The Epsilon Calculus

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What is the Epsilon Calculus?

The ε-calculus is a formalization of logic without quantifiers but with the ε-operator.

If $A(x)$ is a formula, then $\varepsilon x A(x)$ is an ε-term.

Intuitively, $\varepsilon x A(x)$ is an indefinite description: $\varepsilon x A(x)$ is some x for which $A(x)$ is true.

ε can replace $\exists$: $\exists x A(x) \Leftrightarrow A(\varepsilon x A(x))$

Axioms of ε-calculus:

- Propositional tautologies
- (Equality schemata)
- $A(t) \rightarrow A(\varepsilon x A(x))$

Predicate logic can be embedded in ε-calculus.

Why Should You Care?

- Epsilon calculus is of significant historical interest.
  - Origins of proof theory
  - Hilbert's Program
- Alternative basis for fruitful proof-theoretic research.
  - Epsilon Theorems and Herbrand's Theorem: proof theory without sequents
  - Epsilon Substitution Method: yields functionals, e.g.,

$$\vdash \forall x \exists y A(x,y) \rightsquigarrow \forall n\colon \vdash A(n, f(n))$$

- Interesting Logical Formalism
  - Trade logical structure for term structure.
  - Suitable for proof formalization.
  - Choice in logic.
  - Inherently classical (but see work of Bell, DeVidi, Fitting, Mostowski).
  - Expressive power?
- Other Applications:
  - Use of choice functions in provers (e.g., HOL, Isabelle).
  - Applications in linguistics (choice functions, anaphora).

Outline

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7. Generalizations of the Epsilon Theorems
8. (Intermediate) Conclusion
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Historical Remarks

The epsilon calculus was first introduced by Hilbert in 1922, as the basis for his formulation of mathematics for which Hilbert's Program was supposed to be carried out.

Motivation: Logical choice function; ε-terms represent "ideal elements" in proofs.

Original work in proof theory (pre-Gentzen) concentrated on ε-calculus and ε-substitution method (Ackermann, von Neumann, Bernays)

First correct proof of Herbrand's Theorem via Epsilon Calculus

Epsilon substitution method used by Kreisel for no-counterexample interpretation leading to work on analysis of proofs by Kreisel, Luckhardt, Kohlenbach.

Recent work on ordinal analysis of subsystems of analysis by Arai, Avigad, Buchholz, Mints, Tupailo, Tait.

The Epsilon Calculus: Syntax

Language of the Elementary Calculus L_{EC}:

- Free variables: a, b, c, ...
- Bound variables: x, y, z, ...
- Constant and function symbols: f, g, h, ... with arities $\mathrm{ar}(f)$, ...
- Predicate symbols: P, Q, R, ... with arities $\mathrm{ar}(P)$, ...
- Equality: =
- Propositional connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$

  Language of the Predicate Calculus L_{PC}:
- Quantifiers: $\forall$, $\exists$

  Language of the Epsilon Calculus L_{ε}:
- Epsilon: ε

The Epsilon Calculus: Syntax (cont'd)

Semi-formulas and Semi-terms:

1. Any free variable a is a semi-term.
2. Any bound variable x is a semi-term.
3. If f is a function symbol with $\mathrm{ar}(f) = 0$, then f is a semi-term.
4. If f is a function symbol with $\mathrm{ar}(f) = n > 0$, and t_1, ..., t_n are semi-terms, then $f(t_1, \ldots, t_n)$ is a semi-term.
5. If P is a predicate symbol with $\mathrm{ar}(P) = n > 0$, and t_1, ..., t_n are terms, then $P(t_1, \ldots, t_n)$ is an (atomic) semi-formula.
6. If A and B are semi-formulas, then $\neg A$, $A \wedge B$, $A \vee B$ and $A \rightarrow B$ are semi-formulas.
7. If $A(a)$ is a semi-formula containing the free variable a and x is a bound variable not occurring in $A(a)$, then $\forall x A(x)$ and $\exists x A(x)$ are semi-formulas.
8. If A is a semi-formula containing the free variable a and x is a bound variable not occurring in $A(a)$, then $\varepsilon x A(x)$ is a semi-term (an ε-*expression*).

The Epsilon Calculus: Syntax (cont'd)

Terms and formulas:

Terms and formulas are defined exactly as above except that clause (2) is dropped. In other words: a semi-term is a term and a semi-formula a formula if it contains no bound variables without a matching $\forall$, $\exists$, or ε.

Subterms and sub-semi-terms:

1. The only sub(semi)terms of a free variable a is a itself. It has no immediate sub(semi)terms.
2. A bound variable x has no subterms or immediate sub-semi-terms. Its only subsemiterm is x itself.
3. If $f(t_1, \ldots t_n)$ is a semi-term, then its immediate sub-semi-terms are t_1, ..., t_n, and its immediate subterms are those among t_1, ..., t_n which are terms, plus the immediate subterms of those among t_1, ..., t_n which aren't terms. Its sub-semi-terms are $f(t_1, \ldots, t_n)$ and the sub-semi-terms of t_1, ..., t_n; its subterms are those of its sub-semi-terms which are terms.
4. If $P(t_1, \ldots, t_n)$ is an atomic semi-formula, then its immediate sub-semi-terms are t_1, ..., t_n. Its immediate subterms are those among t_1, ..., t_n which are terms, plus the immediate subterms of those among t_1, ..., t_n which aren't terms. Its sub(semi)terms are the sub(semi)terms of t_1, ..., t_n.

5. If A is a semi-formula of the form $B \wedge C$, $B \vee C$, $B \to C$, $\neg B$, $\forall x B$ or $\exists x B$, then its immediate sub(semi) formulas and its sub(semi)formulas are those of B and C.
6. If $\varepsilon x A(x)$ is an ε-expression then its immediate sub(semi) formulas and its sub(semi)formulas are those of $A(x)$.

The Epsilon Calculus: Syntax (cont'd)

Epsilon matrices

An ε-term $\varepsilon x A(x)$ is *an ε-matrix*—or simply, a *matrix*—if all terms occurring in A are free variables, each of which occurs exactly once.

Denote ε-matrices as $\varepsilon x A(x; a_1, \ldots, a_k)$ where the variables $a_1, \ldots, a_k$ displayed are all the free variables occurring in it.

Two matrices $\varepsilon x A(x; a_1, \ldots, a_k)$, $\varepsilon x A(x; b_1, \ldots, b_k)$ that differ only in the indicated tuples of variables $\overline{a}$ and $\overline{b}$, respectively, are considered to be equal. The set of all matrices is denoted Mat.

Corresponding to each ε-term $\varepsilon x A(x)$ there exists a unique matrix $\varepsilon x A(x; a_1, \ldots, a_k)$ and a unique sequence $t_1, \ldots, t_k$ of terms such that $\varepsilon x A(x, t_1, \ldots, t_k) = \varepsilon x A(x)$.

Example: $\varepsilon x A(s, \underbrace{\varepsilon y B(y)}_{e}, \varepsilon z C(x, t))$

Its matrix is: $\varepsilon x A(a, b, \varepsilon z C(x, c))$

The matrix of $\varepsilon x A(x)$ is obtained by replacing all immediate subterms of $\varepsilon x A(x)$ by distinct new free variables. In this newly obtained term we replace distinct occurrences of the same variable by different variables.

The Epsilon Calculus: Intensional Semantics

An *intensional choice function* Φ for a set X is a function $\Phi\colon 2^X \times \mathrm{Mat} \times X^{<\omega} \to X$ so that for $S \subseteq X$, a matrix $\varepsilon x A(x; a_1, \ldots, a_n)$ and $d_1, \ldots, d_n \in S$, $\Phi(S, \varepsilon x A(x; a_1, \ldots, a_n), \langle d_1, \ldots, d_n \rangle) \in S$ if $S \neq \emptyset$.

An *intensional* ε*-structure* $\mathfrak{M} = \langle |\mathfrak{M}|, (\cdot)^{\mathfrak{M}}, \Phi_{\mathfrak{M}} \rangle$ consist of a *domain* $|\mathfrak{M}| \neq \emptyset$, an *interpretation function* $(\cdot)^{\mathfrak{M}}$, and an *intensional choice function* $\Phi_{\mathfrak{M}}$ for $|\mathfrak{M}|$, where

1. If $\mathrm{ar}(f) = 0$, then $f^{\mathfrak{M}} \in |\mathfrak{M}|$
2. If $\mathrm{ar}(f) = n > 0$, then $f^{\mathfrak{M}}\colon |\mathfrak{M}|^n \to |\mathfrak{M}|$
3. If $\mathrm{ar}(P) = n > 0$, then $P^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$

A *variable assignment* s for $\mathfrak{M}$ is a function $s\colon \mathrm{FV} \to |\mathfrak{M}|$. We write $s \sim_c s'$ if $s(a) = s'(a)$ for all free variables a other than c.

The Epsilon Calculus: Intensional Semantics (cont'd)

The *value* $t^{\mathfrak{M},s}$ *of a term* t and the *satisfaction relation* $\mathfrak{M}, s \models A$ is defined by:

1. If a is a free variable, then $a^{\mathfrak{M},s} = s(a)$.
2. If f is a constant symbol, then $f^{\mathfrak{M},s} = f^{\mathfrak{M}}$.
3. If $f(t_1, \ldots, t_n)$ is a term, then $(f(t_1, \ldots, t_n))^{\mathfrak{M},s} = f^{\mathfrak{M}}(t_1^{\mathfrak{M},s}, \ldots, t_n^{\mathfrak{M},s})$.
4. If $P(t_1, \ldots, t_n)$ is an atomic formula, then $\mathfrak{M}, s \models P(t_1, \ldots, t_n)$ iff $\langle t_1^{\mathfrak{M},s}, \ldots, t_n^{\mathfrak{M},s} \rangle \in P^{\mathfrak{M}}$.
5. If A and B are formulas, then:
   - $\mathfrak{M}, s \models A \wedge B$ iff $\mathfrak{M}, s \models A$ and $\mathfrak{M}, s \models B$.
   - $\mathfrak{M}, s \models A \vee B$ iff $\mathfrak{M}, s \models A$ or $\mathfrak{M}, s \models B$.
   - $\mathfrak{M}, s \models A \to B$ iff $\mathfrak{M}, s \not\models A$ or $\mathfrak{M}, s \models B$.
   - $\mathfrak{M}, s \models \neg A$ iff $\mathfrak{M}, s \not\models A$.
6. If $\exists x A(x)$ is a formula and c a free variable not in $A(x)$, then $\mathfrak{M}, s \models \exists x A(x)$ iff $\mathfrak{M}, s' \models A(c)$ for some $s' \sim_c s$.
7. If $\forall x A(x)$ is a formula and c a free variable not in $A(x)$, then $\mathfrak{M}, s \models \forall x A(x)$ iff $\mathfrak{M}, s' \models A(c)$ for all $s' \sim_c s$.
8. If e is an ε-term with matrix $\varepsilon x A(x; a_1, \ldots, a_n)$ so that $e = \varepsilon x A(x; t_1, \ldots, t_n)$, and

   $$X = \{d \in |\mathfrak{M}| : \mathfrak{M}, s' \models A(c; t_1, \ldots, t_n), s' \sim_c s, s'(c) = d\},$$

   then

   $$e^{\mathfrak{M},s} = \Phi_{\mathfrak{M}}(X, \varepsilon x A(x; a_1, \ldots, a_n), \langle t_1^{\mathfrak{M},s}, \ldots, t_n^{\mathfrak{M},s} \rangle).$$

Axiomatization of the Epsilon Calculus: Axioms

Let $\mathrm{Ax}_{\mathrm{EC}}$ be the set of formulas containing all formulas which are either substitution instances of propositional tautologies, or substitution instances of

$$\begin{aligned} a &= a \\ a = b &\to (A(a) \to A(b)). \end{aligned}$$

$\mathrm{Ax}_{\mathrm{PC}}$ is $\mathrm{Ax}_{\mathrm{EC}}$ together with all substitution instances of

$$\begin{aligned} A(a) &\to \exists x A(x) \\ \forall x A(x) &\to A(a). \end{aligned}$$

Any substitution instance of

$$A(a) \to A(\varepsilon x A(x))$$

is called a *critical formula*.

We obtain $\mathrm{Ax}^{\varepsilon}_{\mathrm{EC}}$ and $\mathrm{Ax}^{\varepsilon}_{\mathrm{PC}}$ by adding the critical formulas to $\mathrm{Ax}_{\mathrm{EC}}$ and $\mathrm{Ax}_{\mathrm{PC}}$, respectively.

Axiomatization of the Epsilon Calculus: Deductions

A *deduction* in EC or EC^ε is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in $AxEC$ or $\mathrm{Ax}^\varepsilon_{\mathrm{EC}}$ or follows from A_j, A_k with $j,k < i$ by modus ponens:

$$B, B \to C \vdash C$$

A *deduction* in PC or PC^ε is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in $\mathrm{Ax}_{\mathrm{PC}}$ or $\mathrm{Ax}^\varepsilon_{\mathrm{PC}}$ or follows from A_j, A_k with $j,k < i$ by modus ponens: $B, B \to C \vdash C$, or follows from A_j with $j < i$ by generalization:

$$\begin{array}{lcl} B \to A(a) & \vdash & B \to \forall x A(x) \\ A(a) \to B & \vdash & \exists x A(x) \to B \end{array}$$

A formulas A is called *deducible* (in EC, EC^ε, PC, PC^ε), $\vdash A$, if there is a deduction (in $\mathrm{Ax}_{\mathrm{EC}}$, $\mathrm{Ax}^\varepsilon_{\mathrm{EC}}$, $\mathrm{Ax}_{\mathrm{PC}}$, $\mathrm{Ax}^\varepsilon_{\mathrm{PC}}$, respectively) which has A as its last formula.

Extensional Epsilon Calculus

Alternative semantics: Choice function maps just *sets* to elements:

$$\Phi_{\mathfrak{M}}(S) \in S$$

Then if $e = \varepsilon x A(x, t_1, \ldots, t_n)$

$$e^{\mathfrak{M},s} = \Phi_{\mathfrak{M}}(\{d \in |M| : \mathfrak{M}, s' \models A(c; t_1, \ldots, t_n), s' \sim_c s', s'(c) = d\})$$

In extensional ε-semantics, equivalent ε-terms have same value, i.e., it makes valid the ε-*extensionality axiom*

$$\forall x(A(x) \leftrightarrow B(x)) \to \varepsilon x A(x) = \varepsilon x B(x)$$

Embedding PC in $\mathrm{EC}^{\varepsilon}$

The epsilon operator allows the treatment of quantifiers in a quantifier-free system: using ε terms it is possible to define $\exists x$ and $\forall x$ as follows:

$$\begin{array}{lcl} \exists x A(x) & \Leftrightarrow & A(\varepsilon x A(x)) \\ \forall x A(x) & \Leftrightarrow & A(\varepsilon x \neg A(x)) \end{array}$$

Define A^{ε} by:

1. $x^{\varepsilon} = x$, $a^{\varepsilon} = a$
2. $[\varepsilon x A(x)]^{\varepsilon} = \varepsilon x A(x)^{\varepsilon}$
3. $f(t_1, \ldots, t_n)^{\varepsilon} = f(t_1^{\varepsilon}, \ldots, t_n^{\varepsilon})$, $P(t_1, \ldots, t_n)^{\varepsilon} = P(t_1^{\varepsilon}, \ldots, t_n^{\varepsilon})$.
4. $(A \wedge B)^{\varepsilon} = A^{\varepsilon} \wedge B^{\varepsilon}$, $(A \vee B)^{\varepsilon} = A^{\varepsilon} \vee B^{\varepsilon}$, $(A \to B)^{\varepsilon} = A^{\varepsilon} \to B^{\varepsilon}$, $(\neg A)^{\varepsilon} = \neg A^{\varepsilon}$.
5. $(\exists x A(x))^{\varepsilon} = A^{\varepsilon}(\varepsilon x A^{\varepsilon}(x)')$.
6. $(\forall x A(x))^{\varepsilon} = A^{\varepsilon}(\varepsilon x \neg A^{\varepsilon}(x)')$.

where $A^{\varepsilon}(x)'$ is $A^{\varepsilon}(x)$ with all variables bound by quantifiers or ε's in $A^{\varepsilon}(x)$ renamed by new bound variables (to avoid collision of bound variables when $\varepsilon x A^{\varepsilon}(x)$ is substituted into $A^{\varepsilon}(x)$ where x may be in the scope of a quantifier or epsilon binding a variable occurring in $A^{\varepsilon}(x)$.

Embedding PC in $\mathrm{EC}^{\varepsilon}$: Examples

$$\begin{aligned}
\exists x(P(x) \quad \vee \quad & \forall y\, Q(y))^{\varepsilon} = \\
= \quad & [P(x) \vee \forall y\, Q(y)]^{\varepsilon} \quad \{x \leftarrow \varepsilon x\, [P(x) \vee \forall y\, Q(y)]^{\varepsilon}\} \\
& \quad [P(x) \vee \forall y\, Q(y)]^{\varepsilon} = P(x) \vee Q(\underbrace{\varepsilon y \neg Q(y)}_{e_1}) \\
= \quad & P(x) \vee Q(\underbrace{\varepsilon y \neg Q(y)}_{e_1}) \quad \{x \leftarrow \underbrace{\varepsilon x\, [P(x) \vee Q(\underbrace{\varepsilon y \neg Q(y)}_{e_1})]}_{e_2}\} \\
= \quad & P(\underbrace{\varepsilon x\, [P(x) \vee Q(\underbrace{\varepsilon y \neg Q(y)}_{e_1})]}_{e_2}) \vee Q(\underbrace{\varepsilon y \neg Q(y)}_{e_1})
\end{aligned}$$

$$\begin{aligned}
[\exists x \quad \exists y \quad & A(x,y)]^{\varepsilon} = \\
= \quad & [\exists y A(x,y)]^{\varepsilon} \quad \{x \leftarrow \varepsilon x\, [\exists y A(x,y)]^{\varepsilon}\} \\
& \quad [\exists y A(x,y)]^{\varepsilon} = A(x, \underbrace{\varepsilon y A(x,y)}_{e_1(x)}) \\
= \quad & A(x, \underbrace{\varepsilon y A(x,y)}_{e_1(x)})\{x \leftarrow \underbrace{\varepsilon x A(x, \varepsilon z A(x,z))}_{e_2}\} \\
= \quad & A(\underbrace{\varepsilon x A(x, \varepsilon z A(x,z))}_{e_2}, \underbrace{\varepsilon y A(\underbrace{\varepsilon x A(x, \varepsilon z A(x,z))}_{e_2}, y)}_{e_1(e_2)})
\end{aligned}$$

Embedding PC in EC^ε (cont'd)

Prop. If $\mathrm{PC}^\varepsilon \vdash A$ then $\mathrm{EC}^\varepsilon \vdash A^\varepsilon$.

Translations of quantifier axioms provable from critical formulas:

$$[A(t) \to \exists x A(x)]^\varepsilon = A^\varepsilon(t) \to A^\varepsilon(\varepsilon x A(x)^\varepsilon)$$

Suppose $\mathrm{PC}^\varepsilon \vdash B \to A(a)$, and a does not occur in B. By IH, we have a proof π in EC^ε of $B^\varepsilon \to A(a)^\varepsilon$. Replacing a everywhere in π by $\varepsilon x \neg A(x)$ results in a proof of

$$[B \to \forall x A(x)]^\varepsilon = B^\varepsilon \to A^\varepsilon(\varepsilon x \neg A(x)^\varepsilon).$$

The Epsilon Theorems

First Epsilon Theorem. If A is a formula without bound variables (no quantifiers, no epsilons) and $\mathrm{PC}^{\varepsilon} \vdash A$ then $\mathrm{EC} \vdash A$.

Extended First Epsilon Theorem. If $\exists x_1 \dots \exists x_n A(x_1, \dots, x_n)$ is a purely existential formula containing only the bound variables $x_1, \dots, x_n$, and

$$\mathrm{PC}^{\varepsilon} \vdash \exists x_1 \dots \exists x_n A(x_1, \dots, x_n),$$

then there are terms t_{ij} such that

$$\mathrm{EC} \vdash \bigvee_i A(t_{i1}, \dots, t_{in}).$$

Second Epsilon Theorem. If A is an ε-free formula and $\mathrm{PC}^{\varepsilon} \vdash A$ then $\mathrm{PC} \vdash A$.

Degree and Rank

Degree of an ε-Term

- $\deg(\varepsilon x A(x)) = 1$ if $A(x)$ contains no ε-subterms.
- If $e_1, \ldots, e_n$ are all immediate ε-subterms of $A(x)$, then $\deg(\varepsilon x A(x)) = \max\{\deg(e_1), \ldots, \deg(e_n)\} + 1$.

Rank of an ε-Expression

An ε-expression e is *subordinate* to $\varepsilon x A$ if e is a proper sub-expression of A and contains x.

- $\mathrm{rk}(e) = 1$ if no sub-ε-expression of e is subordinate to e.
- If $e_1, \ldots, e_n$ are all the ε-expressions subordinate to e, then $\mathrm{rk}(e) = \max\{\mathrm{rk}(e_1), \ldots, \mathrm{rk}(e_n)\} + 1$

Examples

$$P(\underbrace{\varepsilon x\,[P(x) \lor Q(\underbrace{\varepsilon y \neg Q(y)}_{e_1})]}_{e_2}) \lor Q(\underbrace{\varepsilon y \neg Q(y)}_{e_1})$$

$\mathrm{deg}(e_1) = 1$, $\mathrm{deg}(e_2) = 2$

$\mathrm{rk}(e_1) = \mathrm{rk}(e_2) = 1$

$$A(\underbrace{\varepsilon x A(x, \varepsilon z A(x,z))}_{e_2}, \underbrace{\varepsilon y A(\underbrace{\varepsilon x A(x, \varepsilon z A(x,z))}_{e_2}, y)}_{e_1(e_2)})$$

$\mathrm{deg}(e_2) = 1$, $\mathrm{deg}(e_1(e_2)) = 2$

$\mathrm{rk}(e_2) = 2$, $\mathrm{rk}(e_1(e_2)) = 1$

Rank of Critical Formulas and Derivations

Rank of a critical formula $A(t) \to A(\varepsilon x A(x))$ is $\mathrm{rk}(\varepsilon x A(x))$.

Rank of a derivation $\mathrm{rk}(\pi)$: maximum rank of its critical formulas.

Critical ε-term of a derivation: ε-term e so that $A(t) \to A(e)$ is a critical formula.

Order of a derivation $o(\pi, r)$ wrt. rank r: number of different critical ε-terms of rank r.

The First Epsilon Theorem

(Proof for case without $=$)

Suppose $\mathrm{PC}^{\varepsilon} \vdash_{\pi} E$ and E contains no bound variables. We show that $\mathrm{EC} \vdash E$ by induction on the rank and degree of π.

First, w.l.o.g. we assume π is actually a derivation in $\mathrm{EC}^{\varepsilon}$. Since E contains no bound variables, $E^{\varepsilon} = E$.

Second, w.l.o.g. we assume π doesn't contain any free variables (replace free variables by new constants—may be resubstituted later).

Lemma. Let e be a critical ε-term of π of maximal degree among the critical ε-terms of maximal rank. Then there is π_e with end formula A so that $\mathrm{rk}(\pi_e) \leq \mathrm{rk}(\pi)$, $\deg(\pi_e) \leq \deg(\pi)$ and $o(\pi_e, \mathrm{rk}(e)) = o(\pi, \mathrm{rk}(e)) - 1$.

The First Epsilon Theorem: Main Lemma

Proof. Construct π_e as follows:

1. Suppose $A(t_1) \to A(e), \ldots, A(t_n) \to A(e)$ are all the critical formulas belonging to e. For each critical formula

$$A(t_i) \to A(e),$$

we obtain a derivation

$$\pi_i \vdash A(t_i) \to E :$$

- Replace e everywhere it occurs by t_i. Every critical formula $A(t) \to A(e)$ belonging to e turns into a formula of the form $B \to A(t_i)$.
- Add $A(t_i)$ to the axioms. Now every such formula is derivable using the propositional tautology

$$A(t_i) \to (B \to A(t_i)) ,$$

and modus ponens.
- Apply the deduction theorem for the propositional calculus to obtain π_i.

The First Epsilon Theorem: Main Lemma

2. Obtain a derivation π' of $\bigwedge \neg A(t_i) \to E$ by:
   - Add $\bigwedge \neg A(t_i)$ to the axioms. Now every critical formula $A(t_i) \to A(e)$ belonging to e is derivable using the propositional tautology $\neg A(t_i) \to (A(t_i) \to A(e))$.
   - Apply the deduction theorem.
3. Combine the proofs

   $$\pi_i \vdash A(t_i) \to E \ ,$$

   and

   $$\pi' \vdash \bigwedge \neg A(t_i) \to E \ ,$$

   to get $\pi_e \vdash E$ (case distinction)

Why is this correct?

Verify that the resulting derivation is indeed a derivation in $\mathrm{EC}^{\varepsilon}$ with the required properties.

We started with critical formulas of the form

$$A(t_i) \to A(e) \,.$$

Facts:

- The proof π' does not contain any critical formulas belonging to e. Hence e is no longer a *critical* ε-term in π'. All other critical formulas (and the critical ε-terms they belong to) remain unchanged. Thus $o(\pi', \mathrm{rk}(e)) = o(\pi, \mathrm{rk}(e)) - 1$.
- In the construction of π_i, we substituted e by t throughout the proof. Such uniform substitutions of a term by another are proof-preserving.
- Replacing e by t_i in $A(e)$ indeed results in $A(t_i)$, since e cannot occur in $A(x)$—else $e = \varepsilon x A(x)$ would be a proper subterm of itself, which is impossible.
- If e appears in another critical formula $B(s) \to B(\varepsilon y B(y))$, we have three cases.

Case I

Case: e occurs only in s.

Replacing e by t_i results in a critical formula $B(s') \to B(\varepsilon y B(y))$.

The new critical critical formula belongs to the same ε-term as the original formula.

Hence $o(\pi_i, \mathrm{rk}(e)) = o(\pi, \mathrm{rk}(e)) - 1$.

Case II

Case: e may occur in $B(y)$ and perhaps also in s, but contains neither s nor $\varepsilon y B(y)$.

In other words, the critical formula has the form

$$B'(s'(e),e) \to B'(\varepsilon y B'(y,e),e).$$

But then the ε-term belonging to this critical formula

$$e' = \varepsilon y B'(y,e) ,$$

is of higher degree than e.

By our assumptions, this implies that $\mathrm{rk}(\varepsilon y B'(y,e)) < \mathrm{rk}(e)$.

Replacing e by t_i results in a different critical formula

$$B'(s'(t_i),t_i) \to B'(\varepsilon y B'(y,t_i),t_i) ,$$

belonging to the ε-term $\varepsilon y B'(y,t_i)$ which has the same rank as e' and hence a lower rank than e itself.

Hence again $o(\pi_i, \mathrm{rk}(e)) = o(\pi, \mathrm{rk}(e)) - 1$.

Case III

Case: e does contain s or $\varepsilon y B(y)$.

Then e is of the form $e'(s)$ or $e'(\varepsilon y B(y))$, and

$B(a)$ is really of the form $B'(e'(a))$ where $e'(a)$ is an ε-term of the same rank as e.

Then $\varepsilon y B(y)$ has the form $\varepsilon y B'(e'(y))$, to which the ε-expression $e'(y)$ is subordinated.

But then $\varepsilon y B'(e'(y))$ has higher rank than $e'(y)$, which has the same rank as e. This cannot happen.

Finally the proof of the lemma follows: In all of the cases considered one ε-critical term of $\mathrm{rk}(e)$ was removed and other ε-critical terms of $\mathrm{rk}(e)$ remained equal. Thus $o(\pi_e, \mathrm{rk}(e)) = o(\pi, \mathrm{rk}(e)) - 1$ holds.

The First Epsilon Theorem: Proof

By induction on $\mathrm{rk}(\pi)$.

If $\mathrm{rk}(\pi) = 0$, there is nothing to prove (no critical formulas).

If $\mathrm{rk}(\pi) > 0$ and the order of π wrt. $\mathrm{rk}(\pi)$ is m, then m-fold application of the lemma results in a derivation π' of rank $< \mathrm{rk}(\pi)$.

The Extended First Epsilon Theorem

Theorem. If $\exists x_1 \ldots \exists x_k A(x_1, \ldots, x_k)$ is a purely existential formula containing only the bound variables x_1, ..., x_k, and

$$\mathrm{PC}^{\varepsilon} \vdash \exists x_1 \ldots \exists x_k A(x_1, \ldots, x_k),$$

then there are terms t_{ij} such that

$$\mathrm{EC} \vdash \bigvee_i A(t_{i1}, \ldots, t_{ik}).$$

Consider proofs in $\mathrm{PC}^{\varepsilon}$ of $\exists x_1 \ldots \exists x_k A(x_1, \ldots, x_k)$, where $A(a_1, \ldots, a_k)$ contains no bound variables.

Employing embedding we obtain a derivation π of $A(s_1, \ldots, s_k)$, where s_1, ..., s_k are terms (containing ε's).

Proof Sketch. We employ the same sequence of elimination steps as in the proof of the First Epsilon Theorem. The difference being that now the end-formula $A(s_1, \ldots, s_k)$ may contain ε-terms.

Hence the first elimination step transform the end-formula into a disjunction.

$$A(s_{01}, \ldots, s_{0k}) \vee \ldots \vee A(s_{n1}, \ldots, s_{nk}) \text{ .}$$

The Second Epsilon Theorem

Theorem. If A is an ε-free formula and $\mathrm{PC}^{\varepsilon} \vdash A$ then $\mathrm{PC} \vdash A$.

Assume A has the form

$$\exists x \forall y \exists z B(x, y, z) \ ,$$

with $B(x, y, z)$ quantifier-free and no other than the indicated variables occur in A.

Herbrand Normal Form. Suppose $A = \exists x \forall y \exists z B(x, y, z)$. If f is a new function symbol, then the Herbrand normal form A^H of A is $\exists x \exists z B(x, f(x), z)$.

Lemma. Suppose $\mathrm{PC}^{\varepsilon} \vdash A$. Then $\mathrm{PC}^{\varepsilon} \vdash A^H$.

Second Epsilon Theorem: Proof

The Strong First Epsilon Theorem yields:

There are ε-free terms r_i, s_i so that

$$\mathrm{EC} \vdash \bigvee_i B(r_i, f(r_i), s_i) \tag{1}$$

We now can replace the t_i by new free variables a_i and obtain from (1), that

$$\bigvee_i B(r_i, a_i, s_i) \ , \tag{2}$$

is deducible in EC.

Then the original prenex formula A can be obtained from (5) if we employ the following rules (deducible in PC)

(μ) : $F \vee G(t) \vdash F \vee \exists y G(y)$

(ν) : $F \vee G(a) \vdash F \vee \forall z G(z)$, provided a appears only in $G(a)$ at the displayed occurrences.

Corollaries

Conservative Extension. Due to the Second Epsilon Theorem the Epsilon Calculus (with equality) is a conservative extension of pure predicate logic.

Equivalence. Due to the Embedding Lemma we have $\mathrm{PC}^\varepsilon \vdash A$ implies $\mathrm{EC}^\varepsilon \vdash A^\varepsilon$. Due to the Second Epsilon Theorem we obtain $\mathrm{EC}^\varepsilon \vdash A^\varepsilon$ implies $\mathrm{PC}^\varepsilon \vdash A$.

Herbrand's Theorem. Assume $A = \exists x \forall y \exists z B(x,y,z)$. Iff $\mathrm{PC}^\varepsilon \vdash A$, then there are terms r_i, s_i such that $\mathrm{EC} \vdash \bigvee_i B(r_i, f(r_i), s_i)$.

Generalizations

First Epsilon Theorem. Let A be a formula without bound variables (no quantifiers, no epsilons) but possible including $=$. Then

$$\mathrm{PC}^{\varepsilon} \cup \mathrm{Ax} \vdash A \text{ implies } \mathrm{EC} \cup \mathrm{Ax} \vdash A\ ,$$

where Ax includes instances of quantifier-free (and ε-free) axioms.

Extended First Epsilon Theorem. Let $\exists \bar{x} A(\bar{x})$ be a purely existential formula (possibly containing $=$). Then

$$\mathrm{PC}^{\varepsilon} \cup \mathrm{Ax} \vdash \exists \bar{x} A(\bar{x}) \text{ implies}$$
$$\mathrm{EC} \cup \mathrm{Ax} \vdash \bigvee_i A(t_{i1}, \ldots, t_{in})\ ,$$

where Ax is defined as above.

Generalizations (cont'd)

Second Epsilon Theorem. If A is an ε-free formula (possibly containing $=$) and

$$\mathrm{PC}^{\varepsilon} \cup \mathrm{Ax} \vdash A \text{ implies } \mathrm{PC} \cup \mathrm{Ax} \vdash A\,,$$

where Ax includes instances of ε-free axioms.

(Intermediate) Conclusion

Some facts in favour of the Epsilon Calculus:

- The input parameter for the proof of Herbrand's Theorem is the collection of critical formulas $\mathcal{C}$ used in the derivation. E.g. this gives a bound depending only on $\mathcal{C}$.
- The Epsilon Calculus allows a condensed representation of proofs.
  Why: Assume $\mathrm{EC}^{\varepsilon} \vdash A^{\varepsilon}$. Then there exists a tautology of the form
  $$\bigwedge_{i,j} (B_i(t_j) \to B_i(\varepsilon x B_i(x))) \to A^{\varepsilon} \,. \tag{3}$$
  Thus as soon as the critical formulas $B_i(t_j) \to B_i(\varepsilon x B_i(x))$ are known, we only need to verify that (3) is a tautology to infer that A^{ε} is provable in $\mathrm{EC}^{\varepsilon}$.
- Formalization of proofs should be simpler in the Epsilon Calculus.

A bluffer's guide to Hilbert's "Ansatz"

Assume we work within number theory and let $\mathfrak{N}$ denote the standard model of number theory.

(For conciseness we ignore induction.)

- The *initial substitution* S_0: Assign the constant function 0 to all ε-terms (by assigning the constant function 0 to all ε-matrices).
- Assume the substitution S_n has already been defined. Define S_{n+1}: Pick a false critical axiom, e.g.

$$P(t) \to P(\varepsilon x P(x)) \; .$$

  (False means wrt. to $\mathfrak{N}$ and the current substitution S_n.)
- Let $z \in \mathbb{N}$ denote the value of t under S_n. Then the next substitution S_{n+1} is obtained by assigning the value z to $\varepsilon x P(x)$.

Note that the critical axiom $P(t) \to P(\varepsilon x P(x))$ is true wrt. to $\mathfrak{N}$ and the current substitution S_{n+1}.

Peano Arithmetic: Results

1**-consistency.** Every purely existential formula derivable in $\mathrm{PA}^{\varepsilon}$ is true.

Provable Recursive Functions. The numerical content of proofs of purely existential formulas in $\mathrm{PA}^{\varepsilon}$ is extractable.

Put differently: The provable recursive functions of $\mathrm{PA}^{\varepsilon}$ are exactly the $< \varepsilon_0$-recursive functions.

Assume $\mathrm{PA}^{\varepsilon} \vdash \forall x \exists y A(x,y)$ with $A(a,b)$ quantifier-free and without free variables other than the shown. Then we can find a $< \varepsilon_0$-recursive function f such that $\forall x A(x, f(x))$ holds.

Peano Arithmetic: Results (cont'd)

Non-counter example interpretation. Let

$$\exists x \forall y \exists z A(a,x,y,z)$$

be deducible in $\mathrm{PA}^{\varepsilon}$ such that only the indicated free variable a occurs. Let $\exists x \exists z A(a,x,f(x),z)$ denote the Herbrand normal form of A.

Then there exists $< \varepsilon_0$-recursive functionals G and H such that for all functions f,

$$A(n,G(f,n),f(G),H(f,n)) \ ,$$

holds.

The transformation $()^{\varepsilon} \colon \mathrm{L}_{\varepsilon} \to \mathrm{L}_{\mathrm{PC}}$ defined yesterday, can be employed to show that Peano Arithmetic embeds into $\mathrm{PA}^{\varepsilon}$:

Then if $\mathrm{PA} \vdash A$, then $\mathrm{PA}^{\varepsilon} \vdash A^{\varepsilon}$.

The Substitution Method

An ε-expression is *canonical* iff it does not contain proper ε-subterms.

That is an ε-term e is canonical if it can be obtained by instantiating the ε-matrix of e by ε-free terms.

Epsilon Substitution:

A finite list of equations

$$\varepsilon x F_1(x) = t_1 \quad \cdots \quad \varepsilon x F_k(x) = t_k\ , \tag{4}$$

such that $\varepsilon x F_1(x)$ are canonical ε-terms and t_i are ε-free terms. The $\varepsilon x F_1(x)$ are the *main* terms of the corresponding equations; the t_i are their values.

The ε-substitution (4) is finite, but it is extended to all ε-terms by assigning a *default* value to all ε-terms outside the domain of (4). This default value is set equal to a constant (or variable) of L_{PC} and denoted as 0.

Definitions

Let S denote an ε-substitution.

S-value of an expression:

- $|e|_{\mathrm{S}} := e$ if e is a variable or a constant.
- $|f(t_1,\ldots,t_n)|_{\mathrm{S}} := f(|t_1|_{\mathrm{S}},\ldots,|t_n|_{\mathrm{S}})$.
- $|\neg A|_{\mathrm{S}} := |A|_{\mathrm{S}}$.
- $|A \odot B|_{\mathrm{S}} := |A|_{\mathrm{S}} \odot |B|_{\mathrm{S}}$ if $\odot; \in \{\wedge, \vee, \rightarrow\}$.
- $|e|_{\mathrm{S}} = t$ if e is a main term of S and t its value.
- $|e|_{\mathrm{S}} = 0$ if e is a canonical ε-expression not in the domain of S.
- If e is an ε-expression which is not canonical, let $t_1,\ldots,t_n$ be its immediate subterms, and let e' results from e by replacing t_i by $|t_i|_{\mathrm{S}}$. Then $|e|_{\mathrm{S}} = |e'|_{\mathrm{S}}$.

For any expression e its value $|e|_{\mathrm{S}}$ exists and is unique.

Definitions (cont'd)

We fix a term model $\mathfrak{M}$ of L_{PC}. Assume v is a variable assignment for $\mathfrak{M}$.

An ε-substitution S is *correct* under $(\mathfrak{M}, v)$ iff for every equation $\varepsilon x F(x) = t$ occurring in S we have $\mathfrak{M}, s \models |F(t)|_{\mathrm{S}}$.

Suppose $\mathrm{PC}^{\varepsilon} \vdash_{\pi} E$ and E contains no bound variables. Let $\mathcal{C}$ denote the collection of all critical formulas in π.

Employing $\mathfrak{M}$, the substitution method successively defines ε-substitutions. The substitution S is *solving* if for all critical formulas $C \in \mathcal{C}$; $\mathfrak{M}, v \models |C|_{\mathrm{S}}$.

Approximation Steps

Initial Substitution:

$$S_0 := \emptyset \; .$$

Approximation Step:

Assume S_n has already been defined.

Pick an arbitrary false critical formula form C (if there exists any):

$$A(t; u_1, \ldots, u_n) \to A(\varepsilon x A(x; \overline{u}); u_1, \ldots, u_n) \; ,$$

where the u_i are immediate subterms of A.

S_{n+1} is obtained by adding

$$\varepsilon x A(x; |u_1|_{\mathrm{S}}, \ldots, |u_n|_{\mathrm{S}}) = |t|_{\mathrm{S}} \; .$$

Furthermore all equations in S_{n+1} such that its main term has higher rank than $\varepsilon x A(x; |\overline{u}|_{\mathrm{S}})$ are removed.

Correctness and Termination

Correctness. In all approximation steps the ε-substitutions are correct.

Termination. All approximation steps for $\mathcal{C}$ (with respect to $\mathfrak{M}, v$) terminate with a solving substitution.

Extended First Epsilon Theorem. If $\exists\bar{x}A(x_1,\ldots,x_n)$ is a purely existential formula containing only the bound variables $\bar{x}$, and $\mathrm{PC}^{\varepsilon} \vdash_{\pi} \exists\bar{x}A(\bar{x})$ so that $\mathcal{C}$ collects all critical axioms in π. Then there is a finite set of ε-substitutions

$$S_1,\ldots,S_p\ ,$$

such that

$$S_1(\bigwedge \mathcal{C}) \vee \cdots \vee S_p(\bigwedge \mathcal{C})\ ,$$

is a tautology.

Peano Arithmetic

To formalize number theory in the ε-calculus we add the following axiom.

$$A(t) \to \varepsilon x A(x) \leq t \ ,$$

called *minimality axiom* or *critical axiom of 2nd kind*.

The presence of this critical axioms requires some changes in the definition of the substitution method.

The method assigns *finite* functions to ε-matrices.

Peano Arithmetic: Approximation Steps

The *initial substitution* S_0: Assign the constant function 0 to all ε-matrices.

The *substitution step* $S_n \to S_{n+1}$: Pick a false (in the standard model $\mathfrak{N}$) critical axiom, e.g.

$$A(t, u(t), v) \to A(\varepsilon x A(x, u(x), v), u(\varepsilon x A(x, u(x), v)), v)$$

Let $|t|_{\mathrm{S}} = z$ and $|v|_{\mathrm{S}} = m$. Consider the first formula $A(k, u(k), m)$ in the sequence

$$A(1, u(1), m), \ldots, A(z, u(z), m) \; ,$$

such that $\mathfrak{N} \models A(k, u(k), m)$.

With respect to S_n some function ϕ was assigned to $\varepsilon x A(x, u(x), a)$. We define a new function ψ as follows:

$$\psi(a) := \begin{cases} \phi(a) & a \neq m \\ k & a = m \end{cases}$$

Correctness and Termination

Correctness has to be changed.

The ε-substitution is correct, if for the matrix $\varepsilon x A(x, u(x), a)$ and the assignment $\varepsilon x A(x, u(x), m) = k$ we have

$$\mathfrak{N} \models |A(k, u(k), m)|_{\mathrm{S}} \,,$$

and for all $\ell \leq k$ $\mathfrak{N} \not\models |A(k, u(k), m)|_{\mathrm{S}}$.

Correctness. In all approximation steps the ε-substitutions are correct.

Termination. All approximation steps for C (with respect to M) terminate with a solving substitution.

The transformation $()^{\varepsilon} \colon \mathrm{L}_{\varepsilon} \to \mathrm{L}_{\mathrm{PC}}$ defined yesterday can be employed to show that Peano Arithmetic embeds into $\mathrm{PA}^{\varepsilon}$:

$\mathrm{PA}^{\varepsilon} \vdash A$ iff $\mathrm{PA} \vdash A^{\varepsilon}$.

The Extended First Epsilon Theorem: Lemma

Lemma. Let e be a critical ε-term of π of of maximal degree among the critical ε-terms of maximal rank, and let m be the number of critical formulas belonging to e. Then there are terms s_{ij} ($1 \leq i \leq n$, $0 \leq j \leq m$) and a derivation π_e with end formula

$$E(s_{01}, \ldots, s_{0m}) \vee \ldots \vee E(s_{n1}, \ldots, s_{nm})$$

so that

$$\mathrm{rk}(\pi_e) \leq \mathrm{rk}(\pi), \quad \deg(\pi_e) \leq \deg(\pi), \quad o(\pi_e, \mathrm{rk}(e)) = o(\pi, \mathrm{rk}(e)) - 1.$$

Employing the lemma the Extended First Epsilon Theorem follows as before my an induction on the rank of the derivation.

Once all critical formulas have been eliminated, we can replace all outermost ε-terms by new free variables.

Note that this already yields Herbrand's Theorem for purely existential, equality-free formulas.

Obviously we can derive $\exists x_1 \ldots \exists x_m E(x_1, \ldots, x_m)$ from $\bigvee_m E(s_{i1}, \ldots, s_{im})$.

Proof of Extended First Epsilon Theorem

By induction on $\mathrm{rk}(\pi)$.

If $\mathrm{rk}(\pi) = 0$, we have a proof of $E(s_1, \ldots, s_m)$ without critical formulas.

If $\mathrm{rk}(\pi) > 0$ and the order of π wrt. $\mathrm{rk}(\pi)$ is m, then m-fold application of the lemma results in a derivation π' of rank $< \mathrm{rk}(\pi)$ with end-formula $\bigvee_{i=0}^{n} E(s_{i1}, \ldots, s_{im})$.

But this again is a formula

$$E'(s_{01}, \ldots, s_{0m}, \ldots, s_{n1}, \ldots, s_{nm})$$

to which the Lemma applies.

And a disjunction of instances of E' is itself a disjunction of instances of E.

Preparations (cont'd)

Let $u_1, \ldots, u_p$ denote terms occurring in the disjunction (1).

Let p_i be the number of occurrences of f in u_i. By possibly reordering and leaving out terms, we may assume that the sequence

$$u_1, \ldots, u_p \ ,$$

is ordered such that $p_i \leq p_{i+1}$.

Now let a_i be new free variables. Replace each occurrence of t_i which does not occur as a subterm of another $t_{i'}$ by a_i.

Then (1) becomes

$$\bigvee_i B(r_i, a_{h_i}, s_i) \ , \tag{5}$$

where the h_i are indices from $[1, p]$.

Preparations (cont'd)

Facts:

Property (A): $h_i = h_{i'}$ iff $r_i = r_i'$.

Proof. If: obvious.

Only if: W.l.o.g. we assume $i < i'$. We show that h_i and $h_{i'}$ must be different. Assume $h_i = h_{i'}$. This implies the terms t_i is equal to $t_{i'}$ in (1) i.e., $f(r_i) = f(r_{i'})$. Hence $r_i = r_{i'}$. Contradiction.

Property (B): If a_i is equal to or a subterm of $r_{i'}$, then $h_i < h_{i'}$.

Proof. $r_{i'}$ occurs in $t_{i'}$. Hence, if a_i is equal to or a subterm of $r_{i'}$, the term t_i is a subterm of $t_{i'}$ in(1).

Thus the number of occurrences of f's in $t_{i'}$ is larger than that in t_i.

It is easy to see that (5) is tautology: Pairs of equal atomic formulas remain pairs of equal atomic formulas.

Because of property (B), a_{h_i} does not occur in $r_{i'}$ for $i' \leq i$.

I.e.,a_{h_i} appears only in the designated positions in the i-th disjunct or anywhere else in disjuncts to the right of the i-th.

Main Part (cont'd)

Thus from (5) we obtain that

$$\bigvee_i \exists z B(r_i, a_i, z) , \tag{6}$$

is derivable in PC.

We consider those a_{h_i} occurring in terms r_i in (6).

Assume a_m is the variable among these with highest index m. Suppose a_m occurs in a disjunct

$$\exists z B(r_j, a_{h_j}, z) .$$

Then a_{h_j} does not occur elsewhere in (6), due to property (A).

Thus rule (ν) is applicable to prove (in PC)

$$\bigvee_i \exists z B(r_i, a_i, z) \vee \forall y \exists z B(r_i, y, z)$$

Iterating these steps we eventually obtain a deduction of A in PC.

Remarks on Proof Lengths

The *length* of a deduction is the number of steps in the deduction.

A deduction in $\mathrm{PC}_0^\varepsilon$ is defined similarly to a deduction in $\mathrm{PC}_0^\varepsilon$ but instead of identity schemas, identity axioms plus formulas of ε-equality are employed.

The Herbrand Complexity:

Let A be a valid prenex formula. If A^H denotes the Herbrand Normal form of A. Then $\mathrm{HC}(A)$ denotes the minimal length, i.e. the minimal number of disjunctions of Herbrand disjunctions of A^H.

Proposition Let A be a prenex formula. No function f can exist, depending only on the length k of the deduction and the logical complexity d of the endformula, such that $f(k,d)$ limits $\mathrm{HC}(A)$.

Main Lemmas

We employ Yukami's trick [**?**].

Set $0 \times k := \underbrace{0 + (0 + \cdots (0 + 0))}_{k \text{ times } 0}$.

Yukami's trick. Using two instances of the following restricted scheme of identity

$$s = 0 \to g(s) = g(0) \tag{7}$$

we can derive $0 \times k = 0$ from (i) $0 + 0 = 0$, (ii) $\forall x, y, z\, x = y \wedge y = z \to x = z$, and (iii) $\forall x, y\ x + y = y \to x = 0$ in constant length for any k.

Main Lemmas (cont'd)

Lemma. The restricted scheme of identity (7) is derivable in $\mathrm{PC}_0^{\varepsilon}$.

Proposition. Hence $\mathrm{PC}_0^{\varepsilon} \vdash 0 \times k = 0$ from (i) $0+0=0$ and (ii) $\forall x, y\ (x+y=y \to x=0)$ in constant number of steps for any k.

Main Lemmas: Proof

Assume g is an arbitrary term.

From $g(a) = g(a)$ and

$$g(a) = g(a) \to \varepsilon x\,(x = g(a)) = g(a)\ .$$

we derive

$$\varepsilon x\,(x = g(a)) = g(a)\ ,$$

Using

$$s = t \to \varepsilon x\,(x = g(s)) = \varepsilon x\,(x = g(t))\ ,$$

together with reflexivity and transitivity, we obtain

$$s = t \to g(s) = g(t)\ ,$$

and hence

$$0 = t \to g(0) = g(t)\ ,$$

is derivable.

Proof. Assume the existence of a bound on $\mathrm{HC}(A)$ in the length of these proofs N and the complexity D of the end-formula.

Hence the term-depth of the Herbrand disjunction of A is bounded in N and D. To see this we employ unification.

Hence, the formula

$$\forall x(x = x) \wedge (i) \wedge (ii) \wedge (iii) \to 0 + (0 + \cdots a \cdots) = 0 \,,$$

for some free variable has to be provable, too. Contradiction.

Relation to PC

Note that cut-elimination implies the existence of a bound on $\mathrm{HC}(A)$ in the length of the proof of A and the complexity of A.

Thus the restricted scheme of identity (7) is not fast provable in PC_0.