KBOs, Ordinals, Subrecursive Hierarchies and All That

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Overview

- Knuth-Bendix Orders
- Ordinals
    - Transfinite Knuth-Bendix Orders
- Subrecursive Hierarchies
    - A Remark on TKBO for Rewriting
- Conclusion

Knuth-Bendix Orders (KBOs for short)

Definitions

$\mathrm{w}(x) = w_0 \geqslant 1$

- let $\succ$ denote a precedence on some signature
- let w denote an admissible weight function into natural numbers

Definition

extension of w to terms

$s \succ_{\mathrm{kbo}} t$ if $\forall\, x$: $|s|_x \geqslant |t|_x$ and

1. $\mathrm{weight}(s) > \mathrm{weight}(t)$, or
2. $\mathrm{weight}(s) = \mathrm{weight}(t)$, and one of the following alternatives holds:
   1. t is a variable, $s = f^n(t)$, $n > 0$,
   2. $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$,
      $\exists\, i$ such that $s_i \succ_{\mathrm{kbo}} t_i$ $\forall\, 1 \leqslant j < i$: $s_j = t_j$
   3. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, $f \succ g$.

Equational Systems

Example

let $\mathsf{F} = \{\mathsf{i}, \circ, \mathsf{e}\}$ and consider the following rules

$$\begin{aligned} (x \circ y) \circ z &= x \circ (y \circ z) \\ \mathsf{e} \circ x &= x \\ \mathsf{i}(x) \circ x &= \mathsf{e} \\ \mathsf{i}(\mathsf{i}(x)) &= x \end{aligned}$$

$$\mathsf{w}(\mathsf{i}) = 0 \quad \mathsf{w}(\circ) = 0 \quad \mathsf{w}(\mathsf{e}) = 1 \quad \mathsf{i} \succ \circ \succ \mathsf{e}$$

Example

let the signature be as above and consider the following rules

$$\begin{aligned} \mathsf{i}(x) \circ (y \circ z) &= x \circ (\mathsf{i}^2(y) \circ z) \\ \mathsf{i}(x) \circ (y \circ (z \circ w)) &= x \circ (z \circ (y \circ w)) \\ \mathsf{i}(x) &= x \end{aligned}$$

$$\mathsf{w}(\mathsf{i}) = 0 \quad \mathsf{w}(\circ) = 0 \quad \mathsf{i} \succ \circ \succ \mathsf{e}$$

Theorem (Knuth, Bendix)

KBOs are simplification orders, for example that means that any TRS whose rules can be oriented with some KBO is terminating

Definition

let $(A, \succ)$ denote a partial ordered set

$$\text{otype}_\succ(a) = \sup\{\text{otype}_\succ(b) + 1 \mid b \in A \text{ and } a \succ b\}$$
$$\text{otype}(\succ) = \sup\{\text{otype}_\succ(a) + 1 \mid a \in A\}$$

Example

ω^ω is the order type of the lexicographic order on finite sequences of numbers

Theorem (Lepper)

there is a low upper bound on the order typ of KBO: $\text{otype}(\succ_{\text{kbo}}) \leqslant \omega^\omega$

Transfinite Knuth-Bendix Orders

A Generalisation of KBOs

let $\mathcal{O}$ denote the ordinals $< \epsilon_0$; recall that any $\alpha \in \mathcal{O}$ is uniquely representable in Cantor Normal Form (CNF):

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

where $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ and α_i in CNF

Definitions

let F be a signature

- w: $\mathsf{F} \to \mathcal{O}$ is weight function for F
- s: $\mathsf{F} \times \mathbb{N} \to \mathcal{O}$ is subterm coefficient function for F, whenever $\forall\ f \in \mathsf{F}$, $\forall\ i$ argument position of f: $\mathsf{s}(f, i) > 0$

Definition

the weight of t, weight: $\mathsf{T}(\mathsf{F},\mathsf{V}) \to \mathcal{O}$, is defined as follows:

$$\mathrm{weight}(t) = \begin{cases} w_0 & \text{if } t \text{ is a variable} \\ \mathrm{w}(f) \oplus (\mathrm{s}(f,1) \odot \mathrm{weight}(t_1)) \\ \qquad \oplus \cdots & \text{if } t = f(t_1,\dots,t_n) \\ \qquad \oplus (\mathrm{s}(f,n) \odot \mathrm{weight}(t_n)) \end{cases}$$

Definition

- let $\succ$ denote a precedence on signature F
- let w denote an admissible weight function into ordinals
- let $\succ_{\mathsf{tkbo}}$ denote the KBO induced by $\succ$,
  if terms get ordinals assigned as weights
- a TKBO is called finite, if all ordinals used are finite

Example

let $\mathsf{F} = \{\mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k}\}$ and consider the following rules:

$$\begin{aligned} \mathsf{f}(x) &\to \mathsf{g}(x) & \mathsf{w}(\mathsf{f}) &= 5 \\ \mathsf{h}(x) &\to \mathsf{f}(\mathsf{f}(x)) & \mathsf{w}(\mathsf{g}) &= 0 \\ \mathsf{k}(x, y) &\to \mathsf{h}(\mathsf{f}(x), \mathsf{f}(y)) & \mathsf{w}(\mathsf{h}) &= \omega \\ & & \mathsf{w}(\mathsf{k}) &= \omega \cdot 2 \end{aligned}$$

subterm coeffiencts are $= 1$

Theorem (Ludwig, Waldmann, 2004)

TKBOs form a simplification order

Theorem (Kovacs, M., Voronkov, 2011)

let F *be finite; any TKBO is equivalent to a TKBO restricting ordinal weights to ordinals* $< \omega^{\omega^\omega}$

A Remark on TKBO for Rewriting

Theorem (Winkler, Zankl, Middeldorp, 2012)

if a finite TRS $\mathcal{R}$ is compatible with a TKBO, then $\mathcal{R}$ is compatible with a finite TKBO

NB: this does not imply that TKBOs are equivalent to finite TKBOSs

Example

let $\mathsf{F} = \{\mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k}\}$ and consider the following rules:

$$\begin{aligned} \mathsf{f}(x) &\to \mathsf{g}(x) & \mathsf{w}(\mathsf{f}) &= 5 \\ \mathsf{h}(x) &\to \mathsf{f}(\mathsf{f}(x)) & \mathsf{w}(\mathsf{g}) &= 0 \\ \mathsf{k}(x, y) &\to \mathsf{h}(\mathsf{f}(x), \mathsf{f}(y)) & \mathsf{w}(\mathsf{h}) &= \omega \\ & & \mathsf{w}(\mathsf{k}) &= \omega \cdot 2 \end{aligned}$$

subterm coeffichents are $= 1$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x \in \mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x+1 & \text{if } \lambda = \omega \\ \beta + \omega^{\alpha} \cdot (x+1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ \beta + \omega^{\alpha[x]} & \text{if } \lambda = \beta + \omega^{\alpha}, \ \alpha \text{ limit} \end{cases}$$

Definition

the family of slow-growing functions $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{G}_0(x) = 0 \qquad \mathsf{G}_{\alpha+1}(x) = \mathsf{G}_\alpha(x) + 1 \qquad \mathsf{G}_\lambda(x) = \mathsf{G}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Example

$$\mathsf{G}_\omega(x) = x+1 \qquad \mathsf{G}_{\omega^{\omega^\omega}}(x) = (x+1)^{x+1^{x+1}} \qquad \mathsf{G}_{\omega \cdot 2}(10) = (10+1) \cdot 2 = 22$$

Remark

the family $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ forms a hierarchy, that is, for $\alpha > \beta$:

$$\exists\ c \text{ such that } \forall\ x \geqslant c\colon\ \mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$$

Example

$\forall x \geqslant 1 \qquad \mathsf{G}_{\omega^\omega}(x) = (x+1)^{x+1} > (x+1) = \mathsf{G}_\omega(x)$

$\forall x \geqslant 1 \qquad \mathsf{G}_\omega(x) = x + 1 \not> y = \mathsf{G}_y(x) \qquad$ whenever $y > x$

Definition

let $>_{(x)}$ denote the transitive closure of the fundamental sequence $\cdot[\cdot]$, defined as follows:

$$\alpha >_{(x)} \beta \qquad \text{if } \beta = \alpha[x] \text{ or } \alpha[x] \geqslant_{(x)} \beta$$

where we set $0[x] = 0$, $(\alpha + 1)[x] = \alpha$

Proof of Theorem on Finite TKBOs.

- it suffices to show that $\exists\ x\ \forall\ l \to r \in \mathcal{R}$: $\mathsf{weight}(l) \geqslant_{(x)} \mathsf{weight}(r)$ as then $\forall\ l \to r \in \mathcal{R}$: $\mathsf{G}_{\mathsf{weight}(l)}(k) \geqslant \mathsf{G}_{\mathsf{weight}(r)}(k)$
- set $k = \max\{\mathsf{N}(\mathsf{weight}(r)) \mid l \to r \in \mathcal{R}\}$
- then $\mathsf{weight}(l) \geqslant \mathsf{weight}(r)$ implies $\mathsf{weight}(l) \geqslant_{(k)} \mathsf{weight}(r)$ ∎

Definition

we define the derivational complexity with respect to $\mathcal{R}$:

$$\mathsf{dheight}(t) = \max\{n \mid \exists u\ t \to^n u\}$$
$$\mathsf{dc}(n) = \max\{\ \mathsf{dheight}(t) \mid \text{size of } t\ \leqslant n\}$$

Theorem

for any TRS $\mathcal{R}$, compatible with a TKBO, the derivational complexity is bounded by a 2-recursive function, that is, $\mathsf{dc}(n) \in \mathsf{Ack}(n, 0)$

Conclusion and Future Work

Summary

1. KBO can be equipped with ordinal weights, where weights $< \omega^{\omega^\omega}$ suffice
2. TKBOs for finite TRS collapse to finite TKBOs
3. TKBOs induce 2-recursive derivational complexity (for finite TRS)

Future Work

- generalised KBOs compute weights based on weakly monote simple algebras $\mathcal{A}$
- clarify restrictions on $\mathcal{A}$ so that ordinal weights again collapse to numbers

Thank Your for Your Attention!