Encoding the Hydra Battle as a rewrite system

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Abstract. In rewriting theory, termination of a rewrite system by Kruskal's theorem implies a theoretical upper bound on the complexity of the system. This bound is, however, far from having been reached by *known* examples of rewrite systems. All known orderings used to establish termination by Kruskal's theorem yield a multiply recursive bound. Furthermore, the study of the order types of such orderings suggests that the class of multiple recursive functions constitutes the least upper bound. Contradicting this intuition, we construct here a rewrite system which reduces by Kruskal's theorem and whose complexity is not multiply recursive. This system is even *totally terminating*. This leads to a new lower bound for the complexity of totally terminating rewrite systems and rewrite systems which reduce by Kruskal's theorem. Our construction relies on the *Hydra battle* using classical tools from ordinal theory and subrecursive functions.

Introduction

One of the main questions in rewriting theory is that of termination, which has long been known to be undecidable. Most of the termination proof techniques developed in term rewriting theory take advantage of a powerful combinatorial result, Kruskal's tree theorem. Kruskal's theorem furnishes a sufficient syntactic condition for termination: every rewrite system which is compatible with the homeomorphic embedding relation is terminating. This theorem has given rise to the definition of several proof methods, such as the multiset path ordering, the lexicographic path ordering, the Knuth-Bendix orderings, polynomial interpretations. All these methods yield the existence of a total strictly monotone ordering compatible with the homeomorphic embedding relation. This corresponds to the concept of *total termination*, introduced by Ferreira and Zantema in [4]. It seems that any reasonable effective method used to establish termination by Kruskal's theorem implies total termination.

For practical purposes, termination is not enough. It is worth knowing the complexity of a given rewrite system, by measuring the number of rewrite steps necessary to reach a normal form. We call this the *derivation length*. The complexity of the total termination orderings mentioned above has been characterised: termination under the multiset path ordering implies primitive recursive derivation length (Hofbauer [5]), termination under the Knuth-Bendix ordering implies multiply recursive derivation length (Hofbauer [6]), and termination under the lexicographic path ordering implies multiply recursive derivation length (Weiermann

[11]). What is known in the general case of total termination ? More generally, what is the expressivity of Kruskal's theorem when applied to finite rewrite systems ? Weiermann has produced a theoretical upper bound for the complexity of terminating rewrite systems by Kruskal's theorem, using the Hardy hierarchy: the length of a derivation is dominated by the Hardy function $(s^{\omega})^{\bar{\phi}_{\Omega^{\omega}}(0)}$, where $\bar{\phi}_{\Omega^{\omega}}(0)$ is an ordinal notation from Bachmann's system for the small Veblen ordinal. To give a proof theoretic intuition about this measure, primitive recursion corresponds to the provably total functions of the $\Sigma_0^1 - Ind$ fragment of Peano arithmetic and multiple recursion corresponds to the $\Sigma_0^2 - Ind$ fragment. However $(s^{\omega})^{\bar{\phi}_{\Omega^{\omega}}(0)}$ is not even provably total in ATR_0. So there is a huge gap between the upper bound formulated by Weiermann and the observed complexity of common rewrite systems. Weiermann concluded his article by emphasising that "it is an *open problem* to prove or disprove that there are always multiply recursive bounds on the derivation lengths of a finite rewrite system $\mathcal{R}$ over a finite signature, for which the rewrite relation $\rightarrow_{\mathcal{R}}$ is contained in a simplification ordering (...)."

In addition to the practical interest of knowing the expressivity of total termination orderings, there is a theoretical issue. The study of known total termination orderings tells us that it is possible to classify the derivation lengths with the order type of the ordering. More precisely, the derivation length is connected to the order type through the so called slow-growing hierarchy. Can this result extend to all totally terminating rewrite systems, or even to all systems reducing by Kruskal's theorem, as suggested by Cichon in [2] ? For the homeomorphic embedding of Kruskal's theorem, the maximal order type was studied by Schmidt [9]: it corresponds to the multiply recursive functions in the slow-growing hierarchy.

The purpose of this article is to present a "negative" result. We produce an example of a totally terminating finite rewrite system, which goes above multiple recursion. So this furnishes a new lower bound for the complexity of totally terminating rewrite systems and for rewrite systems that reduces by Kruskal's theorem. This contradicts Cichon's conjecture too. Our construction relies on the famous combinatorial game of the *Hydra battle* [7], which can be seen as a geometrical representation of the Hardy hierarchy. The paper is organised as follows: in the first section, we recall standard notions of term rewriting theory and termination. The second section is devoted to the presentation of the Hydra battle and the third section to the construction the rewrite system $\mathcal{H}$ which encodes the Hydra battle. The proof of total termination for $\mathcal{H}$ is based on a new characterisation of total termination.

1 Rewriting background

This article assumes some familiarity with term rewriting theory. We recall here some useful basic notions. A comprehensive survey is to be found in Dershowitz-Jouannaud [3].

Let $\mathcal{F}$ be a finite signature whose function symbols have fixed arity. Given a set of variables $\mathcal{V}$, $\mathcal{T}(\mathcal{F},\mathcal{V})$ denotes the term algebra built up from $\mathcal{V}$ and $\mathcal{F}$, and $\mathcal{T}(\mathcal{F})$ the set of closed terms of $\mathcal{T}(\mathcal{F},\mathcal{V})$. For a rewrite system $\mathcal{R}$, we write $\xrightarrow{+}_{\mathcal{R}}$ for the associated rewrite relation. $\mathcal{R}$ *terminates* if $\xrightarrow{+}_{\mathcal{R}}$ is Noetherian. The complexity of a terminating rewrite system is measured by the *derivation length* function $Dl_{\mathcal{R}}$, which is the longest derivation allowed by the rewrite system.

Definition 1 (Derivation length). Let $\mathcal{T}(\mathcal{F},\mathcal{V})$ be a term algebra and $\mathcal{R}$ a terminating rewrite system over $\mathcal{T}(\mathcal{F},\mathcal{V})$. Define the *derivation length* functions $dl_{\mathcal{R}}$ and $Dl_{\mathcal{R}}$:

$$\begin{aligned} dl_{\mathcal{R}} :\ & \mathcal{T}(\mathcal{F}) \to \mathbb{N} \\ & t \mapsto \max\{dl_{\mathcal{R}}(u),\ t \to_{\mathcal{R}} u\} + 1 \end{aligned}$$

$$\begin{aligned} Dl_{\mathcal{R}} :\ & \mathbb{N} \to \mathbb{N} \\ & m \mapsto \max\{n \in \mathbb{N},\ \exists\, t \in \mathcal{T}(\mathcal{F}), dl_{\mathcal{R}}(t) = n\ \wedge\ |t| \leq m\} \end{aligned}$$

where $|t|$ is the height of t.

Given a well-ordered set $(\mathcal{A},\prec)$, an *interpretation* for a rewrite system $\mathcal{R}$ on $\mathcal{A}$ is a morphism $[\,] : \mathcal{T}(\mathcal{F}) \to \mathcal{A}$ such that

$$\forall u \forall v \in \mathcal{T}(\mathcal{F})\ \ u \xrightarrow{+}_{\mathcal{R}} v\ \Rightarrow [u] \succ [v].$$

Since $(\mathcal{A},\prec)$ is well-founded, the interpretation ensures termination.

Definition 2. Let $\mathcal{T}(\mathcal{F},\mathcal{V})$ be a term algebra and $(\mathcal{A},\prec)$ be a well-ordered set. For any morphism $[\,]$ of $\mathcal{T}(\mathcal{F}) \to \mathcal{A}$, we say that

$[\,]$ is *strictly monotone* if for all $u, v, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F})$, for all $f \in \mathcal{F}$

$$[u] \prec [v] \Rightarrow [f(t_1,\ldots,u,\ldots,t_n)] \prec [f(t_1,\ldots,v,\ldots,t_n].$$

$[\,]$ is *monotone* if for all $u, v, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F})$, for all $f \in \mathcal{F}$

$$[u] \preceq [v]\ \Rightarrow\ [f(t_1,\ldots,u,\ldots,t_n)] \preceq [f(t_1,\ldots,v,\ldots,t_n],$$

where $\preceq$ is the reflexive closure of $\prec$.
$[\,]$ *has the subterm property* if for all $u_1 \ldots u_n \in \mathcal{T}(\mathcal{F})$, for all $f \in \mathcal{F}$

$$\forall i\ 1 \leq i \leq n\ \ [u_i] \prec [f(u_1,\ldots,u_n)].$$

Most of the time, interpretations are defined in a compositional way: each symbol of the signature is assigned a function on $\mathcal{A}$ of the same arity. In this case, the interpretation is monotone if each function is increasing, strictly monotone if each function is strictly increasing and it has the subterm property if the result of each function is strictly greater than each of its arguments. We now come to the definition of total termination, due to Ferreira and Zantema [4].

Definition 3 (Total termination). A rewrite system is *totally terminating* if there exists a well-ordered algebra $(\mathcal{A},\prec)$ and a strictly monotone interpretation for $\mathcal{R}$ on $(\mathcal{A},\prec)$.

In other words, if there exists a well-ordered algebra $(\mathcal{A}, \prec)$ and a strictly monotone morphism $[\,] : \mathcal{T}(\mathcal{F}) \to \mathcal{A}$ such that

$$\forall l \to r \in \mathcal{R}\ \forall \sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F})\ \ [l\sigma] \succ [r\sigma]$$

then $\mathcal{R}$ is totally terminating. It is a well-known result that any totally terminating rewrite system on a finite signature with fixed arity is compatible with the homeomorphic embedding relation of Kruskal's theorem (see [4] for instance). We now give another characterisation of total termination, which requires only monotonicity, instead of strict monotonicity.

Proposition 4. *Let $\mathcal{T}(\mathcal{F}, \mathcal{V})$ be a term algebra and let $\mathcal{R}$ be a rewrite system on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. If there exists a well-ordered algebra $(\mathcal{A}, \prec)$ and a morphism $[\,] : \mathcal{T}(\mathcal{F}) \to \mathcal{A}$ such that*

(i) *for all $l \to r$ in $\mathcal{R}$, for all substitutions $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F})$, $[l\sigma] \succ [r\sigma]$,*

(ii) $[\,]$ *has the subterm property,*

(iii) $[\,]$ *is monotone,*

then $\mathcal{R}$ is totally terminating.

Proof. We construct a strictly monotone interpretation $\mathcal{I}$ for $\mathcal{R}$ on the well-ordered algebra $(\mathsf{mul}(\mathcal{A}), \mathsf{mul}(\prec))$ (we write $\mathsf{mul}(\mathcal{A})$ to mean the set of finite multisets on $\mathcal{A}$ and $\mathsf{mul}(\prec)$ the multiset extension of $\prec$ on $\mathsf{mul}(\mathcal{A})$). Let $\cup$ denote the union of multisets. For each term u in $\mathcal{T}(\mathcal{F})$, define $\mathcal{I}(u)$ as the multiset of $\mathsf{mul}(\mathcal{A})$ containing the interpretations of u and its subterms:

$$\begin{aligned} \mathcal{I}(c) &= \{[c]\} \ \text{ whenever } c \text{ is a constant symbol,} \\ \mathcal{I}(f(t_1, \ldots, t_n)) &= \{[f(t_1, \ldots, t_n)]\} \cup \mathcal{I}(t_1) \ldots \cup \mathcal{I}(t_n). \end{aligned}$$

– $\mathcal{I}$ is compatible with $\mathcal{R}$: let $l \to r$ in $\mathcal{R}$ and $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F})$, a substitution. By (i), $[l\sigma] \succ [r\sigma]$, which with (ii) implies $\{[l\sigma]\}\mathsf{mul}(\succ)\mathcal{I}(r\sigma)$. Hence $\mathcal{I}(l\sigma)\mathsf{mul}(\succ)\mathcal{I}(r\sigma)$.

– $\mathcal{I}$ is strictly monotone: let $u, v \in \mathcal{T}(\mathcal{F})$ such that $\mathcal{I}(u)\mathsf{mul}(\prec)\mathcal{I}(v)$ and let $f \in \mathcal{F}$ of arity $n+1$. For all $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F})$, we have

$$\begin{aligned} \mathcal{I}(f(t_1, \ldots, u, \ldots, t_n)) &= \{[f(t_1, \ldots, u, \ldots, t_n)]\} \cup \mathcal{I}(u) \cup \mathcal{I}(t_1) \cup \ldots \cup \mathcal{I}(t_n) \\ \mathcal{I}(f(t_1, \ldots, v, \ldots, t_n)) &= \{[f(t_1, \ldots, v, \ldots, t_n)]\} \cup \mathcal{I}(v) \cup \mathcal{I}(t_1) \cup \ldots \cup \mathcal{I}(t_n). \end{aligned}$$

By hypothesis, we have $\mathcal{I}(u)\mathsf{mul}(\prec)\mathcal{I}(v)$, which implies

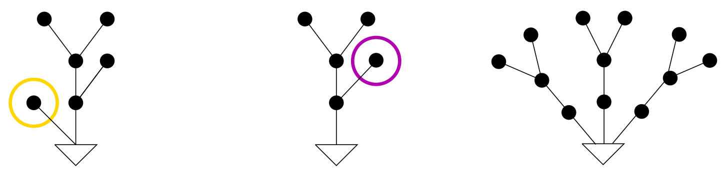
$$\mathcal{I}(u) \cup \mathcal{I}(t_1) \cup \ldots \cup \mathcal{I}(t_n)\ \ \mathsf{mul}(\prec)\ \ \mathcal{I}(v) \cup \mathcal{I}(t_1) \cup \ldots \cup \mathcal{I}(t_n).$$

So it remains to show that $[f(t_1, \ldots, u, \ldots, t_n)] \preceq [f(t_1, \ldots, u, \ldots, t_n)]$. Suppose $[u] \succ [v]$. By hypothesis (ii) on $[\,]$, this would imply $\{[u]\}\mathsf{mul}(\succ)\mathcal{I}(v)$, which contradicts the hypothesis $\mathcal{I}(u)\mathsf{mul}(\prec)\mathcal{I}(v)$. So $[u] \preceq [v]$, which with (iii) ensures $[f(t_1, \ldots, u, \ldots, t_n)] \preceq [f(t_1, \ldots, v, \ldots, t_n)]$. This concludes the proof. □

2 The Hydra battle

The rewrite system we present is based on the *Battle of Hercules and the Hydra* of [7]. Let us recall the general principle. A Hydra is a finite tree, each leaf corresponding to a head. At each step of the game, Hercules chops off one of the heads of the Hydra and the monster grows in turn: if the cut leaf has a grandparent in the tree, then the branch issued from this grandparent is multiplied. The number of copy equals the rank of the step in the game. This implies that the multiplication rate of the Hydra is increasing during the game. Hercules wins when the Hydra is reduced to the empty tree.

A battle may easily be interpreted by a decreasing sequence of ordinals. Associate to each node $\mathtt{n}$ in the tree the ordinal $\langle \mathtt{n} \rangle = \omega^{\langle \mathtt{n}_1 \rangle \oplus \cdots \oplus \langle \mathtt{n}_i \rangle}$, where $\mathtt{n}_1, \ldots, \mathtt{n}_i$ are the children of $\mathtt{n}$ and $\oplus$ denotes the ordinal natural sum. The whole tree is interpreted by $\langle \mathtt{r}_1 \rangle \oplus \cdots \oplus \langle \mathtt{r}_n \rangle$, where $\mathtt{r}_1, \ldots, \mathtt{r}_n$ are the children of the root. Here is an example of battle, with the associated ordinal labelling. So every strategy is a winning strategy for Hercules.



step $2 : \omega^{\omega^2+1} + 1$ step $3 : \omega^{\omega^2+1}$ step $4 : \omega^{\omega^2} 3$

We now concentrate on a particular strategy, which we call "standard". We describe it from the ordinal point of view. Let $\mathcal{CNF}(\varepsilon_0)$ denote the set of notations in Cantor Normal Form for ordinals below ε_0. Given a limit ordinal λ, a *fundamental sequence* $(\lambda_n)_{n \in \mathbb{N}}$ for λ is simply a strictly increasing sequence whose supremum is λ. A canonical assignment of fundamental sequences for $\mathcal{CNF}(\varepsilon_0)$ is defined recursively as follows:

$$\begin{aligned} \omega_n &= n \\ (\alpha + \lambda)_n &= \alpha + \lambda_n \\ (\omega^{\beta+1})_n &= \omega^\beta n \\ (\omega^\lambda)_n &= \omega^{\lambda_n}. \end{aligned}$$

Definition 5 (The standard Hydra battle). For all n in $\mathbb{N}$, define the function $h_n : \mathcal{CNF}(\varepsilon_0) \to \mathcal{CNF}(\varepsilon_0)$ by

$$\begin{aligned} h_n(0) &= 0 \\ h_n(\alpha + 1) &= \alpha \\ h_n(\lambda) &= \lambda_n, \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Given an initial ordinal α_0, the battle is a sequence $(\alpha_n, n)_{n \in \mathbb{N}}$ of $\mathcal{CNF}(\varepsilon_0) \times \mathbb{N}$ such that for all n in $\mathbb{N}$ $\alpha_{n+1} = h_n(\alpha_n)$. In a pair (α_n, n), the ordinal α_n is the Hydra. The second element n is the rank of the step in the game.

For any initial configuration, the standard battle is finite. This fact is however not provable in Peano arithmetic. Indeed, given an initial configuration α, the length of this battle is greater than $s^\alpha(0)$, the αth element of the Hardy hierarchy applied to 0. This can be established using standard tools of number theoretic functions. We do not go into technical details here and we invite the interested reader to consult some classical texts, such as Cichon [1] and Wainer [10]. The only result we need for our construction is the following proposition.

Proposition 6. *The function of* $\mathbb{N} \to \mathbb{N}$ *which associates to each integer* n *the length of the Hydra Battle starting from* $(\omega^{\omega^\omega}, n)$ *with standard strategy is not multiply recursive.*

Proof. The Hardy function $s^{\omega^{\omega^\omega}}$ is not multiply recursive (Robbin [8]). □

It follows that any rewrite system $\mathcal{R}$ encoding the Hydra Battle for trees of height 4, that is ordinals below ω^{ω^ω}, admits a derivation length function which is not multiply recursive. For each n in $\mathbb{N}$, $(\omega^{\omega^\omega}, n)$ reduces in $(\omega^{\omega^n}, n{+}1)$ in one step. For each $\alpha < \omega^{\omega^\omega}$ and for each $m \in \mathbb{N}$, $\mathcal{R}$ encodes the battle with initial configuration (α, m). In particular, it encodes the battle with initial configuration $(\omega^{\omega^n}, n+1)$.

3 Encoding the Hydra battle as a rewrite system

3.1 Construction of the rewrite system $\mathcal{H}$

We now model the process of the Hydra battle by the rewrite system $\mathcal{H}$. A first system for the Hydra battle appears in [3], but its termination cannot be established by Kruskal's theorem. The version we present here is totally terminating. The underlying idea for the transcription is very different. The intuition is as follows. The ordinals of ω^{ω^ω} in Cantor Normal Form are interpreted by terms built up from the constant 0 and the binary function symbol H. For this, define $\mathcal{O}$ by

$$\begin{aligned} \mathcal{O}: \quad \omega^{\omega^\omega} &\to \mathcal{T}(0, \mathsf{H}) \\ 0 &\mapsto 0 \\ \omega^\alpha &\mapsto \mathsf{H}(\mathcal{O}(\alpha), 0) \\ \beta + \omega^\alpha &\mapsto \mathsf{H}(\mathcal{O}(\alpha), \mathcal{O}(\beta)) \end{aligned}$$

To deal with the rank of the step in a battle, we introduce two unary function symbols, $[]$ and $\bullet$. Each step (α, n) of the battle will be encoded by the term $\bullet\, []^n\, \mathcal{O}(\alpha)$. For each ordinal α in ω^{ω^ω}, the system $\mathcal{H}$ should then allow us to derive

$$\bullet\, []^n\, \mathcal{O}(\alpha) \xrightarrow{+}_{\mathcal{H}} \bullet\, []^{n+1}\, \mathcal{O}(h_n(\alpha)).$$

Let's have a closer look on the definition of the ordinal function h_n. Given an ordinal α in ω^{ω^ω}, we distinguish three cases for the computation of $h_n(\alpha)$:

Case 1 : if α is a successor ordinal of the form $s(\beta)$, then $h_n(\alpha) = \beta$,
Case 2 : if α is a limit ordinal of the form $\gamma + \omega^{s(\beta)}$, then $h_n(\alpha) = \gamma + \omega^\beta n$.

Case 3 : if α is a limit ordinal of the form $\gamma + \omega^{\beta+\omega^{s(a)}}$, $h_n(\alpha) = \gamma + \omega^{\beta+\omega^{a}n}$.

So if we write $t = \mathcal{O}(\beta)$, $u = \mathcal{O}(\gamma)$ and $v = \mathcal{O}(a)$, $\mathcal{H}$ should allow us to derive

Case 1 : $\bullet\,[]^n\ \mathsf{H}(0,t) \xrightarrow{+}_{\mathcal{H}} \bullet\,[]^{n+1}\ t$,

Case 2 : $\bullet\,[]^n\ \mathsf{H}(\mathsf{H}(0,t),u) \xrightarrow{+}_{\mathcal{H}} \bullet\,[]^{n+1}\ \mathsf{H}(t,\mathsf{H}(t,\ldots\mathsf{H}(t,u)\ldots))$ (n occurrences of t),

Case 3 : $\bullet\,[]^n\ \mathsf{H}(\mathsf{H}(\mathsf{H}(0,t),u),v) \xrightarrow{+}_{\mathcal{H}} []^{n+1}\mathsf{H}(\mathsf{H}(t,\ldots\mathsf{H}(t,u)\ldots),v)$ (n occurrences of t).

The first case can be handled directly by a single rewrite rule. For the two last cases, we need to introduce three intermediate function symbols: $\circ$, c^1 (for case 2) and c^2 (for case 3). Consider finally the signature $\mathcal{F} = \{\circ, \bullet, [], 0, \mathsf{H}, c^1, c^2\}$, where 0 is a constant symbol, $\circ$, $\bullet$, $[]$ are unary function symbols, H, c_1 are of arity 2 and c^2 is of arity 3. $\mathcal{H}$ is defined on $\mathcal{T}(\mathcal{F},\mathcal{V})$ by

$$\mathcal{H}\left\{\begin{array}{rcll}
\circ x & \to & \bullet\,[]\ x & (1)\\
\bullet\,[]\ x & \to & []\,\bullet\bullet x & (2)\\
\mathsf{H}(0,x) & \to & \circ x & (3)\\
\bullet\mathsf{H}(\mathsf{H}(0,y),z) & \to & c^1(y,z) & (4)\\
\bullet\mathsf{H}(\mathsf{H}(\mathsf{H}(0,x),y),z) & \to & c^2(x,y,z) & (5)\\
\bullet c^1(x,y) & \to & c^1(x,\mathsf{H}(x,y)) & (6)\\
\bullet c^2(x,y,z) & \to & c^2(x,\mathsf{H}(x,y),z) & (7)\\
c^1(y,z) & \to & \circ z & (8)\\
c^2(x,y,z) & \to & \circ\mathsf{H}(y,z) & (9)\\
[]\circ x & \to & \circ\,[]\ x & (10)\\
\bullet x & \to & x & (11)
\end{array}\right.$$

3.2 Complexity of $\mathcal{H}$

We verify that $\mathcal{H}$ simulates the Hydra battle.

Lemma 7. *Let $\alpha \in \omega^{\omega^\omega}$. For all $n \geq 1$, $\bullet\,[]^n\ \mathcal{O}(\alpha) \xrightarrow{+}_{\mathcal{H}} \bullet\,[]^{n+1}\ \mathcal{O}(h_n(\alpha))$.*

Proof. We consider the three cases mentioned above.
Case 1 :

$$\begin{array}{rll}
\bullet\,[]^n\ \mathsf{H}(0,t) & \xrightarrow{+}_{\mathcal{H}} []^n\mathsf{H}(0,t) & (11)\\
& \xrightarrow{+}_{\mathcal{H}} []^n \circ t & (3)\\
& \xrightarrow{+}_{\mathcal{H}} \circ\,[]^n\ t & (10)^{\mathrm{n}}\\
& \xrightarrow{+}_{\mathcal{H}} \bullet\,[]^{n+1}\ t & (1)
\end{array}$$

Case 2 :

$$\begin{array}{rll}
\bullet\,[]^n\ \mathsf{H}(\mathsf{H}(0,t),u) & \xrightarrow{+}_{\mathcal{H}} []^n \bullet^{2^n}\ \mathsf{H}(\mathsf{H}(0,t),u) & (2)^n\\
& \xrightarrow{+}_{\mathcal{H}} []^n \bullet^{n+1}\ \mathsf{H}(\mathsf{H}(0,t),u) & (11)^*\\
& \xrightarrow{+}_{\mathcal{H}} []^n \bullet^n\ c^1(t,u) & (4)\\
& \xrightarrow{+}_{\mathcal{H}} []^n c^1(t,\mathsf{H}(t,\mathsf{H}(t,\ldots\mathsf{H}(t,u)\ldots))) & (6)^n\\
& \xrightarrow{+}_{\mathcal{H}} []^n \circ \mathsf{H}(t,\mathsf{H}(t,\ldots\mathsf{H}(t,u)\ldots)) & (8)\\
& \xrightarrow{+}_{\mathcal{H}} \circ\,[]^n\ \mathsf{H}(t,\mathsf{H}(t,\ldots\mathsf{H}(t,u)\ldots)) & (10)^n\\
& \xrightarrow{+}_{\mathcal{H}} \bullet\,[]^{n+1}\ \mathsf{H}(t,\mathsf{H}(t,\ldots\mathsf{H}(t,u)\ldots)) & (1)
\end{array}$$

Case 3 :

$$\begin{array}{lll}
\bullet\, []^n\, \mathsf{H}(\mathsf{H}(\mathsf{H}(0,t),u),v) & \xrightarrow{+}_{\mathcal{H}} []^n \bullet^{2^n} \mathsf{H}(\mathsf{H}(\mathsf{H}(0,t),u),v) & (2)^n \\
& \xrightarrow{+}_{\mathcal{H}} []^n \bullet^{n+1} \mathsf{H}(\mathsf{H}(\mathsf{H}(0,t),u),v) & (11)^* \\
& \xrightarrow{+}_{\mathcal{H}} []^n \bullet^n c^2(t,u,v) & (5) \\
& \xrightarrow{+}_{\mathcal{H}} []^n c^2(t,\mathsf{H}(t,\ldots \mathsf{H}(t,u)\ldots),v) & (7)^n \\
& \xrightarrow{+}_{\mathcal{H}} []^n \circ \mathsf{H}(\mathsf{H}(t,\ldots \mathsf{H}(t,u)\ldots),v) & (9) \\
& \xrightarrow{+}_{\mathcal{H}} \circ []^n\, \mathsf{H}(\mathsf{H}(t,\ldots \mathsf{H}(t,u)\ldots),v) & (10)^n \\
& \xrightarrow{+}_{\mathcal{H}} \bullet []^{n+1}\, \mathsf{H}(\mathsf{H}(t,\ldots \mathsf{H}(t,u)\ldots),v) & (1)
\end{array}$$

□

Corollary 8. *$Dl_{\mathcal{H}}$ is not multiply recursive.*

Proof. Consequence of proposition 6. □

3.3 $\mathcal{H}$ is totally terminating

The proof of total termination is based on proposition 4: we associate to each function symbol appearing in $\mathcal{H}$ a monotone function which enjoys the subterm property. Our starting point is the intentional meaning of the symbols 0 and H: each term t built up from 0 and H may simply be interpreted by the ordinal $\mathcal{O}^{-1}(t)$. For c_1 and c_2, we shall use the function f, defined by

$$\begin{aligned} f : \mathcal{CNF}(\varepsilon_0) \times \mathcal{CNF}(\varepsilon_0) &\to \mathcal{CNF}(\varepsilon_0) \\ (x,y) &\mapsto y + \omega^{x+1} \end{aligned}$$

Note that the definition of f uses the ordinal sum $+$, which is not strictly monotonic. For instance, $f(2, \omega^2 3 + \omega + 7) = \omega^3 + \omega + 7$.

Lemma 9. *For all α, β in $\mathcal{CNF}(\varepsilon_0)$*

(i) $f(\alpha, \beta \oplus \omega^\alpha) = f(\alpha,\beta)$,
(ii) $f(\alpha,\beta) \leq \beta \oplus \omega^\alpha \omega$,
(iii) $f(\alpha,\beta) > \alpha$ *and* $f(\alpha,\beta) > \beta$,
(iv) *f is an increasing function.*

Proof. Straightforward. □

For the symbols $\bullet$, $\circ$ and $[]$, consider the sub-system

$$\begin{cases} \circ x \to \bullet [] \, x \\ \bullet [] \, x \to [] \bullet \bullet x \\ [] \circ x \to \circ [] \, x \end{cases}$$

This admits an interpretation on $\omega \times \omega$: interpret $\circ$ by $(m,n) \mapsto (2m+3, n)$, $[]$ by $(m,n) \mapsto (2m+2, n)$ and $\bullet$ by $(m,n) \mapsto (m, n+m+1)$. Combining the interpretations for 0, H, c_1 and c_2 on $\mathcal{CNF}(\varepsilon_0)$ and the interpretations for $[]$, $\circ$ and $\bullet$ on $\omega \times \omega$, we finally define $[\,]$ on $\mathcal{CNF}(\varepsilon_0) \times \omega \times \omega$ as follows:

$$
\begin{aligned}
[0] &= (0,0,0) \\
[\mathsf{H}] &= (\alpha, m, n), (\beta, m', n') \mapsto (\omega^\alpha \oplus \beta, 0, 0) \\
[c^1] &= (\alpha, m, n), (\beta, m', n') \mapsto (f(\alpha,\beta), 0, 0) \\
[c^2] &= (\alpha, m, n), (\beta, m', n'), (\gamma, m'', n'') \mapsto (\gamma \oplus \omega^{f(\alpha,\beta)}, 0, 0) \\
[\bullet] &= (\alpha, m, n) \mapsto (\alpha, m, n+m+1) \\
[\circ] &= (\alpha, m, n) \mapsto (\alpha, 2m+3, n) \\
[[]] &= (\alpha, m, n) \mapsto (\alpha, 2m+2, n)
\end{aligned}
$$

$\mathcal{CNF}(\varepsilon_0) \times \omega \times \omega$ is ordered by the lexicographic combination of $(\mathcal{CNF}(\varepsilon_0), \in)$ and $(\omega \times \omega, \in)$. We write $<$ for this ordering.

Lemma 10.

(i) $[\,]$ *has the subterm property,*

(ii) $[\,]$ *is monotone,*

(iii) *for all* $l \to r \in \mathcal{H}$*, for all substitutions* $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F})$*,* $[l\sigma] > [r\sigma]$.

Proof. (i) and (ii) are direct, using lemma 9 for f. For (iii), we examine each rule:

(1) $\mathsf{H}(0,t) \to \circ t : (\alpha+1, 0, 0) > (\alpha, 2m+3, n)$

(2) $c^1(u,v) \to \circ v : (f(\beta,\gamma), 0, 0) > (\gamma, 2m''+3, n'')$

(3) $c^2(t,u,v) \to \circ\mathsf{H}(u,v) : (\gamma \oplus \omega^{f(\alpha,\beta)}, 0, 0) > (\gamma \oplus \omega^\beta, 3, 0)$

(4) $\circ t \to \bullet [] \, t : (\alpha, 2m+3, n) > (\alpha, 2m+2, n+2m+3)$

(5) $\bullet [] \, t \to [] \bullet \bullet t$: $(\alpha, 2m+2, n+2m+3) > (\alpha, 2m+2, n+2m+2)$

(6) $[] \circ t \to \circ [] \, t : (\alpha, 4m+8, n) > (\alpha, 4m+7, n)$

(7) $\bullet\mathsf{H}(\mathsf{H}(0,u),v) \to c^1(u,v) : (\gamma \oplus \omega^{\beta+1}, 0, 1) > (f(\beta,\gamma), 0, 0)$

(8) $\bullet\mathsf{H}(\mathsf{H}(\mathsf{H}(0,t),u),v) \to c^2(t,u,v) : (\gamma \oplus \omega^{\beta+\omega^{\alpha+1}}, 0, 1) > (\gamma \oplus \omega^{f(\alpha,\beta)}, 0, 0)$

(9) $\bullet c^1(t,u) \to c^1(t, \mathsf{H}(t,u)) : (f(\alpha,\beta), 0, 1) > (f(\alpha,\beta), 0, 0)$

(10) $\bullet c^2(t,u,v) \to c^2(t, \mathsf{H}(t,u), v)$: $(\gamma \oplus \omega^{f(\alpha,\beta)}, 0, 1) > (\gamma \oplus \omega^{f(\alpha,\beta)}, 0, 0)$

(11) $\bullet t \to t$: $(\alpha, m, n+n+1) > (\alpha, m, n)$.

(t, u, v are terms of $\mathcal{T}(\mathcal{F})$ whose interpretations are (α, m, n), (β, m', n') and (γ, m'', n'') respectively). □

Proposition 11. $\mathcal{H}$ *is totally terminating.*

Proof. Consequence of lemma 10 and proposition 4. □

3.4 Extension of $\mathcal{H}$

The rewrite system $\mathcal{H}$ models a restrained version of Hydra battle with ordinals below ω^{ω^ω}. It may easily be extended to deal with higher ordinals, below ε_0. To reach $\omega^{\omega^{\omega^\omega}}$, one adds a 4-ary function symbol c^3 and so on. In this way one exhausts the provably total functions of Peano arithmetic.

Perspectives

We have exhibited a totally terminating rewrite system which departs from multiple recursion. What still remains open is what complexity can be achieved via total termination or termination by Kruskal's theorem. Moreover, our example rekindles the debate on the relationship between order type and length of derivation for a rewrite system. Our construction is interesting from a proof-theoretical point of view. We have shown that it is possible to encode the Hardy hierarchy by a finite rewrite system. So it can be directly connected with the work of Weiermann in [12], which uses the Hardy hierarchy too. Unfortunately, our construction is restrained to ordinals below ε_0. Is it possible to describe higher ordinals and reach $\phi_{\Omega^\omega}(0)$, the maximal order type of homeomorphic embedding of Kruskal's theorem ? This then would imply that the bound formulated by Weiermann is, surprisingly, a least upper bound.

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