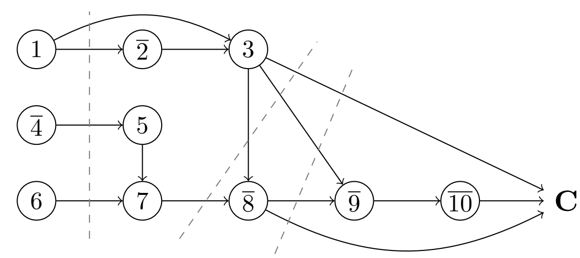
SAT and SMT Solving | SS 2018 | LVA 703048

Solutions to Test Exercises | June 21, 2018

1 (a) The implication graph looks as follows:



The UIPs are 6, 7, and $\overline{8}$. The indicated cuts lead to the implied clauses $\overline{1} \vee 4 \vee \overline{6}$, $\overline{3} \vee \overline{7}$, and $\overline{3} \vee 8$, from left to right.

(b) Both $\overline{3} \vee \overline{7}$ and $\overline{3} \vee 8$ are minimal. Using resolution they can be derived as follows: The conflict clause is $\overline{3} \vee 8 \vee 10$, its literal whose complement was assigned last is 10. The clause responsible for this assignment is $9 \vee \overline{10}$. We thus resolve

$$\frac{\overline{3} \vee 8 \vee 10 \qquad 9 \vee \overline{10}}{\overline{3} \vee 8 \vee 9}$$

The literal in the resulting clause whose complement was assigned last is 9. The clause responsible for this assignment is $\overline{3} \vee 8 \vee \overline{9}$. We hence get

$$\frac{\overline{3} \vee 8 \vee 9 \qquad \overline{3} \vee 8 \vee \overline{9}}{\overline{3} \vee 8}$$

The last-assigned literal is now 8, the responsible clause $\overline{3} \vee \overline{7} \vee \overline{8}$. We thus get $\overline{3} \vee \overline{7}$ in a final resolution step.

$$\frac{\overline{3} \vee 8 \qquad \overline{3} \vee \overline{7} \vee \overline{8}}{\overline{3} \vee \overline{7}}$$

2 (a) For example, substituting literals as follows:

$$1\colon \mathsf{a} \approx \mathsf{b} \qquad 2\colon \mathsf{f(a)} \approx \mathsf{f(b)} \qquad 3\colon \mathsf{f(a)} \approx \mathsf{c} \qquad 4\colon \mathsf{f(c)} \approx \mathsf{c} \qquad 5\colon \mathsf{f(b)} \approx \mathsf{f(f(a))}$$

the propositional skeleton is $1 \wedge (\overline{2} \vee \overline{3}) \wedge (\overline{1} \vee \overline{4}) \wedge 5$.

(b) We apply DPLL(T) as follows:

	$\parallel 1,\ (\overline{2} \vee 3),\ (\overline{1} \vee \overline{4}),\ 5$	
$\Longrightarrow$	$1 \parallel 1,\ (\overline{2} \vee 3),\ (\overline{1} \vee \overline{4}),\ 5$	unit propagate
$\Longrightarrow$	$1\ \overline{4} \parallel 1,\ (\overline{2} \vee 3),\ (\overline{1} \vee \overline{4}),\ 5$	unit propagate
$\Longrightarrow$	$1\ \overline{4}\ 5 \parallel 1,\ (\overline{2} \vee 3),\ (\overline{1} \vee \overline{4}),\ 5$	unit propagate
$\Longrightarrow$	$1\ \overline{4}\ 5\ 3^d \parallel 1,\ (\overline{2} \vee 3),\ (\overline{1} \vee \overline{4}),\ 5$	decide

At this point the SAT solver claims satisfiability with model $1\,\overline{4}\,5\,\overline{3}$, corresponding to

$$\mathsf{a} \approx \mathsf{b} \wedge \mathsf{f(a)} \not\approx \mathsf{c} \wedge \mathsf{f(c)} \not\approx \mathsf{c} \wedge \mathsf{f(b)} \approx \mathsf{f(f(a))}$$

We apply congruence closure to check that the model is T-consistent. This is the case if the positive literals $P = \{\mathsf{a} \approx \mathsf{b},\ \mathsf{f(b)} \approx \mathsf{f(f(a))}\}$ do not imply one of the literals $N = \{\mathsf{f(c)} \approx \mathsf{c},\ \mathsf{f(a)} \approx \mathsf{c}\}$ occurring negatively in the model. We start by putting all subterms into different sets:

1: {a} 2: {b} 3: {c} 4: {f(a)} 5: {f(b)} 6: {f(c)} 7: {f(f(a))}

Merging sets according to equations in P yields

1: {a, b} 3: {c} 4: {f(a)} 5: {f(b), f(f(a))} 6: {f(c)}

Since $\mathsf{a} \approx \mathsf{b}$ implies $\mathsf{f(a)} \approx \mathsf{f(b)}$ the sets 4 and 5 must be merged:

1: {a, b} 3: {c} 4: {f(a), f(b), f(f(a))} 6: {f(c)}

No more merge steps are posible. As neither f(c) and c, nor f(a) and c are in the same set the model is consistent. Hence the given formula φ is satisfiable.

3 We start by putting all subterms into different sets:

1: {a} 2: {b} 3: {c} 4: {f(a)} 5: {f(b)} 6: {f(c)}
7: {f(f(a))} 8: {g(a, b)} 9: {g(a, c)} 10: {g(f(f(a)), b)} 11: {g(c, a)} 12: {f(g(a, c))}

Next term sets are merged according to the equations in E:

1: {a, f(a)} 2: {b, g(a, b)} 3: {c, g(c, a)} 5: {f(b), f(f(a))} 6: {f(c), g(a, c)}
10: {g(f(f(a)), b)} 12: {f(g(a, c))}

Sets 1 and 5 are merged because $\mathsf{a} \approx \mathsf{f(a)}$ implies $\mathsf{f(a)} \approx \mathsf{f(f(a))}$:

1: {a, f(a), f(b), f(f(a))} 2: {b, g(a, b)} 3: {c, g(c, a)} 6: {f(c), g(a, c)}
10: {g(f(f(a)), b)} 12: {f(g(a, c))}

Sets 2 and 10 are merged because $\mathsf{f(f(a))} \approx \mathsf{a}$ implies $\mathsf{g(a,b)} \approx \mathsf{g(f(f(a)),b)}$:

1: {a, f(a), f(b), f(f(a))} 2: {b, g(a, b), g(f(f(a)), b)} 3: {c, g(c, a)} 6: {f(c), g(a, c)}
12: {f(g(a, c))}

No more merge operations are possible. So we can conclude that

(a) $E \models \mathsf{g(f(f(a)),b)} \approx \mathsf{b}$ holds because the two terms are both in set 2.

(b) $E \models \mathsf{f(c)} \approx \mathsf{f(g(a,c))}$ does not hold because the two terms end up in different sets.

4 (a) The only variable violating its (upper) bound is s_1, so we want to decrease its value. Because the coefficients for (s_1, s_2) and (s_1, s_3) are positive, this implies that the value for one of s_3 or s_2 will to be decreased as well. But since neither of s_3 and s_2 have a lower bound, both are suitable.

According to Bland's rule the lexicographically smallest suitable variable pair should be chosen. Here this would be (s_1, s_3).

(b) We pivot s_1 with s_3. This yields the following updated tableau:

$$\begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 1 & -1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} \end{array}$$

The variable s_1 is set to its upper bound 1, and the nonbasic variable s_2 is still assigned 4. The remaining variables are updated to $x = 3$, $y = 4$, $s_3 = -7$, $s_4 = 5$, which satisfies the constraints.

5 (a) The following propositional formula encodes $\mathbf{a}_3 >_u (\mathbf{b}_3 \,\&\, \mathbf{c}_3)$:

$$\begin{aligned} &(a_2 \land \neg(b_2 \land c_2)) \lor \\ &((a_2 \leftrightarrow (b_2 \land c_2)) \land a_1 \land \neg(b_1 \land c_1)) \lor \\ &((a_2 \leftrightarrow (b_2 \land c_2)) \land (a_1 \leftrightarrow (b_1 \land c_1)) \land a_0 \land \neg(b_0 \land c_0)) \end{aligned}$$

(b) The multiplication $\mathbf{a}_3 \times \mathbf{b}_3$ produces an overflow if $(a_2 \land (b_1 \lor b_2)) \lor (b_2 \land (a_1 \lor a_2)) \lor (a_1 \land a_0 \land b_1 \land b_0)$ is satisfied.

6 (a) The formula $(z = 0) \land (x + z \geqslant y) \land (y \leqslant x) \land (\mathsf{f}(z) > z) \land (\mathsf{f}(x) = \mathsf{f}(z) + \mathsf{f}(y))$ can be purified to

$$\begin{aligned} \psi_1 &= (z = 0) \land (x + z \geqslant y) \land (y \leqslant x) \land (c > z) \land (a = c + b) \\ \psi_2 &= (a = \mathsf{f}(x)) \land (b = \mathsf{f}(y)) \land (c = \mathsf{f}(z)) \end{aligned}$$

using fresh variables a, b, and c.

(b) The formulas ψ_1 and ψ_2 are compatible with the guessed equivalence relation $y = z$, $b = c = x$. So ψ is satisfiable.