Interactive Theorem Proving

Week 3

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Summary

So far

Proof Assistants (history, foundations, basic properties)
HOL Light (kernel, connectives, rules)

- $\lambda_{\rightarrow}$, STT
- à la Church, à la Curry
- Type Assignment
- Curry-Howard Isomorphism and example derivations

Today

- Gentzen Style Natural Deduction
- Properties of $\lambda_{\rightarrow}$
- Principal Types
- TCP, TSP, TIP
- Lambda Cube

Gentzen style natural deduction

assumption

$$\frac{\vdots}{A} \quad \rightarrow \quad [A]^{\mathtt{H}}$$

conjunction introduction

$$\frac{\vdots}{A \wedge B} \quad \rightarrow \quad \dfrac{\dfrac{\vdots}{A} \qquad \dfrac{\vdots}{B}}{A \wedge B}\,\wedge\mathrm{i}$$

Gentzen style natural deduction

conjunction elimination left

$$\begin{array}{c}\vdots\\ \hline A\end{array} \quad \color{red}{\rightarrow} \quad \dfrac{\begin{array}{c}\vdots\\ \hline A \wedge B\end{array}}{A}\wedge\mathsf{e}_1$$

conjunction elimination right

$$\begin{array}{c}\vdots\\ \hline B\end{array} \quad \color{red}{\rightarrow} \quad \dfrac{\begin{array}{c}\vdots\\ \hline A \wedge B\end{array}}{B}\wedge\mathsf{e}_2$$

Gentzen style natural deduction

disjunction introduction left

$$\frac{\vdots}{A \lor B} \quad \rightarrow \quad \frac{\dfrac{\vdots}{A}}{A \lor B}\ \lor\mathrm{i}_1$$

disjunction introduction right

$$\frac{\vdots}{A \lor B} \quad \rightarrow \quad \frac{\dfrac{\vdots}{B}}{A \lor B}\ \lor\mathrm{i}_2$$

Gentzen style natural deduction

disjunction elimination

$$\dfrac{\vdots}{C} \quad \rightarrow \quad \dfrac{\dfrac{\vdots}{A \lor B} \qquad \dfrac{\begin{array}{c}[A]^{\mathtt{H1}}\\ \vdots\end{array}}{C} \qquad \dfrac{\begin{array}{c}[B]^{\mathtt{H2}}\\ \vdots\end{array}}{C}}{C}\ \lor\mathrm{e}\ [\mathtt{H1},\mathtt{H2}]$$

implication introduction

$$\dfrac{\vdots}{A \rightarrow B} \quad \rightarrow \quad \dfrac{\dfrac{\begin{array}{c}[A]^{\mathtt{H}}\\ \vdots\end{array}}{B}}{A \rightarrow B}\ \rightarrow\mathrm{i}\ [\mathtt{H}]$$

Gentzen style natural deduction

implication elimination

$$\dfrac{\vdots}{B} \quad \color{red}{\longrightarrow} \quad \dfrac{\dfrac{\vdots}{A \to B} \qquad \dfrac{\vdots}{A}}{B}\,{\to}\mathrm{e}$$

negation introduction

$$\dfrac{\vdots}{\neg A} \quad \color{red}{\longrightarrow} \quad \dfrac{\dfrac{\begin{matrix}[A]^{\mathtt{H}} \\ \vdots\end{matrix}}{\bot}}{\neg A}\,\neg\mathrm{i}\ [\mathtt{H}]$$

Gentzen style natural deduction

negation elimination

$$\begin{array}{c}\vdots\\ \hline \bot\end{array} \quad \rightarrow \quad \cfrac{\begin{array}{c}\vdots\\ \hline \neg A\end{array} \qquad \begin{array}{c}\vdots\\ \hline A\end{array}}{\bot}\ \neg\mathrm{e}$$

bottom elimination

$$\begin{array}{c}\vdots\\ \hline A\end{array} \quad \rightarrow \quad \cfrac{\begin{array}{c}\vdots\\ \hline \bot\end{array}}{A}\ \bot\mathrm{e}$$

Gentzen style natural deduction

universal introduction

$$\dfrac{\vdots}{\forall x\, A} \quad \textcolor{red}{\rightarrow} \quad \dfrac{\begin{array}{c}\vdots\\ \hline A[y/x]\end{array}}{\forall x\, A}\ \forall\text{i}$$

universal elimination

$$\dfrac{\vdots}{A[t/x]} \quad \textcolor{red}{\rightarrow} \quad \dfrac{\begin{array}{c}\vdots\\ \hline \forall x\, A\end{array}}{A[t/x]}\ \forall\text{e}$$

Gentzen style natural deduction

existential introduction

$$\frac{\vdots}{\exists x\, A} \quad \rightarrow \quad \dfrac{\dfrac{\vdots}{A[t/x]}}{\exists x\, A}\,\exists\mathrm{i}$$

existential elimination

$$\frac{\vdots}{B} \quad \rightarrow \quad \dfrac{\dfrac{\vdots}{\exists x\, A} \qquad \dfrac{\begin{matrix}[A[y/x]]^{\mathtt{H}} \\ \vdots\end{matrix}}{B}}{B}\,\exists\mathrm{e}\,[\mathtt{H}]$$

Example Derivation

$$
\dfrac{\dfrac{[\exists x(P(x) \lor \neg Q(a))]^{\mathtt{H1}} \qquad \dfrac{[P(b) \lor \neg Q(a)]^{\mathtt{H3}} \qquad \dfrac{[P(b)]^{\mathtt{H4}}}{\exists x\, P(x)}\,\exists\text{i} \qquad \dfrac{\dfrac{[\neg Q(a)]^{\mathtt{H5}} \quad [Q(a)]^{\mathtt{H2}}}{\bot}\,\neg\text{e}}{\exists x\, P(x)}\,\bot\text{e}}{\exists x\, P(x)}\,\lor\text{e}\ [\mathtt{H4,H5}]}{\dfrac{\exists x\, P(x)}{Q(a) \to \exists x\, P(x)}\,\to\text{i}\ [\mathtt{H2}]}\,\exists\text{e}\ [\mathtt{H3}]}{\exists x(P(x) \lor \neg Q(a)) \to Q(a) \to \exists x\, P(x)}\,\to\text{i}\ [\mathtt{H1}]
$$

Corresponding Box-style Proof

1		$\exists x(P(x) \vee \neg Q(a))$	assumption
2		$Q(a)$	assumption
3	b	$P(b) \vee \neg Q(a)$	assumption
4		$P(b)$	assumption
5		$\exists x\, P(x)$	$\exists$i 4
6		$\neg Q(a)$	assumption
7		$\bot$	$\neg$e 6,2
8		$\exists x\, P(x)$	$\bot$e 7
9		$\exists x\, P(x)$	$\vee$e 3,4—5,6—8
10		$\exists x\, P(x)$	$\exists$e 1,3—9
11		$Q(a) \rightarrow \exists x\, P(x)$	$\rightarrow$i 2—10
12		$\exists x(P(x) \vee \neg Q(a)) \rightarrow Q(a) \rightarrow \exists x\, P(x)$	$\rightarrow$i 1—11

Properties of $\lambda_{\rightarrow}$

- Uniqueness of Types

$$\text{If } \Gamma \vdash M : \sigma \text{ and } \Gamma \vdash M : \tau \text{ , then } \sigma = \tau.$$

- Subject Reduction

$$\text{If } \Gamma \vdash M : \sigma \text{ and } M \rightarrow_{\beta\eta} N \text{ , then } \Gamma \vdash N : \sigma.$$

- Substitution Property

$$\text{If } \Gamma, x : \tau, \Delta \vdash M : \sigma, \Gamma \vdash P : \tau, \text{ then } \Gamma, \Delta \vdash M[x := P] : \sigma.$$

- Thinning

$$\text{If } \Gamma \vdash M : \sigma \text{ and } \Gamma \subset \Delta, \text{ then } \Delta \vdash M : \sigma.$$

- Strengthening

$$\text{If } \Gamma, x : \tau \vdash M : \sigma \text{ and } x \notin FV(M), \text{ then } \Gamma \vdash M : \sigma.$$

- Strong Normalization

$$\text{If } \Gamma \vdash M : \sigma, \text{ then all } \beta\eta\text{-reductions from } M \text{ terminate.}$$

Consequences

- Subterm property
- Condensing: $\Gamma|_{FV(M)}$
- Permutation
- No self application
- β-normal forms
- Some terms do not have fixed points

Intuitionistic Logic

Drawbacks of classical logic

- There are $x \notin \mathbb{Q}$ and $y \notin \mathbb{Q}$ st. $x^y \in \mathbb{Q}$.
  - Proof: by cases $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$
- There are seven 7s in a row in the decimal representation of π.

Brouwer, beginning of 20th century

Intuitionistic logic developed later around 1930

- $A \to \neg\neg A$ has an intuitionistic interpretation
- but $\neg\neg A \to A$ does not

Easier correspondence to $\lambda_?$-calculi

Constructive proofs have computational content

Brouwer-Heyting-Kolmogorov interpretation

Proof of $A \to B$

Function that maps proofs of A to proofs B

Proof of $A \wedge B$

Pair of proofs of A and B

Proof of $A \vee B$

Either a proof of A or a proof of B

Proof of $\forall x.P(x)$

Function that maps an object x to a proof of $P(x)$

Proof of $\bot$

Does not exist.Negation of A turns a proof of A into a nonexistant object

Summary

Today

- Gentzen Style Natural Deduction
- Properties of $\lambda \rightarrow$
- Intuitionistic Logic

Next time

- Principal Types
- TCP, TSP, TIP
- Lambda Cube
- Polymorphism
- HOL Light subgoal package