Computational Logic

Interactive Theorem Proving

Week 4

Cezary Kaliszyk

Mar 24, 2015

Summary

So far

Proof Assistants, HOL Light, $\lambda_{\rightarrow}$

- Gentzen Style Natural Deduction
- Intuitionistic Logic
- Properties of $\lambda_{\rightarrow}$
- BHK interpretation

Today

- Principal types
- λ-cube
- Dependent types

Definitions

Definition: type substitution

A map from type variables to types

Definition: unifier

For two given types σ and τ their unifier, is such a type substitution s that $s(\sigma) = s(\tau)$

Definition: mgu (most general unifier)

Given types σ and τ their mgu is a type substitution s, such that:

- $s(\sigma) = s(\tau)$
- $\forall t.t(\sigma) = t(\tau) \rightarrow \exists r.t = r \circ s$

Definitions

Definition: type substitution

A map from type variables to types

Definition: unifier

For two given types σ and τ their unifier, is such a type substitution s that $s(\sigma) = s(\tau)$

Definition: mgu (most general unifier)

Given types σ and τ their mgu is a type substitution s, such that:

- $s(\sigma) = s(\tau)$
- $\forall t.t(\sigma) = t(\tau) \rightarrow \exists r.t = r \circ s$

The above notions generalize to lists of types

Algorithm computing mgu

In $\lambda_{\to}$ only one function

Input: $\sigma_1, \ldots, \sigma_n$, output: mgu or "not unifiable".

$$\frac{E_1; g(\tau_1, \ldots, \tau_n) \approx g(\tau'_1, \ldots, \tau'_n); E_2}{E_1; \tau_1 \approx \tau'_1; \ldots; \tau_n \approx \tau'_n; E_2}\, d_1$$

$$\frac{E_1; \tau_1 \to \tau_2 \approx \tau'_1 \to \tau'_2; E_2}{E_1; \tau_1 \approx \tau'_1; \tau_2 \approx \tau'_2; E_2}\, d_2$$

$$\frac{E_1; \alpha \approx \tau; E_2 \quad \alpha \notin V(\tau)}{(E_1; E_2)\{\alpha/\tau\}}\, v_1$$

$$\frac{E_1; \tau \approx \alpha; E_2 \quad \alpha \notin V(\tau)}{(E_1; E_2)\{\alpha/\tau\}}\, v_2$$

$$\frac{E_1; \tau \approx \tau; E_2}{E_1; E_2}\, t$$

Principal Types

Definition: Principal type

σ is a principal type for an untyped λ-term M if:

- $M : \sigma$ in STT à la Curry
- $\forall \tau, M : \tau \rightarrow \exists s.\tau = s(\sigma)$

Principal Types: example

$$\lambda x^{\alpha}.\lambda y^{\beta}.y^{\beta}(\lambda z^{\gamma}.y^{\beta}x^{\alpha})$$

1. Assign type variables to all variables: $x : \alpha, y : \beta, z : \gamma$.
2. Assign type variables to all applicative subterms: $y\,x : \delta$, $y(\lambda z.y\,x) : \epsilon$.
3. Generate equations between types, necessary for the term to be typable:
$$\beta = \alpha \to \delta \quad \beta = (\gamma \to \delta) \to \epsilon$$
4. Find a most general unifier that solves the above equations:
$$\alpha := \gamma \to \delta, \beta := (\gamma \to \delta) \to \epsilon, \delta := \epsilon$$
5. The principal type of $\lambda x.\lambda y.y(\lambda z.xy)$ is now:
$$(\gamma \to \epsilon) \to ((\gamma \to \epsilon) \to \epsilon) \to \epsilon$$

Typical questions in Type Theory

TCP (type checking problem)

$M : \sigma$?

TSP (type synthesis problem) (variant of well-typedness)

$M :$?

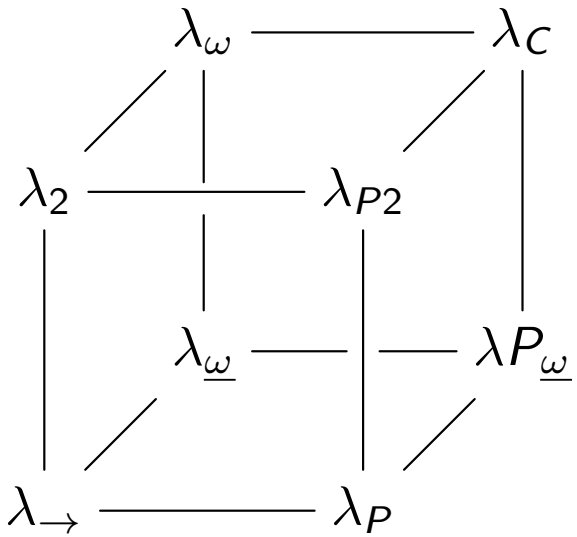
TIP (type inhabitation problem) (variants: closed term, $\varnothing$)

$? : \sigma$

- For $\lambda_{\to}$ all are decidable
  - both in Curry and in Church style
- TCP and TSP are usually equivalent
  - application typing rule is to blame
- TCP and TSP quickly become undecidable in Curry style
  - TIP corresponds to provability in some logic

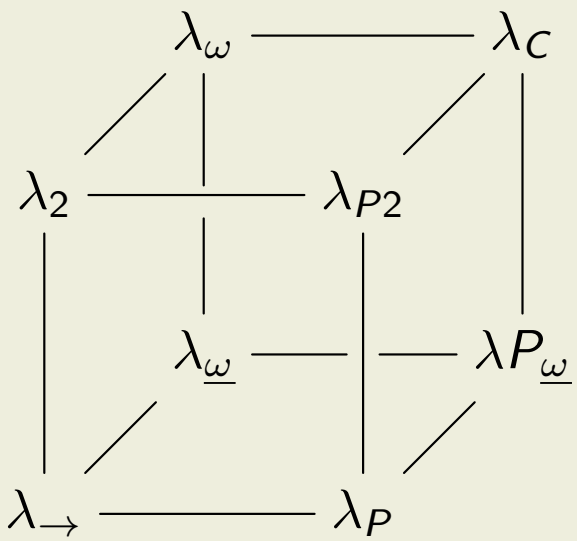
Lambda cube (Barendregt, 1991)

propositional logic	$\longleftrightarrow$	$\lambda_{\rightarrow}$
predicate logic	$\longleftrightarrow$	λ_P (dependent types)
2nd order propositional logic	$\longleftrightarrow$	λ_2, System F (2nd order typed λ-calc)
		$\lambda_{\underline{\omega}}$ (type operators)



Lambda cube (again)

Four kinds of dependencies



- terms on terms (already in $\lambda_\rightarrow$)
- dependent types (λ_P)
- polymorphism (λ_2)
- terms depend on types
- Combining the three is hard! (Girard's paradox)

Dependent types vs Polymorphism vs CoC

Printf

What type does it have?

Bit-strings of length n

- Type of bit-strings: $bs : \mathbb{N} \to \star$
- Bit-string made of zeros: $0_{bs} : (\forall n : \mathbb{N}) bs(n)$
- $\mathbb{R}^{\mathbb{N}}$

Vectors (later polymorphic)

- Type of hd?

Constructive Division

$$a/b/\!\!/P \qquad \approx \qquad \frac{a}{b \neq 0} \quad .$$

Summary

Today

- Principal types
- λ-cube
- Dependent types

Next time

- λ_P
- $\lambda_{\underline{\omega}}$, polymorphism
- More advanced features in HOL-Light