.

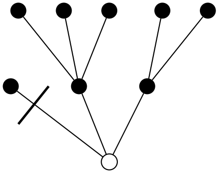
Theorem [Kirby, Paris] Every strategy is a winning strategy.

In proof, associate with each Hydra an ordinal $< \epsilon_0$:

- ➡ To each leaf assign 0.
- ➡ To each other node v assign $\omega^{\alpha_1} \oplus \cdots \oplus \omega^{\alpha_n}$, if α_i are the ordinals assigned to the successors of v.
- ➡ The ordinal representing the Hydra, is the ordinal assigned to the root.

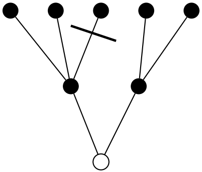
$\oplus$ denotes the natural sum.

Example



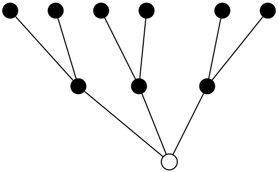
$(H_1, 1)$

$(\omega^3 \oplus \omega^2 \oplus 1, 1)$



$(H_2, 2)$

$(\omega^3 \oplus \omega^2, 2)$



$(H_3, 3)$

$(\omega^2 \cdot 3, 3)$

$$\omega^3 + \omega^2 + 1 \quad > \quad \omega^3 + \omega^2 \quad > \quad \omega^2 \cdot 3$$

The Standard Hydra Battle

- define a specific recursive strategy
- associate with $\alpha \in$ Cantor Normal Form, $\alpha_n \in$ Cantor Normal Form:

$$\alpha_n = \begin{cases} 0 & \text{if } \alpha = 0 \\ \beta & \text{if } \alpha = \beta + 1 \\ \beta + \omega^\gamma \cdot n & \text{if } \alpha = \beta + \omega^{\gamma+1} \\ \beta + \omega^{\gamma_n} & \text{if } \alpha = \beta + \omega^\gamma \text{ and } \gamma \in \text{Lim} \end{cases}$$

Definition The Standard Hydra Battle

A Hydra is an ordinal in CNF. The Hydra battle is a sequence of configurations (α, n):

$$(\alpha, n) \underbrace{\Longrightarrow}_{\text{one step}} (\alpha_n, n+1) .$$

Challenge Set 1

Challenge Code the (standard) Hydra battle as a TRS

TRS $(\mathcal{F}, \mathcal{H})$

$$h(e(x), y) \to h(d(x, y), S(y))$$

$$d(g(0,0), y) \to e(0)$$
$$d(g(x, y), z) \to g(e(x), d(y, z))$$
$$d(g(g(x, y), 0), S(z)) \to$$
$$\to g(d(g(x, y), S(z)), d(g(x, y), z))$$
$$g(e(x), e(y)) \to e(g(x, y))$$

$$h(x, y) \to h(d(x, y), S(y))$$

$$d(g(0,0), y) \to 0$$

$$d(g(x, y), z) \to g(x, d(y, z))$$

$$(\alpha, n) \Longrightarrow (\alpha_n, n+1)$$

definition of α_n ?

Solution: Correct the TRS

Dershowitz proposed the following TRS as a rectification of the originally proposed system $\mathcal{H}$:

TRS $(\mathcal{F}, \mathcal{D})$

$$h(e(x), y) \to h(d(x, y), S(y))$$

$$\begin{aligned}
d(g(g(0, x), y), S(z)) &\to g(e(x), d(g(g(0, x), y), z)) \\
d(g(g(0, x), y), 0) &\to e(y) \\
d(g(0, x), y) &\to e(x) \\
d(g(x, y), z) &\to g(d(x, z), e(y))
\end{aligned}$$

$$g(e(x), e(y)) \to e(g(x, y))$$

$h(x, y) \to h(d(x, y), S(y))$ $\qquad (\alpha, n) \Longrightarrow (\alpha_n, n+1)$

$d(g(g(0, x), y), S(z)) \to$

Goal (intermediate)

$$h(e(x), y) \rightarrow h(d(x, y), S(y))$$

one step

$$\begin{aligned}
d(g(g(0, x), y), S(z)) &\rightarrow g(e(x), d(g(g(0, x), y), z)) \\
d(g(g(0, x), y), 0) &\rightarrow e(y) \\
d(g(0, x), y) &\rightarrow e(x) \\
d(g(x, y), z) &\rightarrow g(d(x, z), e(y))
\end{aligned}$$

defines strategy

$$g(e(x), e(y)) \rightarrow e(g(x, y))$$

auxilliary

Goal

➡ find reduction order $\succ$ (with order type ϵ_0) compatible with $\mathcal{D}$:

Challenge Set 2

Challenge $\mathcal{D}$ is not simply terminating:

$$h(e(x), e(x)) \to_{\mathcal{D}} h(d(x, e(x)), S(e(x))) \to^{*}_{\mathcal{E}\mathrm{mb}} h(e(x), e(x))$$

Challenge the order type of $\succ$ has be estimated

Solution construct well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ so that order type of $\succ$ equals ϵ_0

reminder:

➡ a TRS is terminating iff it is compatible with a well-founded, monotone algebra

Challenge Set 3

Challenge interpretation of d

- let the domain of $\mathcal{A}$ equal $\mathrm{CNF}(\epsilon_0)$
- set $d_{\mathcal{A}}(\alpha, n) = \alpha_n$, as suggested by the semantics
- as $\succ$ employ the usual comparison $>$ of ordinals
- then for any m: $\omega > m$, but $\omega_n = n \not> m - 1 = (m)_n$, if $n < m$

Solution cook up a suitable ordinal notation system: i.e., consider ordinal terms OT, partially ordered by $\succ$, so that $\mathrm{otype}(\succ) = \epsilon_0$

Challenge Set 4

Challenge interpretation of second argument of d and h

- let $\mathcal{A} = (\mathsf{OT}, \succ)$ as above
- how to define $d_{\mathcal{A}}(\alpha, \beta)$, if $\beta \succcurlyeq \omega$?
- how to define $h_{\mathcal{A}}(\alpha, \beta)$, if $\beta \succcurlyeq \omega$?

Solution make use of collapsing functions: $\mathrm{C}_\alpha \colon \mathbb{N} \to \mathbb{N}$, i.e. for an approximation $\sqsupset$ of $\succ$:

$$\text{for all } n: \ \alpha \sqsupset \beta \text{ implies } \mathrm{C}_\alpha(n) \sqsupset \mathrm{C}_\beta(n)$$

Ordinal Notation System

➡ $\succ$: smallest partial order on OT such that

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m} \succ \omega^{\beta_1} + \cdots + \omega^{\beta_n} = \beta$$

if either $\alpha_i \succ \beta_i$ for some i and for all j: $\alpha_j \sim \beta_j$, or $m > n$ and for all i: $\alpha_i \sim \beta_i$.

➡ $\sim$: smallest equivalence on OT such that if $\alpha_i \preccurlyeq \alpha_{i+1}$, then

$$\alpha = \cdots + \omega^{\alpha_i} + \omega^{\alpha_{i+1}} + \cdots \sim$$
$$\sim \cdots + \omega^{\alpha_{i+1}} + \cdots = \beta$$

➡ N: for $\alpha \in$ OT, N(α) denotes the number of ωs in α

Lemma the relation $\succ$ is a well-founded, partial order on OT, and $\mathsf{otype}(\succ) = \epsilon_0$

- approximation $\sqsupset$:

$$\alpha \sqsupset \beta \quad \text{if} \quad (\alpha \succ \beta \land \mathsf{N}(\alpha) \geq \mathsf{N}(\beta)) \lor (\alpha \sim \beta \land \mathsf{N}(\alpha) > \mathsf{N}(\beta))$$
$$\alpha \equiv \beta \quad \text{if} \quad (\alpha \sim \beta \land \mathsf{N}(\alpha) = \mathsf{N}(\beta))$$

- n-predecessors (induced by polynomial p):

$$\alpha[n] := \{\beta \mid \alpha \succ \beta \text{ and } p(\mathsf{N}(\alpha), n) \geq \mathsf{N}(\beta)\}$$

- the $\sqsupset$-maximal element of $\alpha[n]$ is, up to the equivalence $\equiv$, unique; fix an arbitrary $\sqsupset$-maximal element and denote it as $P_n(\alpha)$ ($\approx \alpha_n$)

Solution to Challenge Set 3, 4

Lemma if $\alpha, \beta \neq 0$ and $\alpha \sqsupset \beta$, then $P_n(\alpha) \sqsupset P_n(\beta)$

➡ collapsing functions:

$$C_\alpha(n) := \max(\{2^{n+1}\} \cup \{C^2_\beta(n) \mid \alpha \succ \beta \wedge p(N(\alpha), n) \geq N(\beta)\})$$

Lemma if $\alpha \sqsupset \beta$, then $C_\alpha(n) \sqsupset C_\beta(n)$

a monotone $\mathcal{F}$-algebra for the TRS $(\mathcal{F}, \mathcal{D})$

- carrier of $\mathcal{A}$ is set

$$\{(\alpha, m, 1) \mid \alpha \in \mathsf{OT}, m \in \mathbb{N}\} \cup \{(0, m, 0) \mid m \in \mathbb{N}\}$$

- triples are related as

$$(\alpha, m, p) \rhd (\beta, n, q) \quad \text{iff}$$
$$((\alpha \sqsupset \beta \wedge m \geq n) \vee (\alpha \equiv \beta \wedge m > n)) \wedge (p \geq q)$$

➡ interpretations

$d_{\mathcal{A}}$ $(\alpha, m, p), (\beta, n, q) \mapsto (P_n(\alpha), \mathsf{C}_{P_n(\alpha)}(\mathsf{C}_\beta(0) + m + n), 1)$ $\alpha \neq 0$

$(0, m, p), (\beta, n, q) \mapsto (0, 2^{\mathsf{C}_\beta(0)+m+n}, 0)$

$g_{\mathcal{A}}$ $(\alpha, m, 1), (\beta, n, q) \mapsto (\beta + \omega^\alpha, \mathsf{C}_\beta(0) + m + n, 1)$

$(\alpha, m, 0), (\beta, n, q) \mapsto (0, \mathsf{C}_\beta(0) + m + n, 0)$

$h_{\mathcal{A}}$ $(\alpha, m, 1), (\beta, n, q) \mapsto (0, \mathsf{C}_\alpha(\mathsf{C}_\beta(0) + m + n), 1)$

$(\alpha, m, 0), (\beta, n, q) \mapsto (0, \mathsf{C}_\beta(0) + m + n, 0)$

$e_{\mathcal{A}}$ $(\alpha, m, p) \mapsto (\alpha, m + 1, 1)$

$S_{\mathcal{A}}$ $(\alpha, m, p) \mapsto (\alpha, m + 1, 1)$

$0_{\mathcal{A}}$ $(0, 0, 1)$

Main results

Theorem

- the well-founded, monotone $\mathcal{F}$-algebra $(\mathcal{A}, \rhd)$ is compatible with the TRS $(\mathcal{F}, \mathcal{D})$

Theorem

- $\mathcal{D}$ is terminating and termination can be established by a reduction order of order type ϵ_0

Theorem

- there exists a the well-founded, monotone $\mathcal{F}$-algebra $(\mathcal{B}, \blacktriangleright)$ that is compatible with the TRS $(\mathcal{F}, \mathcal{H})$, hence termination of $\mathcal{H}$ can be established by a reduction order of order type at most ϵ_0

Conclusion

Some observations:

- ➡ The TRSs $\mathcal{H}$, $\mathcal{D}$ are non simply terminating, however the Hydra battle can be represented by a simply (even toally) terminating TRS $\mathcal{R}$. (Touzet)
- ➡ Neither $\mathcal{H}$, $\mathcal{D}$, nor $\mathcal{R}$ can be handled by an automatic termination prover (yet).
- ➡ Touzet's system $\mathcal{R}$ is infinite, when extended to all Hydras
- ➡ The TRSs $\mathcal{R}$ can be extended to show that simple termination is really complex (Lepper)

use Semantic Labelling

Proof use semantic labeling

- Define a quasi-model $(\mathcal{A}, \sqsubset)$ such that $\sqsubset^{\equiv}$ is compatible with $\mathcal{H}$
- Show that a suitable defined labeled TRS $\mathcal{H}_{\mathrm{lab}}$ together with the rules in $\mathcal{D}\mathrm{ec}(\sqsubset)$ is terminating.
- This can be done by using a suitable instance of **RPO!** with the precedence $\succ$:
  - $h_a \succ d \succ g \succ e \succ S \succ 0$ and
  - $h_a \succ h_b$ if $a = (i, 1)$ and $b = (j, 1)$ with $i > j$

□