Relating derivation lengths with the slow-growing hierarchy directly

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Abstract. In this article we introduce the notion of a generalized system of fundamental sequences and we define its associated slow-growing hierarchy. We claim that these concepts are genuinely related to the classification of the complexity—the derivation length— of rewrite systems for which termination is provable by a standard termination ordering. To substantiate this claim, we re-obtain multiple recursive bounds on the the derivation length for rewrite systems terminating under lexicographic path ordering, originally established by the second author.

1 Introduction

To show termination of a rewrite system R one usually shows that the induced reduction relation $\to_R$ is contained in some abstract ordering known to be well-founded. One way to assess the strength of such a termination ordering is to calculate its *order type*, cf. [7]. There appears to be a subtle relationship between these order types and the *complexity* of the rewrite system R considered. Cichon [5] discussed (and investigated) whether the complexity of a rewrite system for which termination is provable using a termination ordering of order type α is eventually dominated by a function from the *slow-growing hierarchy* along α. It turned out that this principle—henceforth referred to as (CP)—is valid for the (i) *multiset path ordering* ($\succ_{\text{MPO}}$) and the (ii) *lexicographic path ordering* ($\succ_{\text{LPO}}$).

More precisely, Hofbauer [9] proved that $\succ_{\text{MPO}}$ as termination ordering implies primitive recursive derivation length, while the second author showed that $\succ_{\text{LPO}}$ as termination ordering implies multiply-recursive derivation length [17]. If one regards the order types of $\succ_{\text{MPO}}$ and $\succ_{\text{LPO}}$, respectively, then these results imply the correctness of (CP) for (i) and (ii). Buchholz [3] has given an alternative proof of (CP) for (i) and (ii). His proof avoids the (sometimes lengthy) calculations with functions from subrecursive hierarchies in [9, 17]. Instead a clever application of proof-theoretic results is used. Although this proof is of striking beauty, one might miss the link to term rewriting theory that is provided in [9, 17].

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The mentioned proofs [9, 17, 3] of (CP)—with respect to (i) and (ii)—are *indirect*. I.e. without direct reference to the slow-growing hierarchy. By now, we know from the work of Touzet [16] and Lepper [10, 11] that (CP) fails to hold in general. However, our interest in (CP) is motivated by our strong belief that there exist reliable ties between *proof theory* and *term rewriting theory*. Ties which become particularly apparent if one studies those termination orderings for which (CP) holds.

To articulate this belief we give yet another *direct* proof of (CP) (with respect to (i) and (ii)). To this avail we introduce the notion of a *generalized system of fundamental sequences* and we define its associated *slow-growing hierarchy*. These concepts are genuinely related to classifying derivation lengths for rewrite systems for which termination is proved by a standard termination ordering. To emphasize this let us present the general outline of the proof method.

Let terms $s = t_0, t_1, \dots, t_n$ be given, such that $s \to_R t_1 \to_R \cdots \to_R t_n$ holds, where t_n is in normal form and term-depth of s ($\tau(s)$) is $\leq m$. Assume $\to_R$ is contained in a termination ordering $\succ$. Hence $s \succ t_1 \succ \cdots \succ t_n$ holds. Assume further the sequence $(s, t_1, \dots, t_n)$ is chosen so that n is maximal. Then in the realm of classifications of derivation lengths one usually defines an *interpretation* $\mathcal{I}\colon \mathcal{T}(\Sigma, \mathcal{V}) \to \mathbb{N}$ such that $\mathcal{I}(s) > \mathcal{I}(t_1) > \cdots > \mathcal{I}(t_n)$ holds. ($\mathcal{T}(\Sigma, \mathcal{V})$ denotes the term algebra over the signature Σ and the set of variables $\mathcal{V}$.) The existence of such an interpretation then directly yields a bound on the derivation length.

The problem with this approach is to guess the right interpretation from the beginning. More often than not this is not at all obvious. Therefore we want to generate the interpretation function directly from the termination ordering in an intrinsic way. To this avail we proceed as follows. We separate $\mathcal{I}$ into an *ordinal interpretation* $\pi\colon \mathcal{T}(\Sigma) \to T$ and an ordinal theoretic function $g\colon T \to \mathbb{N}$. (T denotes a suitable chosen set of terms representing an initial segment of the ordinals, cf. Definition 2.) This works smoothly. Firstly, we can employ the connection between the termination ordering $\succ$ and the ordering on the notation system T. This connection was already observed by Dershowitz and Okada, cf. [7]. Secondly, it turns out that g can be defined in terms of the slow-growing function $G_x\colon T \to \mathbb{N}$; $x \in \mathbb{N}$. (Note that we have swapped the usual denotation of arguments, see Definition 4 and Definition 9.)

To simplify the presentation we restrict our attention to a rewrite system R whose termination can be shown by a *lexicographic path ordering* $\succ_{\text{LPO}}$. It will become apparent later that the proof presented below is (relative) easily adaptable to the case where the rewrite relation $\to_R$ is contained in a *multiset path ordering* $\succ_{\text{MPO}}$. We assume the signature Σ contains at least one constant c.

Let R be a rewrite system over $\mathcal{T}(\Sigma, \mathcal{V})$ such that $\to_R$ is contained in a lexicographic path ordering. Let terms $s = t_0, t_1, \dots, t_n$ be given, such that $s \to_R t_1 \to_R \cdots \to_R t_n$ holds, where t_n is in normal form and $\tau(s) \leq m$. By our choice of R this implies

$$s \succ_{\text{LPO}} t_1 \succ_{\text{LPO}} \cdots \succ_{\text{LPO}} t_n \quad . \tag{1}$$

We define a ground substitution ρ: $\rho(x) = c$, for all $x \in \mathcal{V}$. Let $>$ denote a suitable defined (well-founded) ordering relation on the ordinal notation system T. Let $l, r \in \mathcal{T}(\Sigma, \mathcal{V})$. Depending on m and properties of R, we show the existence of a natural number h such that $l \succ_{\text{LPO}} r$ implies $\pi(l\rho) > \pi(r\rho)$ and $G_h(\pi(l\rho)) > G_h(\pi(r\rho))$, respectively. Employing this form of an *Interpretation Theorem* we conclude from (1) for some $\alpha \in T$

$$\alpha > \pi(s\rho) > \pi(t_1\rho) > \cdots > \pi(t_n\rho) \quad .$$

and consequently

$$G_h(\alpha) > G_h(\pi(s\rho)) > G_h(\pi(t_1\rho)) > \cdots > G_h(\pi(t_n\rho)) \quad .$$

Thus $G_h(\alpha)$ calculates an upper bound for n. Therefore the *complexity* of R can be measured in terms of the *slow-growing hierarchy* along the *order type* of T.

To see that this method calculates an optimal bound, it remains to relate the function $G_x \colon T \to \mathbb{N}$ to the multiply-recursive functions. We employ Girard's Hierarchy Comparison Theorem [8]. Due to (a variant) of this theorem any multiple-recursive function can be majorized by functions from the slow-growing hierarchy and vice versa.[1] (For further details see Section 4.)

Contrary to the original proof in[17], we can thus circumvent technical calculations with the F-hierarchy (the fast-growing hierarchy) and can shed light on the way the slow-growing hierarchy relates the order type of the termination ordering $\succ$ to the bound on the length of reduction sequences along $\to_R$.

2 The Lexicographic Path Ordering

We assume familiarity with the basic concepts of term rewriting. However, we fix some notations. Let $\Sigma = \{f_1, \ldots, f_K\}$ denote a finite signature such that any function symbol $f \in \Sigma$ has a unique *arity*, denoted as $\text{ar}(f)$. The cardinality K is assumed to be fixed in the sequel. To avoid trivialities we demand that Σ is non-empty and contains at least one constant, i.e. a function symbol of arity 0. We set $N := \max\{\text{ar}(f) \colon f \in \Sigma\}$.

The set of terms over Σ and the countably infinite set of variables $\mathcal{V}$ is denoted as $\mathcal{T}(\Sigma, \mathcal{V})$. We will use the meta-symbols $l, r, s, t, u, \ldots$ to denote terms. The set of variables occurring in a term t is denoted as $\text{var}(t)$. A term t is called *ground* or *closed* if $\text{var}(t) = \emptyset$. The set of ground terms over Σ is denoted as $\mathcal{T}(\Sigma)$. If no confusion can arise, the reference to the signature Σ and the set of variables $\mathcal{V}$ is dropped. With $\tau(s)$ we denote the *term depth* of s, defined as $\tau(s) := 0$, if $s \in \mathcal{V}$ or $s \in \Sigma$ and otherwise $\tau(f(s_1, \ldots, s_m)) := \max\{\tau(s_i) \colon 1 \leq i \leq m\} + 1$. A *substitution* $\sigma \colon \mathcal{V} \to \mathcal{T}$ is a mapping from the set of variables to the set of terms. The application of a substitution σ to a term t is (usually) written as $t\sigma$ instead of $\sigma(t)$.

[1] A k-ary function g is said to be *majorized* by a unary function f if there exists a number $n < \omega$ such that $g(x_1, \ldots, x_k) < f(\max\{x_1, \ldots, x_k\})$, whenever $\max\{x_1, \ldots, x_k\} \geq n$.

A *term rewriting system* (or *rewrite system*) R over $\mathcal{T}$ is a finite set of rewrite rules (l, r). The *rewrite relation* $\to_R$ on $\mathcal{T}$ is the least binary relation on $\mathcal{T}$ containing R such that (i) if $s \to_R t$ and σ a substitution, then $s\sigma \to_R t\sigma$ holds, and (ii) if $s \to_R t$, then $f(\ldots, s, \ldots) \to_R f(\ldots, t, \ldots)$. A rewrite system R is *terminating* if there is no infinite sequence $\langle t_i : i \in \mathbb{N} \rangle$ of terms such that $t_1 \to_R t_2 \to_R \cdots \to_R t_m \to_R \cdots$. Let $\succ$ denote a total order on Σ such that $f_j \succ f_i \leftrightarrow j > i$ for $i, j \in \{1, \ldots, K\}$. The *lexicographic path ordering* $\succ_{\text{LPO}}$ on $\mathcal{T}$ (induced by $\succ$) is defined as follows, cf. [1].

Definition 1. *$s \succ_{\text{LPO}} t$ iff*

1. *$t \in \text{var}(s)$ and $s \neq t$, or*
2. *$s = f_j(s_1, \ldots, s_m)$, $t = f_i(t_1, \ldots, t_n)$, and*
   - *there exists k $(1 \le k \le m)$ with $s_k \succeq_{\text{LPO}} t$, or*
   - *$j > i$ and $s \succ_{\text{LPO}} t_l$ for all $l = 1, \ldots, n$, or*
   - *$i = j$ and $s \succ_{\text{LPO}} t_l$ for all $l = 1, \ldots, n$, and there exists an i_0 $(1 \le i_0 \le m)$ such that $s_1 = t_1, \ldots s_{i_0 - 1} = t_{i_0 - 1}$ and $s_{i_0} \succ_{\text{LPO}} t_{i_0}$.*

Proposition 1. *(Kamin-Levy).*

1. *If $s \succ_{\text{LPO}} t$, then $\text{var}(t) \subseteq \text{var}(s)$.*
2. *For any total order $\prec$ on Σ, the induced lexicographic order $\succ_{\text{LPO}}$ is a simplification order on $\mathcal{T}$.*
3. *If R is a rewrite system such that $\to_R$ is contained in a lexicographic path ordering, then R is terminating.*

Proof. Folklore.

3 Ordinal Terms and the Lexicographic Path Ordering

Let N be defined as in the previous section. In this section we define a set of terms T (and a subset $P \subset T$) together with a well-ordering $<$ on T. The elements of T are built from 0, $+$ and the $(N+1)$-ary function symbol ψ. It is important to note that the elements of T are *terms* not ordinals. Although these terms can serve as representations of an initial segment of the set of ordinals On, we will not make any use of this *interpretation*. In particular the reader not familiar with proof theory should have no difficulties to understand the definitions and propositions of this section. However some basic amount of understanding in proof theory may be useful to grasp the origin and meaning of the presented concepts, cf. [7, 11, 15]. For the reader familiar with proof theory: Note that P corresponds to the set of additive principal numbers in T, while ψ represents the (set-theoretical) fixed-point free Veblen function, cf. [15, 11].

Definition 2. *Recursive definition of a set T of ordinal terms, a subset $P \subset T$, and a binary relation $>$ on T.*

1. *$0 \in T$.*
2. *If $\alpha_1, \ldots, \alpha_m \in P$ and $\alpha_1 \ge \cdots \ge \alpha_m$, then $\alpha_1 + \cdots + \alpha_m \in T$.*

3. *If* $\alpha_1, \ldots, \alpha_{N+1} \in T$*, then* $\psi(\alpha_1, \ldots, \alpha_{N+1}) \in P$ *and* $\psi(\alpha_1, \ldots, \alpha_{N+1}) \in T$*.*
4. $\alpha \neq 0$ *implies* $\alpha > 0$*.*
5. $\alpha > \beta_1, \ldots, \beta_m$ *and* $\alpha \in P$ *implies* $\alpha > \beta_1 + \cdots + \beta_m$*.*
6. *Let* $\alpha = \alpha_1 + \cdots + \alpha_m$*,* $\beta = \beta_1 + \cdots + \beta_n$*. Then* $\alpha > \beta$ *iff*
   - $m > n$*, and for all* i *(*$i \in \{1, \ldots, n\}$*)* $\alpha_i = \beta_i$*, or*
   - *there exists* i *(*$i \in \{1, \ldots, m\}$*) such that* $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}$*, and* $\alpha_i > \beta_i$*.*
7. *Let* $\alpha = \psi(\alpha_1, \ldots, \alpha_{N+1})$*,* $\beta = \psi(\beta_1, \ldots, \beta_{N+1})$*. Then* $\alpha > \beta$ *iff*
   - *there exists* k *(*$1 \leq k \leq N+1$*) with* $\alpha_k \geq \beta$*, or*
   - $\alpha > \beta_l$ *for all* $l = 1, \ldots, N+1$ *and there exists an* i_0 *(*$1 \leq i_0 \leq N+1$*) such that* $\alpha_1 = \beta_1, \ldots \alpha_{i_0-1} = \beta_{i_0-1}$ *and* $\alpha_{i_0} > \beta_{i_0}$*.*

We use lower-case Greek letters to denote the elements of T. Furthermore we formally define $\alpha + 0 = 0 + \alpha = \alpha$ for all $\alpha \in T$.

We sometimes abbreviate sequences of (ordinal) terms like $\alpha_1, \ldots, \alpha_n$ by $\overline{\alpha}$. Hence, instead of $\psi(\alpha_1, \ldots, \alpha_{N+1})$ we may write $\psi(\overline{\alpha})$. To relate the elements of T to more expressive ordinal notations, we define $1 := \psi(\overline{0})$, $\omega := \psi(\overline{0}, 1)$, and $\epsilon_0 := \psi(\overline{0}, 1, 0)$. Let Lim be the set of elements in T which are neither 0 nor of the form $\alpha + 1$. Elements of Lim are called *limit* ordinal terms.

Proposition 2. *Let* $(T, <)$ *be defined as above. Then* $(T, <)$ *is a well-ordering.*

Proof. Let $|\alpha|$ denote the number of symbols in the ordinal term α. Exploiting induction on $|\alpha|$ one easily verifies that the ordering $(T, <)$ is well-defined. To show well-foundedness one uses induction on the lexicographic path ordering $\prec_{\text{LPO}}$, exploiting the close connection between Definition 1.2 in Section 2 and Definition 2.7 above. $\square$

In the following proposition we want to relate the *order type* of the well-ordering $(T, <)$ and the well-partial ordering $\prec_{\text{LPO}}$. Concerning the latter it is best to momentarily restrict our attention to the well-ordering $(\mathcal{T}(\Sigma), \prec_{\text{LPO}})$. We indicate the arity of the function symbol ψ employed in Definition 2. We write $(T(N+1), <)$ instead of $(T, <)$. Similarly we write $(\mathcal{T}(\Sigma(N)), \prec_{\text{LPO}})$ to indicate the maximal arity of function symbols in the finite signature Σ. Let $\overline{\Theta}_{\Omega^\omega}(0)$ denote the small Veblen ordinal [15] and let $\mathbf{otyp}(M)$ denote the order type of a well-odering M.

Proposition 3. *1. For any number* k*, there exists an order isomorphic embedding from* $(\mathcal{T}(\Sigma(k)), \prec_{\text{LPO}})$ *into* $(T(k+1), <)$*.*
2. *For any number* $k > 2$*, there exists an order isomorphic embedding from* $(T(k), <)$ *into* $(\mathcal{T}(\Sigma(k)), \prec_{\text{LPO}})$*.*
3. $\sup_{k<\omega}(\boldsymbol{otyp}((T(k), <))) = \sup_{k<\omega}(\boldsymbol{otyp}((\mathcal{T}(\Sigma(k)), \prec_{\text{LPO}}))) = \overline{\Theta}_{\Omega^\omega}(0)$*.*

Proof. The first two assertions are a consequence of the well-ordering proof of $(T, <)$. We only comment on the stated lower bound in the second one. The statement fails for $(T(2), <)$ and $(\mathcal{T}(\Sigma(2)), \prec_{\text{LPO}})$. The presence of the binary function symbol $+$ in $T(2)$ can make the ordering $<$ more expressive than $\prec_{\text{LPO}}$. This difference vanishes for $k \geq 3$. The third assertion follows from [14]. $\square$

4 Fundamental Sequences and Sub-recursive Hierarchies

To each ordinal term $\alpha \in T$ we assign a canonical sequence of ordinal terms $\langle \alpha[x] \colon x \in \mathbb{N} \rangle$, the *fundamental sequence*. The concept of fundamental sequences is a crucial one in (ordinal) proof theory. The main idea of utilizing fundamental sequences in term rewriting, is that the descent along the branches of such a sequence can, informally speaking, code rewriting steps. We have to wade through some technical definitions.

We define the set $\mathrm{IS}_{\overline{\alpha}}(\gamma)$, the set of *interesting subterms* of γ (relative to $\overline{\alpha}$) by induction on γ. We set $\mathrm{IS}_{\overline{\alpha}}(0) := \emptyset$, $\mathrm{IS}_{\overline{\alpha}}(\gamma_1 + \cdots + \gamma_m) := \bigcup_{i=1}^{m} \mathrm{IS}_{\overline{\alpha}}(\gamma_i)$, and finally

$$\mathrm{IS}_{\overline{\alpha}}(\psi(\gamma_1, \ldots, \gamma_{N+1})) := \begin{cases} \{\psi(\overline{\gamma})\} & \text{if } (\gamma_1, \ldots, \gamma_N) \geq_{\text{LEX}} (\alpha_1, \ldots, \alpha_N) \\ \bigcup_{i=1}^{N+1} \mathrm{IS}_{\overline{\alpha}}(\gamma_i) & \text{otherwise.} \end{cases}$$

The (relative to $\overline{\alpha}$) *maximal interesting subterm* $\mathrm{MS}_{\overline{\alpha}}(\gamma_1, \ldots, \gamma_n)$ of a non-empty sequence $(\gamma_1, \ldots, \gamma_n)$ is defined as the maximum of the terms occurring in $\mathrm{IS}_{\overline{\alpha}}(\gamma_i)$. Let $>_{\text{LEX}}$ denote the lexicographic ordering on sequences of ordinal terms induced by $>$. Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_N \in T$ and $\beta \in T$. Then set

$$\textsc{Fix}(\overline{\alpha}) := \{\psi(\overline{\gamma}, \delta) \colon \overline{\gamma} >_{\text{LEX}} \overline{\alpha} \text{ and } \psi(\overline{\gamma}, \delta) > \alpha_i \text{ for all } i = 1, \ldots, N\} \quad .$$

For a unary function symbol f we define the n^{th} iteration f^n inductively as (i) $f^0(x) := x$, and (ii) $f^{n+1}(x) := f(f^n(x))$. We will make use of this notation for functions of higher arity by assuming that all but one argument remain fixed. We use $\cdot$ to indicate the free position. In the sequel λ (possibly extended by a subscript) will always denote a limit ordinal term.

Definition 3. *Recursive definition of $\alpha[x]$ for $x < \omega$.*

$$\begin{aligned}
0[x] &:= 0 \\
(\alpha_1 + \cdots + \alpha_m)[x] &:= \alpha_1 + \cdots + \alpha_m[x] \qquad m > 1, \alpha_1 \geq \cdots \geq \alpha_m \\
\psi(\overline{0})[x] &:= 0 \\
\psi(\overline{0}, \beta + 1)[x] &:= \psi(\overline{0}, \beta) \cdot (x + 1) \\
\psi(\overline{0}, \lambda)[x] &:= \psi(\overline{0}, \lambda[x]) \qquad \lambda \notin \textsc{Fix}(\overline{0}) \\
\psi(\overline{0}, \lambda)[x] &:= \lambda \cdot (x + 1) \qquad \lambda \in \textsc{Fix}(\overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, 0)[x] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \overline{0})^{x+1}(0) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \beta + 1)[x] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \overline{0})^{x+1}(\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \beta)) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \lambda)[x] &:= \psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \lambda[x]) \qquad \lambda \notin \textsc{Fix}(\overline{\alpha}, \overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \lambda)[x] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \overline{0})^{x+1}(\lambda) \qquad \lambda \in \textsc{Fix}(\overline{\alpha}, \overline{0}) \\
\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, 0)[x] &:= \psi(\alpha_1, \ldots, \lambda_i[x], \overline{0}, \mathrm{MS}_{\overline{\alpha}, \lambda_i, \overline{0}}(\overline{\alpha}, \lambda_i)) \\
\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \beta + 1)[x] &:= \psi(\alpha_1, \ldots, \lambda_i[x], \overline{0}, \psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \beta)) \\
\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \lambda)[x] &:= \psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \lambda[x]) \qquad \lambda \notin \textsc{Fix}(\overline{\alpha}, \overline{0}) \\
\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, \lambda)[x] &:= \psi(\alpha_1, \ldots, \lambda_i[x], \overline{0}, \lambda) \qquad \lambda \in \textsc{Fix}(\overline{\alpha}, \overline{0})
\end{aligned}$$

The above definition is given in such a way as to simplify the comparison between the fundamental sequences for T and the fundamental sequences for the set of ordinal terms $T(2)$ (built from 0, +, and a 2-ary function symbol ψ) as presented in [18]. Note that our definition is equivalent to the more compact one presented in [11]. The following proposition is stated without proof. A proof (for a slightly different assignment of fundamental sequences) can be found in [4].

Proposition 4. *Let $\alpha \in T$ be given; assume $x < \omega$. If $\alpha > 0$, then $\alpha > \alpha[x]$. For $\alpha > 1$ we get $\alpha[x] > 0$, and if $\alpha \in \mathrm{L{\scriptsize IM}}$, then $\alpha[x+1] > \alpha[x]$. Finally, if $\beta < \alpha \in \mathrm{L{\scriptsize IM}}$, then there exists $x < \omega$, such that $\beta < \alpha[x]$ holds.*

In the definition of $\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, 0)[x]$ we introduce at the last position of ψ the term $\mathrm{MS}_{\overline{a},\overline{0}}(\overline{\alpha})$. We cannot simply dispense of this term. To see this, we alter the definition of the crucial case. We momentarily consider only 3-ary ψ-functions; we set $\Gamma_0 := \psi(1,0,0)$ and calculate $\psi(0,\Gamma_0,0)[x]$:

$$\begin{aligned}\psi(0,\Gamma_0,0)[x] &= \psi(0,\psi(1,0,0)[x],0)\\ &= \psi(0,\psi(0,\cdot,0)^{x+1}(0),0)\\ &= \psi(0,\cdot,0)^{x+2}(0)\\ &< \psi(1,0,0) \quad .\end{aligned}$$

Hence for every $x < \omega$; $\psi(0,\Gamma_0,0)[x] < \Gamma_0$ holds. This contradicts the last assertion of the proposition as $\Gamma_0 < \psi(0,\Gamma_0,0)$. As a side-remark we want to mention that the given assignment of fundamental sequences even fulfills the *Bachmann* property, see [2]. Utilizing Definition 3 we are now in the position to define sub-recursive hierarchies of ordinal functions.

Definition 4. *(The slow-growing hierarchy). Recursive definition of the function $G_\alpha \colon \omega \to \omega$ for $\alpha \in T$.*

$$\begin{aligned}G_0(x) &:= 0\\ G_{\alpha+1}(x) &:= G_\alpha(x) + 1\\ G_\lambda(x) &:= G_{\lambda[x]}(x) \quad .\end{aligned}$$

Definition 5. *(The fast-growing hierarchy.) Recursive definition of the function $F_\alpha \colon \omega \to \omega$ for $\alpha \in T$.*

$$\begin{aligned}F_0(x) &:= x + 1\\ F_{\alpha+1}(x) &:= F_\alpha^{x+1}(x)\\ F_\lambda(x) &:= F_{\lambda[x]}(x) \quad .\end{aligned}$$

It is easy to see that $G_\alpha(x) < F_\alpha(x)$ for all $\alpha > 0$. To see that the name of the hierarchy $\{G_\alpha \colon \alpha \in T\}$ is appropriate, it suffices to calculate some examples. Take e.g. G_ω: $G_\omega(x) = G_{\psi(\overline{0})\cdot(x+1)}(x) = G_{x+1}(x) = G_x(x) + 1 = x + 1$.

Recall that a function f is *elementary* (in a function g) if f is definable explicitely from 0, 1, +, $\dot{-}$ (and g), using bounded sum and product. $E(g)$ denotes the class of all such functions f. Then G_{ϵ_0} majorizes the elementary functions E. In contrast the function F_ω already majorizes the primitive recursive functions, i.e. its growth rate is comparable to the (binary) Ackermann function. Furthermore the class of multiple recursive functions can be characterized by the hierarchy $\{E(F_\gamma)\colon \gamma < \omega^\omega\}$, cf. [12, 13].

However, the following theorem states a (surprising) connection between the slow- and fast-growing hierarchy. See e.g. [8, 6, 18] for further reading on the Hierarchy Comparison Theorem.

Theorem 1. *(The Hierarchy Comparison Theorem.)*

$$\bigcup_{\alpha \in T} E(G_\alpha) = \bigcup_{\gamma < \omega^{N+1}} E(F_\gamma) \quad .$$

Proof. We do not give a detailed proof, but only state the main idea. In [18] the hierarchy comparison theorem has been established for the set of ordinal terms $T(2)$ (built from 0, +, and the function symbol ψ, where $\mathrm{ar}(\psi) = 2$). To extend the result to T it suffices to follow the pattern of the proof in [18].

The difficult direction is to show that every function in the hierarchy $\{F_\gamma\colon \gamma < \omega^{N+1}\}$ is majorized by some G_α. To show this one in particular needs to extend the proofs of Lemma 5 and Theorem 1 in [18] adequately. The reversed direction follows by standard techniques, cf. [6]. □

5 The Interpretation Theorem

For all $\alpha \in T$ there are uniquely determined ordinal terms $\alpha_1 \geq \cdots \geq \alpha_m \in P$ such that $\alpha = \alpha_1 + \cdots + \alpha_m$ holds. In addition, for every $\alpha \in P$ there exist unique $\alpha_1, \ldots, \alpha_{N+1}$ such that $\alpha = \psi(\alpha_1, \ldots, \alpha_{N+1})$. (This normal form property is trivial by definition.) Now assume $\alpha, \beta \in T$ with $\alpha = \gamma_1 + \cdots + \gamma_{m_0}$, $\beta = \gamma_{m_0+1} + \cdots + \gamma_m$. Then the *natural sum* $\alpha \# \beta$ is defined as $\gamma_{\rho(1)} + \cdots + \gamma_{\rho(m)}$, where ρ denotes a permutation on $\{1, \ldots, m\}$ such that $\gamma_{\rho(1)} \geq \cdots \geq \gamma_{\rho(m)}$ holds.

Let R denote a finite rewrite system whose induced rewrite relation is contained in $\succ_{\text{LPO}}$.

Definition 6. *Recursive definition of the interpretation function $\pi\colon \mathcal{T}(\Sigma) \to T$. Let N denote the maximal arity of a function symbol in Σ. If $s = f_j \in \Sigma$, then set $\pi(s) := \psi(j, \overline{0})$. Otherwise, let $s = f_j(s_1, \ldots, s_m)$ and set*

$$\pi(s) := \psi(j, \pi(s_1), \ldots, \pi(s_m) + 1, \overline{0}) \quad .$$

In the sequel of this section we show that π defines an *interpretation* for R on $(T, <)$; i.e. we establish the following theorem.

Theorem 2. *For all $s, t \in \mathcal{T}(\Sigma)$ we have $s \to_R t$ implies $\pi(s) > \pi(t)$.*

Unfortunately this is not strong enough. The problem being that $\alpha > \beta$ implies that G_α majorizes G_β, only. Whereas to proceed with our general program—see Section 1—we need an interpretation theorem for a binary relation $\succ$ on T, such that $\alpha \succ \beta \Rightarrow G_\alpha(x) > G_\beta(x)$ holds for all x. We introduce a notion of a generalized system of fundamental sequences. Based on this generalized notion, it is then possible to define a suitable ordering $\succ$.

Definition 7. *(Generalized system of fundamental sequences for $(T, <)$.) Recursive definition of $(\alpha)^x$ for $x < \omega$.*

1. $(0)^x := \emptyset$
2. *Assume* $\alpha = \alpha_1 + \cdots + \alpha_m$; $m > 1$. *Then* $\beta \in (\alpha)^x$ *if either*
   – $\beta = \alpha_1 \# \cdots \alpha_i^* \cdots \# \alpha_m$ *and* $\alpha_i^* \in (\alpha_i)^x$ *holds, or*
   – $\beta = \alpha_i$.
3. *Assume* $\alpha = \psi(\overline{\alpha})$. *Then* $\beta \in (\alpha)^x$ *if*
   – $\beta = \psi(\alpha_1, \ldots, \alpha_i^*, \ldots, \alpha_{N+1})$, *and* $\alpha_i^* \in (\alpha_i)^x$, *or*
   – $\beta = \alpha_i + x$, *where* $\alpha_i > 0$, *or*
   – $\beta = \psi(\overline{\alpha})[x]$.

By recursion we define the *transitive closure* of the ownership $(\alpha)^x \ni \beta$: $(\alpha >_{(x)} \beta) \leftrightarrow (\exists \gamma \in (\alpha)^x (\gamma >_{(x)} \beta \vee \gamma = \beta))$. Let $\alpha, \beta \in T$. It is easy to verify that $\alpha >_{(x)} \beta$ (for some $x < \omega$) implies $\alpha > \beta$. If no confusion can arise we write α^x instead of $(\alpha)^x$.

Lemma 1. *(Subterm Property) Let $x < \omega$ be arbitrary.*

1. $\alpha <_{(x)} \gamma_1 \# \cdots \alpha \cdots \# \gamma_m$.
2. $\alpha <_{(x)} \psi(\gamma_1, \ldots, \alpha, \ldots, \gamma_{N+1})$.

Proof. The first assertion is trivial. The second assertion follows by the definition of $<_{(x)}$ and assertion 1. □

Lemma 2. *(Monotonicity Property) Let $x < \omega$ be arbitrary.*

1. *If* $\alpha >_{(x)} \beta$, *then* $\gamma_1 \# \cdots \alpha \cdots \# \gamma_m >_{(x)} \gamma_1 \# \cdots \beta \cdots \# \gamma_m$.
2. *If* $\alpha >_{(x)} \beta$, *then* $\psi(\gamma_1, \ldots, \alpha, \ldots, \gamma_{N+1}) >_{(x)} \psi(\gamma_1, \ldots, \beta, \ldots, \gamma_{N+1})$.

Proof. We employ induction on α to prove 1). We write (ih) for induction hypothesis. We may assume that $\alpha > 0$. By definition of $\alpha >_{(x)} \beta$ we either have (i) that there exist $\delta \in \alpha^x$ and $\delta >_{(x)} \beta$ or (ii) $\beta \in \alpha^x$. Firstly, one considers the latter case. Then $(\gamma_1 \# \cdots \beta \cdots \# \gamma_m) \in (\gamma_1 \# \cdots \alpha \cdots \# \gamma_m)^x$ holds by Definition 7. Therefore $(\gamma_1 \# \cdots \beta \cdots \# \gamma_m) <_{(x)} (\gamma_1 \# \cdots \alpha \cdots \# \gamma_m)$ follows. Now, we consider the first case. By assumption $\delta >_{(x)} \beta$ holds, by (ih) this implies $(\gamma_1 \# \cdots \delta \cdots \# \gamma_m) >_{(x)} (\gamma_1 \# \cdots \beta \cdots \# \gamma_m)^x$. Now $(\gamma_1 \# \cdots \alpha \cdots \# \gamma_m) >_{(x)} (\gamma_1 \# \cdots \delta \cdots \# \gamma_m)$ follows by definition of $>_{(x)}$, if we replace β by δ in the proof of the second case. This completely proves 1).

To prove 2) we proceed by induction on α. By definition of $\alpha >_{(x)} \beta$ we have either (i) $\delta \in \alpha^x$ and $\delta >_{(x)} \beta$ or (ii) $\beta \in \alpha^x$. It is sufficient to consider the latter case, the first case follows from the second as above. By Definition 7, $\beta \in \alpha^x$ implies $\psi(\gamma_1, \ldots, \beta, \ldots, \gamma_{N+1}) \in \psi(\gamma_1, \ldots, \alpha, \ldots, \gamma_{N+1})^x$. □

In the sequel we show the existence of a natural number e, such that for all $s, t \in \mathcal{T}$, and any ground substitution ρ, $s \to_R t$ implies $\pi(s\rho) >_{(e)} \pi(t\rho)$. Theorem 2 follows then as a corollary. The proof is involved, and makes use of a sequence of lemmas.

Lemma 3. *Assume $\alpha, \beta \in \textsc{Lim}$; $x \geq 1$. If $\alpha >_{(x)} \beta$, then $\alpha >_{(x+1)} \beta + 1$ holds.*

To prove the lemma we exploit the following auxiliary lemma.

Lemma 4. *We assume the assumptions and notation of Lemma 3; assume Lemma 3 holds for all $\gamma, \delta \in \textsc{Lim}$ with $\gamma, \delta < \alpha$. Then $\alpha >_{(x+1)} \alpha[x+1] \geq_{(x+1)} \alpha[x] + 1$.*

Proof. The lemma follows by induction on the form of α by analyzing all cases of Definition 3. □

Proof. (of Lemma 3) The proof proceeds by induction on the form of α. We consider only the case where $\alpha = \psi(\alpha_1, \ldots, \alpha_{N+1})$. The case where $\alpha = \alpha_1 + \cdots + \alpha_m$ is similar but simpler.

By definition of $\alpha >_{(x)} \beta$ we have either (i) $\gamma \in \alpha^x$ and $\gamma >_{(x)} \beta$ or (ii) $\beta \in \alpha^x$. Assume for $\gamma \in \alpha^x$ we have already shown that $\gamma + 1 <_{(x+1)} \alpha$. Then for $\beta <_{(x)} \gamma$, we conclude by (ih) and the Subterm Property $\beta + 1 <_{(x+1)} \gamma <_{(x+1)} \gamma + 1 <_{(x+1)} \alpha$. Hence, it suffices to consider the second case. We proceed by case distinction on the form of β.

Case $\beta = \psi(\alpha_1, \ldots, \alpha_i^*, \ldots, \alpha_{N+1})$ where $\alpha_i^* \in (\alpha_i)^x$ for some i ($1 \leq i \leq N+1$). Note that $\alpha_i < \alpha$, hence (ih) is applicable to establish $\alpha_i^* + 1 <_{(x+1)} \alpha_i$.

Furthermore by the Subterm Property follows $\alpha_i^* <_{(x+1)} \alpha_i^* + 1$ and therefore $\psi(\alpha_1, \ldots, \alpha_i^*, \ldots, \alpha_{N+1}) <_{(x+1)} \psi(\alpha_1, \ldots, \alpha_i^* + 1, \ldots, \alpha_{N+1})$ holds with Monotonicity. Applying (ih) with respect to $\psi(\alpha_1, \ldots, \alpha_i^* + 1, \ldots, \alpha_{N+1})$ we obtain

$$\begin{aligned}\psi(\alpha_1, \ldots, \alpha_i^*, \ldots, \alpha_{N+1}) + 1 &<_{(x+1)} \psi(\alpha_1, \ldots, \alpha_i^* + 1, \ldots, \alpha_{N+1}) \\ &<_{(x+1)} \psi(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{N+1}) = \alpha \quad .\end{aligned}$$

The last inequality follows again by an application of the Monotonicity Property.

Case $\beta = \alpha_i + x$: Then $(\alpha_i + x) + 1 = \alpha_i + (x+1) <_{(x+1)} \alpha$.

Case $\beta = \psi(\overline{\alpha})[x]$. Clearly $\beta \in \textsc{Lim}$. Then the auxiliary lemma becomes applicable. Thus $\psi(\overline{\alpha})[x] + 1 \leq_{(x+1)} \alpha[x+1] <_{(x+1)} \alpha$. □

Lemma 5. *Let $t \in \mathcal{T}(\Sigma)$ be given. Assume $\tau(t) \leq d$, and $f_j \in \Sigma$. If $f_j \succ_{\text{LPO}} t$, then $\pi(f_j) >_{(2d)} \pi(t)$.*

Proof. We proceed by induction on $\tau(t)$. In the presentation of the argument, we will frequently employ the Subterm and the Monotonicity Property without further notice. Set $\alpha := \pi(f_j)$, and $\beta := \pi(t)$. Furthermore it is a crucial observation that $0 <_{(x)} \alpha$ holds for any $x < \omega$, $\alpha \in T$. (This follows by a simple induction on α.)

Case $\tau(t) = 0$: Then by assumption $t = f_i \in \Sigma$, $i < j$. Hence $i <_{(2d)} j$ holds and we conclude $\pi(t) = \psi(i, \overline{0}) <_{(2d)} \psi(j, \overline{0}) = \pi(f_j)$.

Case $\tau(t) > 0$: Let $t = f_i(t_1, \ldots, t_n)$. Set $\beta_l := \pi(t_l)$ for all $l = 1, \ldots, n$. By (ih) one obtains $\beta_l <_{(2(d-1))} \alpha$ for all l. For all l, we need only consider the case where $\beta_l \in \alpha^{2(d-1)}$. We consider $\psi(j,\overline{0})[2d]$ and apply the following sequence of descents via $>_{(2d)}$:

$$\begin{array}{rcl}\psi(j,\overline{0})[2d] & = & \psi(j-1,\cdot,\overline{0})^{2d+1}(0) \\ & = & \psi(j-1,\psi(j-1,\cdot,\overline{0})^{2d}(0),\overline{0}) \\ & >_{(2d)} & \psi(j-1,\psi(j-1,\cdot,\overline{0})^{2d-1}(0)+1,\overline{0}) \\ & >_{(2d)} & \psi(j-1,\underbrace{\psi(j-1,\cdot,\overline{0})^{2d-1}(0)}_{\psi(j,\overline{0})[2(d-1)]},\cdot,\overline{0})^{2d+1}(0) \quad .\end{array}$$

We define $\gamma_1 := \psi(j,\overline{0})[2(d-1)]$ and $\gamma_{k+1} := \psi(j-1,\gamma_1,\ldots,\gamma_k+1,\overline{0})[2(d-1)]$. By iteration of the above descent, we see

$$\begin{array}{rcl}\alpha[2d] & = & \psi(j,\overline{0})[2d] \\ & >_{(2d)} & \psi(j-1,\gamma_1,\ldots,\gamma_n+1,\overline{0}) \\ & >_{(2d)} & \psi(j-1,\alpha[2(d-1)],\ldots,\alpha[2(d-1)]+1,\overline{0}) \qquad (\delta) \quad .\end{array}$$

Let l $(1 \leq l \leq n)$ be fixed. By assumption we have $\beta_l \in (\alpha)^{2(d-1)}$. We proceed by case distinction on the definition of β_l.

Assume $\beta_l = \psi(j,\overline{0})[2(d-1)]$. Then $\delta = \psi(j-1,\alpha[2(d-1)],\ldots,\beta_l,\ldots,\alpha[2(d-1)]+1,\overline{0})$. Assume $\beta_l = \psi(j^*,\overline{0})$, where $j^* \in (j)^{2(d-1)}$, i.e. $j^* \leq_{(2d)} j-1 <_{(2d)} j$. Therefore $\alpha[2(d-1)] >_{(2d)} \psi(j-1,\overline{0})$. Hence $\delta >_{(2d)} \psi(j-1,\alpha[2(d-1)],\ldots,\beta_l,\ldots,\alpha[2(d-1)]+1,\overline{0})$. Finally assume $\beta_l = j+2(d-1)$. Then $\beta_l <_{(2(d-1)+1)} \psi(j-1,\overline{0}) <_{(2(d-1)+1)} \psi(j-1,\cdot,\overline{0})^{2d-1}(0) = \alpha[2(d-1)]$. Hence $\beta_l <_{(2d)} \alpha[2(d-1)]$ by Lemma 3 and therefore $\delta >_{(2d)} \psi(j-1,\alpha[2(d-1)],\ldots,\beta_l,\ldots,\alpha[2(d-1)]+1,\overline{0})$.

As l was fixed but arbitrary, the above construction is valid for all l. And the lemma follows. □

Lemma 6. *Let $f_i(t_1,\ldots,t_n), f_j(s_1,\ldots,s_m) \in \mathcal{T}(\Sigma)$ be given; let $d > 0$. Then*

1. *If $i < j$, $\pi(f_j(\overline{s})) >_{(2(d-1))} \pi(t_l)$ for all $l = 1,\ldots,n$. Then $\pi(f_j(\overline{s})) >_{(2d)} \pi(f_i(\overline{t}))$ holds.*
2. *If $s_1 = t_1,\ldots,s_{i_0-1} = t_{i_0-1}$, $\pi(s_{i_0}) >_{(2(d-1))} \pi(t_{i_0})$, and $\pi(f_j(\overline{s})) >_{(2(d-1))} \pi(t_l)$, for all $l = i_0+1,\ldots,n$, then $\pi(f_j(\overline{s})) >_{(2d)} \pi(f_i(\overline{t}))$ holds.*

Proof. The proof of assertion 1) is similar to the proof of assertion 2) but simpler. Hence, we concentrate on 2). Set $\alpha := \pi(f_j(\overline{s}))$; $\beta := \pi(f_i(\overline{t}))$; finally set $\alpha_i := \pi(s_i)$ for all $i = 1,\ldots,m$, and $\beta_i := \pi(t:i)$ for all $i = 1,\ldots,n$. As above, we consider only the case where $\beta_l \in (\alpha)^{2(d-1)}$. The other case follows easily.

$$\begin{array}{rcl}\alpha[2d] & = & \psi(j,\alpha_1,\ldots,\alpha_m+1,\overline{0})[2d] \\ & = & \psi(j,\alpha_1,\ldots,\alpha_m,\psi(j,\alpha_1,\ldots,\alpha_m,\cdot,\overline{0})^{2d}(0),\overline{0}) \\ & >_{(2d)} & \psi(j,\alpha_1,\ldots,\alpha_m,\psi(j,\alpha_1,\ldots,\alpha_m,\cdot,\overline{0})^{2d-1}(0)+1,\overline{0}) \\ & = & \psi(j,\alpha_1,\ldots,\alpha_m,\underbrace{\psi(j,\alpha_1,\ldots,\alpha_m+1,\overline{0})[2(d-1)]}_{\alpha[2(d-1)]}+1,\overline{0}) \quad .\end{array}$$

Similar to above, we define $\gamma_1 := \alpha[2(d-1)] = \psi(j, \alpha_1, \dots, \alpha_m + 1, \overline{0})[2(d-1)]$ and $\gamma_{k+1} := \psi(j, \alpha_1, \dots, \alpha_m, \gamma_1, \dots, \gamma_{k+1} + 1, \overline{0})[2(d-1)]$ and obtain

$$\begin{aligned}\alpha[2d] &>_{(2d)} \psi(j, \alpha_1, \dots, \alpha_m, \gamma_1, \dots, \gamma_{N-m} + 1) \\ &>_{(2d)} \psi(j, \alpha_1, \dots, \alpha_m, \alpha[2(d-1)], \dots, \alpha[2(d-1)] + 1) \\ &>_{(2d)} \psi(j, \alpha_1, \dots, \alpha_{i_0}, \overline{0}, \alpha[2(d-1)] + 1) \quad .\end{aligned}$$

By assumption $\beta_{i_0} <_{(2(d-1))} \alpha_{i_0}$ and by Lemma 3 this implies $\beta_{i_0} + 1 <_{(2d)} \alpha_{i_0}$. We set $\overline{\alpha} := \alpha_1, \dots, \alpha_{i_0 - 1}$, then we obtain

$$\begin{aligned}\psi(j, \overline{\alpha}, \alpha_{i_0}, \overline{0}, \alpha[2(d-1)] + 1) &>_{(2d)} \psi(j, \overline{\alpha}, \beta_{i_0} + 1, \overline{0}, \alpha[2(d-1)] + 1) \\ &>_{(2d)} \psi(j, \overline{\alpha}, \beta_{i_0} + 1, \overline{0}, \alpha[2(d-1)] + 1)[2d] \\ &>_{(2d)} \psi(j, \overline{\alpha}, \beta_{i_0}, \psi(j, \overline{\alpha}, \beta_{i_0} + 1, \overline{0}, \alpha[2(d-1)]), \overline{0}) \\ &>_{(2d)} \psi(j, \overline{\alpha}, \beta_{i_0}, \alpha[2(d-1)] + 1), \overline{0}) \\ &= \psi(j, \beta_1, \dots, \beta_{i_0}, \alpha[2(d-1)] + 1), \overline{0}) \quad .\end{aligned}$$

As in the first part of the proof, we obtain $\alpha[2d] >_{(2d)} \psi(j, \overline{\alpha}, \alpha_{i_0}, \overline{0}, \alpha[2(d-1)] + 1) >_{(2d)}$

$$>_{(2d)} \psi(j, \beta_1, \dots, \beta_{i_0}, \alpha[2(d-1)], \dots, \alpha[2(d-1)] + 1, \overline{0}) \quad .$$

By assumption we have $\beta_l <_{(2(d-1))} \alpha$ for all $l = 1, \dots, n$. It remains to prove that this implies $\beta_l \leq_{(2d)} \gamma$. For this it is sufficient to consider the case where $\beta_l \in (\alpha)^{2(d-1)}$. The proof proceeds by case-distinction on the construction of β_l. The proof is similar to the respective part in the proof of Lemma 5, and hence omitted. □

Lemma 7. *Let $s, t \in \mathcal{T}$ be given. Assume $s = f_j(s_1, \dots, s_m)$, ρ is a ground substitution, $\tau(t) \leq d$. Assume further $s_k \succ_{\text{LPO}} u$ and $\tau(u) \leq d$ implies $\pi(s_k\rho) >_{(2d)} \pi(u\rho)$ for all $u \in \mathcal{T}$. Then $s \succ_{\text{LPO}} t$ implies $\pi(s\rho) >_{(2d)} \pi(t\rho)$.*

Proof. The proof is by induction on d.

Case $d = 0$: Hence $\tau(t) = 0$; therefore $t \in \mathcal{V}$ or $t = f_i \in \Sigma$. Consider $t \in \mathcal{V}$. Then t is a subterm of s. Hence there exists k ($1 \leq k \leq m$) s.t. t is subterm of s_k. Hence $s_k \succeq_{\text{LPO}} t$, and by assumption this implies $\pi(s_k\rho) >_{(2d)} \pi(t\rho)$, and therefore $\pi(s\rho) >_{(2d)} \pi(t\rho)$ by the Subterm Property.

Now assume $t = f_i \in \Sigma$. As $s \succ_{\text{LPO}} t$ by assumption either $i < j$ or $s_k \succeq_{\text{LPO}} t$ holds. In the latter case, the assumptions render $\pi(s_k\rho) \geq_{(2d)} \pi(t\rho)$; hence $\pi(s\rho) >_{(2d)} \pi(t\rho)$. Otherwise, $\pi(s\rho) = \psi(j, \pi(s_1\rho), \dots, \pi(s_m\rho) + 1, \overline{0})$, while $\pi(t\rho) = \pi(t) = \psi(i, \overline{0})$. As $\pi(s_k\rho) >_{(x)} 0$ holds for arbitrary $x < \omega$, we conclude $\pi(s\rho) >_{(2d)} \pi(t\rho)$.

Case $d > 0$: Assume $\tau(t) > 0$. (Otherwise, the proof follows the pattern of the case $d = 0$.) Let $t = f_i(t_1, \dots, t_n)$, and clearly $\tau(t_l) \leq (d-1)$ for all $l = 1, \dots, n$. We start with the following observation: Assume there exists i_0 s.t. $s \succ_{\text{LPO}} t_l$ holds for all $l = i_0 + 1, \dots, n$. Then by (ih) we have $\pi(s\rho) >_{(2(d-1))} \pi(t_l\rho)$.

We proceed by case-distinction on $s \succ_{\text{LPO}} t$. Assume firstly there exists k ($1 \leq k \leq m$) s.t. $s_k \succeq_{\text{LPO}} t$. Utilizing the assumptions of the lemma, we conclude $\pi(s\rho) >_{(2d)} \pi(t\rho)$. Now assume $i < j$ and $s \succ_{\text{LPO}} t_l$ for all $l = 1, \ldots, n$. Clearly $s\rho, t\rho \in \mathcal{T}(\Sigma)$. By the observation $\pi(s\rho) >_{(2(d-1))} \pi(t_l\rho)$ holds. Hence Lemma 6.1 becomes applicable and therefore $\pi(s\rho) >_{(2d)} \pi(t\rho)$ holds true. Finally assume $i = j$; $s_1 = t_1, \ldots, s_{i_0-1} = t_{i_0-1}$; $s_{i_0} \succ_{\text{LPO}} t_{i_0}$; $s \succ_{\text{LPO}} t_l$, for all $l = i_0 + 1, \ldots, m$. Utilizing the observation, we see that Lemma 6.2 becomes applicable and therefore $\pi(s\rho) >_{(2d)} \pi(t\rho)$. □

Lemma 8. *Let $t \in \mathcal{T}(\Sigma)$ be given, assume $\tau(t) \leq d$. Then $\psi(K + 1, \overline{0}) >_{(2d)} \pi(t)$.*

Proof. The proof is by induction on $\tau(t)$ and follows the pattern of the proof of Lemma 5. □

Theorem 3. *Let $l, r \in \mathcal{T}$ be given. Assume ρ is a ground substitution, $\tau(t) \leq d$. Then $l \succ_{\text{LPO}} r$ implies $\pi(l\rho) >_{(2d)} \pi(r\rho)$.*

Proof. We proceed by induction on $\tau(s)$.

CASE $\tau(s) = 0$: Then s can either be a constant or a variable. As $s \succ_{\text{LPO}} t$ holds, we can exclude the latter case. Hence assume $s = f_j$. As $f_j \succ_{\text{LPO}} t$, t is closed. Hence the assumptions of the theorem imply the assumptions of Lemma 5 and we conclude $\pi(s\rho) = \pi(s) >_{(2d)} \pi(t) = \pi(t\rho)$.

CASE $\tau(s) > 0$: Then s can be written as $f_j(s_1, \ldots, s_m)$. By (ih) $s_k \succ_{\text{LPO}} u$ and $\tau(u) \leq d$ imply $\pi(s_k\rho) >_{(2d)} \pi(t\rho)$. Therefore the present assumptions contain the assumptions of Lemma 7 and hence $\pi(s\rho) >_{(2d)} \pi(t\rho)$ follows. □

Theorem 4. *(The Interpretation Theorem.) Let R denote a finite rewrite system whose induced rewrite relation is contained in $\succ_{\text{LPO}}$. Then there exists $k < \omega$, such that for all $l, r \in \mathcal{T}$, and any ground substitution ρ $l \to_R r$ implies $\pi(l\rho) >_{(k)} \pi(r\rho)$.*

Proof. Set d equal to $\max\{\tau(r) \colon \exists l\ (l, r) \in R\}$. Then the theorem follows as a corollary to Theorem 3 if k is set to $2d$. □

6 Collapsing Theorem

We define a variant of the slow-growing hierarchy, cf. Definition 4, suitable for our purposes.

Definition 8. *Recursive definition of the function $\widetilde{G}_\alpha \colon \omega \to \omega$ for $\alpha \in T$.*

$$\widetilde{G}_0(x) := 0$$
$$\widetilde{G}_\alpha(x) := \max\{\widetilde{G}_\beta(x) \colon \beta \in (\alpha)^x\} + 1 \quad .$$

Lemma 9. *Let $\alpha \in T$, $\alpha > 0$ be given. Assume $x < \omega$ is arbitrary.*

1. *$\widetilde{G}_\alpha$ is increasing. (Even strictly if $\alpha > \omega$.)*

2. If $\alpha >_{(x)} \beta$, then $\widetilde{G}_\alpha(x) > \widetilde{G}_\beta(x)$.

Proof. Both assertions follow by induction over $<$ on α. □

We need to know that this variant of the slow-growing hierarchy is indeed slow-growing. We show this by verifying that the hierarchies $\{\widetilde{G}_\alpha \colon \alpha \in T\}$ and $\{G_\alpha \colon \alpha \in T\}$ coincide with respect to growth-rate. It is a triviality to verify that there exists $\beta \in T$ such that $\widetilde{G}_\beta$ majorizes G_α. (Simply set $\beta = \alpha$.) The other direction is less trivial. One first proves that for any $\alpha \in T$ there exists $\gamma < \omega^{N+1}$ such that $\widetilde{G}_\alpha(x) \leq F_\gamma(x)$ for almost all x. Secondly one employs the Hierarchy Comparison Theorem once more to establish the existence of $\beta \in T$ such that $\widetilde{G}_\alpha(x) \leq G_\beta(x)$ holds for almost all x.

Theorem 5.

$$\bigcup_{\alpha \in T} E(G_\alpha) = \bigcup_{\alpha \in T} E(\widetilde{G}_\alpha) = \bigcup_{\gamma < \omega^{N+1}} E(F_\gamma) \quad .$$

7 Complexity Bounds

The *complexity* of a terminating finite rewrite system R is measured by the *derivation length* function.

Definition 9. *The derivation length function* $\mathrm{DL}_R \colon \omega \to \omega$. *Let $m < \omega$ be given.* $\mathrm{DL}_R(m) := \max\{n \colon \exists t_1, \ldots, t_n \in \mathcal{T} \;\; ((t_1 \to_R \cdots \to_R t_n) \wedge (\tau(t_1) \leq m))\}$.

Let R be a rewrite system over $\mathcal{T}$ such that $\to_R$ is contained in a lexicographic path ordering. Now assume that there exist $s = t_0, t_1, \ldots, t_n \in \mathcal{T}$ with $\tau(s) \leq m$ such that

$$s \to_R t_1 \to_R \cdots \to_R t_n$$

holds. By our choice of R this implies $s \succ_{\text{LPO}} t_1 \succ_{\text{LPO}} \cdots \succ_{\text{LPO}} t_n$. By assumption on Σ there exists $c \in \Sigma$, with $\mathrm{ar}(c) = 0$. We define a ground substitution ρ: $\rho(x) = c$, for all $x \in \mathcal{V}$. Let $k < \omega$ be defined as in Theorem 4. Recall that K denotes the cardinality of Σ. We conclude from the Interpretation Theorem and Lemma 8, $\pi(s\rho) >_{(k)} \pi(t_1\rho) >_{(k)} \cdots >_{(k)} \pi(t_n\rho)$ and $\psi(K+1, \overline{0}) >_{(2m)} \pi(s\rho)$. Setting $h := \max\{2m, k\}$ and utilizing Lemma 3, we obtain $\psi(K+1, \overline{0}) >_{(h)} \pi(s\rho) >_{(h)} \cdots >_{(h)} \pi(t_n\rho)$. An application of Lemma 9.2 yields

$$\widetilde{G}_{\psi(K+1,\overline{0})}(h) > \widetilde{G}_{\pi(s\rho)}(h) > \cdots > \widetilde{G}_{\pi(t_n\rho)}(h) \quad .$$

Employing Theorem 5 we conclude the existence of $\gamma < \omega^\omega$, such that

$$F_\gamma(\max\{2m, k\}) \geq \widetilde{G}_{\psi(K+1,\overline{0})}(\max\{2m, k\}) \geq \mathrm{DL}_R(m) \quad .$$

The class of multiply-recursive functions is captured by $\bigcup_{\gamma < \omega^\omega} E(F_\gamma)$, see [13]). Thus we have established a multiply-recursive upper bound for the derivation length of R if $\to_R$ is contained in a lexicographic path ordering. Furthermore, this bound is essentially optimal, cf. [17].

8 Conclusion

The presented proof method is generally applicable. Let R denote a rewrite system whose termination can be shown via $\succ_{\text{MPO}}$. To yield a primitive recursive upper bound for the complexity of R the above proof can be employed. Firstly the definition of the interpretation function π has to be changed as follows. If $s = f_j(s_1, \ldots, s_m)$, then we set

$$\pi(s) := \psi(j, \pi(s_1)\# \cdots \#\pi(s_m)\#1) \quad .$$

Then the presented proof needs only partial changes. It suffices to reformulate (and reprove) Lemma 5, 6, 7, and 8, respectively.

Future work will be concerned with the *Knuth-Bendix ordering*. Due to the more complicated nature of this ordering the statement of the interpretation is not so simple. Still we believe that only mild alterations of the given proof are necessary.

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