# Logic Programming

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# Summary of Last Lecture

## Example (Implementing same_vars)

```
same_var(foo,Y) ← var(Y), !, fail.
same_var(X,Y) ← var(X), var(Y).
```

## Example (Bad Cut)

```
minimum(X,Y,X) ← X ⩽ Y, !.        ← minimum(2,5,5)
minimum(X,Y,Y).                   true
```

## Types of Red Cuts

1. cuts that are built-in (e.g. in the implementation of negation)
2. green cuts that become red, when conditions are fulfilled
3. supposedly green cut that changes the behaviour of the program

# Outline of the Lecture

## Logic Programs

introduction, basic constructs, database and recursive programming, theory of logic programs

## The Prolog Language

programming in pure prolog, arithmetic, structure inspection, meta-logical predicates, cuts, extra-logical predicates, how to program efficiently

## Advanced Prolog Programming Techniques

nondeterministic programming, incomplete data structures, definite clause grammars, meta-programming, constraint logic programming

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# Program Access and Manipulation

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## clause database operations

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  ```
  ← assert(C).
  true
  ```

- side effect: add rule $C$ to program
- *retract/1*

  ```
  ← retract(C).
  false
  ```

- side effect: remove first rule from program that unifies with $C$

## Example (Fibonacci Numbers Revisited)

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:- dynamic(fibonacci/2).
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```
:- dynamic(fibonacci/2).

fibonacci(0,0).
fibonacci(1,1).
fibonacci(N,X) :-
    N > 1,
    N1 is N-1, fibonacci(N1,Y),
    N2 is N-2, fibonacci(N2,Z),
    X is Y+Z,
    assert(fibonacci(N,X)),
    !.
```

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    N1 is N-1, fibonacci(N1,Y),
    N2 is N-2, fibonacci(N2,Z),
    X is Y+Z,
    asserta(fibonacci(N,X)),
    !.
```

## Example

```
edit :- edit(file([],[])).

edit(File) :-
  read(Command),
  edit(File,Command).

edit(File,exit) :- !.
edit(File,Command) :-
  apply(Command,File,File1),
  !,
  edit(File1).
edit(File,Command) :-
  write(Command),
  write(' is not applicable'),
  !,
  edit(File).
```

```
apply(up,file([X|Xs],Ys),
    file(Xs,[X|Ys])).
apply(down,file(Xs,[Y|Ys]),
    file([Y|Xs],Ys)).
apply(insert(Line), file(Xs,Ys),
    file(Xs,[Line|Ys])).
apply(delete,file(Xs,[Y|Ys]),
    file(Xs,Ys)).
apply(print,file([X|Xs],Ys),
    file([X|Xs],Ys)) :-
  write(X), nl.
apply(print(*),file(Xs,Ys),
    file(Xs,Ys)) :-
  reverse(Xs,Xs1),
  write_file(Xs1),
  write_file(Ys).
```

# Operator and Precedences

## Query Operator

```
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P ↦ 400,
A ↦ yfx
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```
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```

```
:- 1*2*3 = 1*(2*3).
false
```

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```

## Define Operator

```
:- op(350, xfy, new).
:- X = *(new(1,*(2,3)),*(4,new(new(4,5),*(6,new(7,8))))).
X ↦ 1 new (2*3) * (4* (4 new 5) new (6*7 new 8))
```

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X ↦ (1 new 2*3)* (4* (4 new 5 new 6* (7 new 8)))
```

## Definition

- if *op*(*Precdence*, *Associativity*, *Name*) is used in program, then it has to be added with `:-`

```
:- op(350,xfy,new)
```

- if in a program `:- query` occurs, then `query` is directly executed when the program is loaded
- precedence: positive number, smaller numbers bind stronger
- five modes of associativity
  - xfy: right-associative, `X o Y o Z = X o (Y o Z)`
  - yfx: left-associative, `X o Y o Z = (X o Y) o Z`
  - xfx: non-associative, `X o Y o Z` will not be parsed
  - fy: prefix-operator, `o X`
  - yf: postfix-operator, `X o`

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## Example

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sublist(Xs,AXBs) :- suffix(XBs,AXBs), prefix(Xs,XBs).
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What is better?

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the first alternative:

- consider

```
sublist([1,2,3,4],[1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4])
```

- the 1st clause iterates over the 2nd list to find a suitable suffix
- then iterates over the first list
- no intermediate data structures are created

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```

- the 1st clause iterates over the 2nd list to find a suitable suffix
- then iterates over the first list
- no intermediate data structures are created
- in the 2nd clause an auxilliary list is created

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- equivalently the number of unifications (performed and attempted) asymptotically bounds the runtime
- on the other hand, if unification needs to be taken into account time complexity analysis is more involved
- in general size of search space and size of input terms needs to be taken into account

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reverse([X|Xs],Zs) :-
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    append(Ys,[X],Zs).
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```

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```
reverse(Xs,Ys) :- reverse(Xs,[],Ys).

reverse([X|Xs],Acc,Ys) :-
    reverse(Xs,[X|Acc],Ys).
reverse([],Ys,Ys).
```

## Suggestion ②

tuning, via:

1. good goal order
2. elimination of (unwanted) nondeterminism by using explicit conditions and cuts
3. exploit clause indexing (order arguments suitably)
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*By default, SWI-Prolog, as most other implementations, indexes predicates on their first argument.*

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- consider a generic clause for $A$

$$A' \leftarrow B_1, \dots, B_n$$

  such that $A$ and $A'$ unify with $\sigma$
- suppose the goal $B_1\sigma, \dots, B_{n-1}\sigma$ is deterministic
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## Definition

clause indexing is used to detect which clauses are applicable for reduction: 2nd clause in append need not be considered

# Howto Implement Functions

## Functions vs Relations

- often, we want to compute functions:
  1. addition: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$
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- that is, we implement functions $f(i_1, \ldots, i_n) = (o_1, \ldots, o_m)$ by relations $f_{rel}/(n+m)$
- result is obtained by query $f_{rel}(i_1, \ldots, i_n, X_1, \ldots, X_m)$
  1. addition: $plus(n, m, Z)$ — $Z = n + m$
  2. sorting: $sort(list, Xs)$ — $Xs =$ sorted version of $list$

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    does not work
- solution: store result of each sub-expression in fresh variable

```
f(X, ) :- times(X,X,Z),
```

$$\underbrace{\underbrace{x^2}_{z} + \underbrace{7 \cdot (\underbrace{\underbrace{x^2}_{z} - 5)}_{v}}_{u}}_{f(x)=y}$$

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plus(Z,U,Y).
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### Example (Ackermann function in Haskell)

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ack 0 m = m + 1
ack (n+1) m = if m == 0 then ack n 1 else
          ack n (ack (n+1) (m-1))
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### Example (Ackermann function as logic program)

```
ack(0,M,s(M)).
ack(s(N),M,R) :- =(M,0,B), cond(B,N,M,R).
cond(true,N,M,R) :- ack(N,s(0),R).
cond(false,N,M,R) :- -(M,s(0),U),ack(s(N),U,V),ack(N,V,R).
```

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  can be programmed as

```
f(X,Y) :- eval(s(X*X) - X*X, Y).
```

- evaluator is simple logic program

```
eval(0,0).
eval(s(E),s(N)) :- eval(E,N).
eval(E+F,K) :- eval(E,N), eval(F,M), plus(N,M,K).
eval(E-F,K) :- eval(E,N), eval(F,M), plus(M,K,N).
eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
```

## Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

`f(s(s(0)),Y)`

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```
eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                    |
              f(s(s(0)),Y)
```

## Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                              |
        eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                              |
                         f(s(s(0)),Y)
```

## Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                          N = s(N1) |
eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                    |
      eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                    |
                             f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(s(X*X) - X*X, Y).)

```
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                             |
          eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                 N = s(N1)   |
          eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                             |
                  eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                             |
                                        f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(s(X*X) - X*X, Y).)

```
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                              N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                         |
             eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                               N = s(N1) |
              eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                                         |
                        eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                                         |
                                                   f(s(s(0)),Y)
```

## Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                              |
          eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                    N = s(N1) |
           eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                              |
                  eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                              |
                                       f(s(s(0)),Y)
```

# Example (f(X,Y) :- eval(s(X*X) - X*X, Y).)

```
eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   |
eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   N = s(N1) |
eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                   |
eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                   |
f(s(s(0)),Y)
```

# Example (f(X,Y) :- eval(s(X*X) - X*X, Y).)

```
        times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                N3 = s(s(0)) ||
    eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                     N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                  N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                  N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                             |
        eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   N = s(N1) |
         eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                             |
                eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                             |
                                      f(s(s(0)),Y)
```

# Example (f(X,Y) :- eval(s(X*X) - X*X, Y).)

```
                  eval(s(s(0))*s(s(0)),M), plus(M,Y,s(s(s(s(0)))))
                                  N1 = s(s(s(s(0)))) ‖
          times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                      N3 = s(s(0)) ‖
     eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                          N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                       N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                       N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                  |
          eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N = s(N1) |
           eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                                  |
                    eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                                  |
                                          f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(s(X*X) - X*X, Y).)

```
                 plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                            M = s(s(s(s(0)))) ‖
         eval(s(s(0))*s(s(0)),M), plus(M,Y,s(s(s(s(s(0))))))
                           N1 = s(s(s(s(0)))) ‖
 times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                N3 = s(s(0)) ‖
 eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                      N5 = 0 |
 eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                  N4 = s(N5) |
 eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                  N2 = s(N4) |
 eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                             |
     eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                   N = s(N1) |
       eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                             |
             eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                             |
                                  f(s(s(0)),Y)
```

## Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
                                   □
                        Y = s(0) ‖
           plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                  M = s(s(s(s(0)))) ‖
      eval(s(s(0))*s(s(0)),M), plus(M,Y,s(s(s(s(s(0))))))
                 N1 = s(s(s(s(0)))) ‖
times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                       N3 = s(s(0)) ‖
eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                             N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                          N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                          N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                     |
  eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                           N = s(N1) |
  eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                     |
          eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                     |
                             f(s(s(0)),Y)
```

# Example (f(X,Y) :- eval(s(X*X) - X*X, Y).)

```
                                   □
                       Y = s(0) ‖
              plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                 M = s(s(s(s(0)))) ‖
        eval(s(s(0))*s(s(0)),M), plus(M,Y,s(s(s(s(s(0))))))
                N1 = s(s(s(s(0)))) ‖
  times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                       N3 = s(s(0)) ‖
  eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                           N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                       N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                       N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                  |
   eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                        N = s(N1) |
   eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                  |
          eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                  |
                            f(s(s(0)),Y)
```

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- consider sub-expression `X*X`

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- `let(X,E,F)` encodes *let* $x = e$ *in* $f$

```
eval(0,0).
eval(s(E),s(N)) :- eval(E,N).
eval(E+F,K) :- eval(E,N), eval(F,M), plus(N,M,K).
eval(E-F,K) :- eval(E,N), eval(F,M), plus(M,K,N).
eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
eval(let(X,E,F),K) :- eval(E,N), X = N, eval(F,K).
```

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eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
eval(let(X,E,F),K) :- eval(E,N), X = N, eval(F,K).
```

## Example

```
f(X,Y) :- eval(s(X*X) - X*X, Y).
f(X,Y) :- eval(let(X2, X*X, s(X2) - X2), Y).
```

## Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

`f(s(s(0)),Y)`

## Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                    |
             f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).)

```
eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                         |
     eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                         |
                    f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).)

```
        X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
                  N = s(s(s(s(0)))) ‖
eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                  |
    eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                  |
                       f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).)

```
        eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
               X2 = s(s(s(s(0)))) |
         X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
                N = s(s(s(s(0)))) ‖
eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                  |
     eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                  |
                         f(s(s(0)),Y)
```

## Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
eval(s(s(s(s(s(0))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                              |
        eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
                 X2 = s(s(s(s(0)))) |
         X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
                  N = s(s(s(s(0)))) ||
     eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                              |
       eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                              |
                         f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).)

```
     eval(s(s(s(s(0)))),M), plus(M,Y,s(s(s(s(s(0))))))
                    N = s(s(s(s(s(0))))) ‖
eval(s(s(s(s(s(0))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                                  |
          eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
                     X2 = s(s(s(s(0)))) |
           X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
                      N = s(s(s(s(0)))) ‖
     eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                  |
         eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                  |
                            f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).)

```
                    plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                            M = s(s(s(s(0)))) ‖
          eval(s(s(s(s(0)))),M), plus(M,Y,s(s(s(s(s(0))))))
                         N = s(s(s(s(s(0))))) ‖
eval(s(s(s(s(s(0))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                                        |
                    eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
                           X2 = s(s(s(s(0)))) |
                     X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
                            N = s(s(s(s(0)))) ‖
          eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                        |
                   eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                        |
                                 f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).)

```
                              □
                   Y = s(0) ‖
plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
          M = s(s(s(s(0)))) ‖
eval(s(s(s(s(0)))),M), plus(M,Y,s(s(s(s(s(0))))))
        N = s(s(s(s(s(0))))) ‖
eval(s(s(s(s(s(0))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                              |
eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
         X2 = s(s(s(s(0)))) |
X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
          N = s(s(s(s(0)))) ‖
eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                              |
eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                              |
                        f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).)

```
                              □
                   Y = s(0) ‖
      plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
             M = s(s(s(s(0)))) ‖
   eval(s(s(s(s(0)))),M), plus(M,Y,s(s(s(s(s(0))))))
            N = s(s(s(s(s(0))))) ‖
eval(s(s(s(s(s(0))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                              |
      eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
            X2 = s(s(s(s(0)))) |
       X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
             N = s(s(s(s(0)))) ‖
   eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                              |
     eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                              |
                        f(s(s(0)),Y)
```

## Example (f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).)

```
                                □
                     Y = s(0) ‖
        plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
              M = s(s(s(s(0)))) ‖
   eval(s(s(s(s(0)))),M), plus(M,Y,s(s(s(s(s(0))))))
           N = s(s(s(s(s(0))))) ‖
eval(s(s(s(s(s(0))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                                |
        eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
             X2 = s(s(s(s(0)))) |
         X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
              N = s(s(s(s(0)))) ‖
   eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                |
       eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                |
                          f(s(s(0)),Y)
```

## Speeding up “let” even further

- detected problems:
  1. after computing $x^2$, result is evaluated again
     `eval(s(s(s(s(0)))),M)`
  2. eval also steps into initial input

## Speeding up "let" even further

- detected problems:
  1. after computing $x^2$, result is evaluated again `eval(s(s(s(s(0)))),M)`
  2. eval also steps into initial input
- solution: add new constructor *num* which states that the argument is a number, and hence, does not have to be evaluated

```
eval(0,0).
eval(s(E),s(N)) :- eval(E,N).
eval(E+F,K) :- eval(E,N), eval(F,M), plus(N,M,K).
eval(E-F,K) :- eval(E,N), eval(F,M), plus(M,K,N).
eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
eval(num(N),N).
eval(let(X,E,F),K) :- eval(E,N),X = num(N), eval(F,K).
```

## Speeding up "let" even further

- detected problems:
  1. after computing $x^2$, result is evaluated again `eval(s(s(s(s(0)))),M)`
  2. eval also steps into initial input
- solution: add new constructor *num* which states that the argument is a number, and hence, does not have to be evaluated

```
eval(0,0).
eval(s(E),s(N)) :- eval(E,N).
eval(E+F,K) :- eval(E,N), eval(F,M), plus(N,M,K).
eval(E-F,K) :- eval(E,N), eval(F,M), plus(M,K,N).
eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
eval(num(N),N).
eval(let(X,E,F),K) :- eval(E,N),X = num(N), eval(F,K).
```

## Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y)`)

```
f(s(s(0)),Y)
```

## Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y)`)

```
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                       |
                  f(s(s(0)),Y)
```

## Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y)`)

```
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                GX = num(s(s(0))) |
 GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                  |
                          f(s(s(0)),Y)
```

## Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                  |
      eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                   GX = num(s(s(0))) |
      GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                  |
                           f(s(s(0)),Y)
```

## Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                        |
          eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                        |
                eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                            GX = num(s(s(0))) |
                GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                        |
                                  f(s(s(0)),Y)
```

## Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y)`)

```
        eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                   N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                                |
          eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                                |
                eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                               GX = num(s(s(0))) |
                GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                                |
                                          f(s(s(0)),Y)
```

# Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                   N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                   N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                |
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                |
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                |
                          f(s(s(0)),Y)
```

# Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
                    X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
                                N = s(s(s(s(0)))) ||
              times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                                   N2 = s(s(0)) |
       eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                   N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                                |
          eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                                |
                eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                                GX = num(s(s(0))) |
                GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                                |
                                         f(s(s(0)),Y)
```

# Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
        X2 = num(s(s(s(s(0))))) |
X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
        N = s(s(s(s(0)))) ||
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
        N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
        N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
        |
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
        |
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
        GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
        |
f(s(s(0)),Y)
```

## Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
eval(s(num(s(s(s(s(0)))))),N), eval(num(s(s(s(s(0))))),M), plus(M,Y,N)
                              |
eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
                              | X2 = num(s(s(s(s(0)))))
X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
                              ‖ N = s(s(s(s(0))))
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                              | N2 = s(s(0))
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                              | N1 = s(s(0))
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                              |
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                              |
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                              | GX = num(s(s(0)))
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                              |
f(s(s(0)),Y)
```

# Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
eval(num(s(s(s(s(0))))),N1), eval(num(s(s(s(s(0))))),M), plus(M,Y,s(N1))
                                N = s(N1) |
eval(s(num(s(s(s(s(0)))))),N), eval(num(s(s(s(s(0))))),M), plus(M,Y,N)
                                          |
eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
                      X2 = num(s(s(s(s(0))))) |
X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
                         N = s(s(s(s(0)))) ||
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                            N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                            N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                          |
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                          |
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                         GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                          |
                                   f(s(s(0)),Y)
```

# Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
eval(num(s(s(s(s(0))))),M), plus(M,Y,s(s(s(s(s(0))))))
                    N1 = s(s(s(s(0)))) |
eval(num(s(s(s(s(0))))),N1), eval(num(s(s(s(s(0))))),M), plus(M,Y,s(N1))
                    N = s(N1) |
eval(s(num(s(s(s(s(0)))))),N), eval(num(s(s(s(s(0))))),M), plus(M,Y,N)
                    |
eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
                    X2 = num(s(s(s(s(0))))) |
X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
                    N = s(s(s(s(0)))) ||
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                    N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                    N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                    |
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                    |
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                    GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                    |
f(s(s(0)),Y)
```

# Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
                    plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                            M = s(s(s(s(0)))) |
          eval(num(s(s(s(s(0))))),M), plus(M,Y,s(s(s(s(s(0))))))
                            N1 = s(s(s(s(0)))) |
 eval(num(s(s(s(s(0))))),N1), eval(num(s(s(s(s(0))))),M), plus(M,Y,s(N1))
                                    N = s(N1) |
  eval(s(num(s(s(s(s(0)))))),N), eval(num(s(s(s(s(0))))),M), plus(M,Y,N)
                                              |
             eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
                         X2 = num(s(s(s(s(0))))) |
                X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
                             N = s(s(s(s(0)))) ||
         times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                                  N2 = s(s(0)) |
 eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                  N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                              |
        eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                              |
             eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                             GX = num(s(s(0))) |
             GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                              |
                                  f(s(s(0)),Y)
```

# Example (f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))

```
□
Y = s(0) ‖
plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
M = s(s(s(s(0)))) |
eval(num(s(s(s(s(0))))),M), plus(M,Y,s(s(s(s(s(0))))))
N1 = s(s(s(s(0)))) |
eval(num(s(s(s(s(0))))),N1), eval(num(s(s(s(s(0))))),M), plus(M,Y,s(N1))
N = s(N1) |
eval(s(num(s(s(s(s(0)))))),N), eval(num(s(s(s(s(0))))),M), plus(M,Y,N)
|
eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
X2 = num(s(s(s(s(0))))) |
X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
N = s(s(s(s(0)))) ‖
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
|
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
|
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
|
f(s(s(0)),Y)
```