Computational Logic	WS 2015/2016	LVA 703607

EXAM 2 — September 30, 2016

name: immatriculation number:

This exam consists of six exercises. The available points for each item are written in the margin. You need at least 50 points to pass.

12 **1** Complete the following table:

formula	$\alpha/\beta/\gamma/\delta$	universal	satisfiable
$\bot \supset (\forall x)P(x)$			✓
$\neg(((A \supset B) \supset A) \supset A)$		✓	
$(\forall x)P(x) \supset (\neg(\exists y)Q(y) \supset (\forall x)P(x))$			
$(\forall x)[(P(x) \supset Q(x)) \uparrow (\neg Q(x) \supset P(x))]$	γ		

2 Give tableau proofs of the following sentences.

5 (a) $(\neg P \supset Q) \supset ((P \supset Q) \supset Q)$

10 (b) $(\forall x)[P(x) \vee Q(x)] \supset [(\forall x)P(x) \vee (\exists x)Q(x)]$

10 (c) $(\forall x)(\exists y)[P(x) \supset Q(y)] \supset (\forall x)[P(x) \supset (\exists y)Q(y)]$

9 **3** Answer **three** of the following five questions.

- State and prove *Hintikka's lemma* for propositional logic.
- State *Lyndon's homomorphism theorem*.
- State *at least seven axiom schemes* of Hilbert systems.
- What is an *interpolant* for a first-order sentence $X \supset Y$?
- Give a *sequent calculus proof* of the sentence

$$(\forall x)[P(x) \supset Q(x)] \supset [(\forall x)P(x) \supset (\forall x)Q(x)]$$

$\boxed{4}$ This exercise is about the propositional compactness theorem.

4 (a) State the propositional compactness theorem.

20 (b) Complete the following proof of the propositional compactness theorem by filling in the missing parts.

Let $\mathcal{C} = \{W \mid$ __________ $\}$. We have $S \in \mathcal{C}$ by assumption. We prove that $\mathcal{C}$ is a __________.

(1) If both $A \in W$ and $\neg A \in W$ then $W \notin \mathcal{C}$ because the subset __________ of W is unsatisfiable.

(2) if $\bot \in W$ or $\neg\top \in W$ then $W \notin \mathcal{C}$ because $\bot$ and $\neg\top$ are unsatisfiable.

(3) Suppose $\neg\neg Z \in W \in \mathcal{C}$ and let V be a finite subset of __________. The set $(V \cap W) \cup \{\neg\neg Z\}$ is a finite subset of W and thus satisfiable. Hence also the set $(V \cap W) \cup \{\neg\neg Z, Z\}$ is satisfiable. Since V is a subset of $(V \cap W) \cup \{\neg\neg Z, Z\}$, V must be satisfiable.

(4) Suppose $\alpha \in W \in \mathcal{C}$. We need to show that __________. So let V be a finite subset of $W \cup \{\alpha_1, \alpha_2\}$. The set $(V \cap W) \cup \{\alpha\}$ is a finite subset of W and thus satisfiable because $W \in \mathcal{C}$. Hence also the set $(V \cap W) \cup \{\alpha, \alpha_1, \alpha_2\}$ is satisfiable because __________. Since V is a subset of $(V \cap W) \cup \{\alpha, \alpha_1, \alpha_2\}$, V is satisfiable.

(5) In the final case we have __________. We need to show that $W \cup \{\beta_1\} \in \mathcal{C}$ or $W \cup \{\beta_2\} \in \mathcal{C}$. For a proof by contradiction, suppose that neither $W \cup \{\beta_1\} \in \mathcal{C}$ nor $W \cup \{\beta_2\} \in \mathcal{C}$. So there exist finite subsets $V_1 \subseteq W \cup \{\beta_1\}$ and $V_2 \subseteq W \cup \{\beta_2\}$ such that __________. The set $((V_1 \cup V_2) \cap W) \cup \{\beta\}$ is a finite subset of W and hence satisfiable. Since β is equivalent to __________, $((V_1 \cup V_2) \cap W) \cup \{\beta_1\}$ or $((V_1 \cup V_2) \cap W) \cup \{\beta_2\}$ is satisfiable. However, this is impossible since V_1 is a subset of the former and V_2 a subset of the latter set.

The proof is concluded by an appeal to __________.

5 This exercise is about Herbrand's theorem.

5 (a) Compute the Herbrand universe of the sentence $(\exists x)[R(f(x), a) \supset \neg(\exists y)R(b, f(y))]$.

5 (b) Define the Herbrand expansion $\mathcal{E}(X, D)$ of an arbitrary sentence X over the domain $D = \{t_1, t_2\}$.

10 (c) Compute a tautologous Herbrand expansion for the valid sentence

$$(\forall z)(\exists w)(\forall x)[(\forall y)R(x, y) \supset R(w, z)]$$

10 **6** Determine whether the following statements are true or false. Every correct answer is worth 2 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

true	false	statement
☐	☐	$\{\neg X \supset \bot\} \vdash_{ph} X$
☐	☐	The set of all satisfiable propositional formulas is a Hintikka set.
☐	☐	The rank of the formula $(\exists x)(\forall y)[R(x, y) \supset \neg(\exists z)R(z, f(y))]$ is 5.
☐	☐	The sequent $X \supset Y, X \wedge Y \rightarrow \neg\neg X$ is an associated sequent of the set $\{X \wedge Y, \neg X, X \supset Y\}$.
☐	☐	The propositional formula $A \wedge (B \vee C)$ is an interpolant of the tautology $[A \wedge ((B \wedge D) \vee \neg C)] \supset \neg[(A \vee E) \supset \neg(C \supset B)]$.