SAT and SMT Solving

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Outline

- Summary of Last Week
- Unsatisfiable Cores
- Algorithm by Fu and Malik
- Unsatisfiable Cores in Practice

Summary of Last Week

Maximum Satisfiability

Consider CNF formulas χ and φ as sets of clauses.

Definitions

- $\mathrm{maxSAT}(\varphi)$ is maximal $|\psi|$ such that $\psi \subseteq \varphi$ and $\bigwedge_{C \in \psi} C$ satisfiable
- $\mathrm{pmaxSAT}(\varphi, \chi)$ is maximal $|\psi|$ such that $\psi \subseteq \varphi$ and $\chi \wedge \bigwedge_{C \in \psi} C$ satisfiable

Definitions

given weights $w_C \in \mathbb{Z}$ for all $C \in \varphi$,

- $\mathrm{maxSAT}_w(\varphi)$ is maximal $\sum_{C \in \psi} w_C$ such that $\psi \subseteq \varphi$ and $\bigwedge_{C \in \psi} C$ satisfiable
- $\mathrm{pmaxSAT}_w(\varphi, \chi)$ is maximal $\sum_{C \in \psi} w_C$ such that $\psi \subseteq \varphi$ and $\chi \wedge \bigwedge_{C \in \psi} C$ satisfiable

Definition

$\mathrm{minUNSAT}(\varphi)$ is minimal $|\psi|$ such that $\psi \subseteq \varphi$ and $\bigwedge_{C \in \psi} \neg C$ is satisfiable

Lemma

$$|\varphi| = |\mathrm{minUNSAT}(\varphi)| + |\mathrm{maxSAT}(\varphi)|$$

Branch & Bound

Idea

- gets list of clauses φ as input return minUNSAT(φ)
- explores assignments in depth-first search

Notation

- φ_x is formula φ with all occurrences of x replaced by T
- $\varphi_{\overline{x}}$ is formula φ with all occurrences of x F
- for list of clauses φ, function $\mathtt{simp}(\varphi)$
  - replace $\neg$T by F and $\neg$F by T
  - drops all clauses which contain T
  - removes F from all remaining clauses
- $\square$ denotes empty clause and $\#\mathtt{empty}(\varphi)$ number of empty clauses in φ

Algorithm (Branch & Bound)

```
function BnB(φ, UB)
  φ = simp(φ)
  if φ contains only empty clauses then
      return #empty(φ)
  if #empty(φ) ⩾ UB then
      return UB
  x = selectVariable(φ)
  UB := min(UB, BnB(φ_x, UB))
  return min(UB, BnB(φ_x̄, UB))
```

Theorem

$\mathtt{BnB}(\varphi, |\varphi|) = \mathsf{minUNSAT}(\varphi)$

Binary Search

Idea

- gets list of clauses φ as input and returns minUNSAT(φ)
- repeatedly call SAT solver in binary search fashion

Definitions

- cardinality constraint is
$$\sum_{x \in X} x \bowtie N$$
where $\bowtie$ is $=$, $<$, $>$, $\leqslant$, or $\geqslant$, X is set of propositional variables, and $N \in \mathbb{N}$
- valuation v satisfies $\sum_{x \in X} x \bowtie N$ iff $k \bowtie N$
where k is number of variables $x \in X$ such that $v(x) = \mathrm{T}$

Remark

cardinality constraints are expressible in CNF

Algorithm (Binary Search)

```
function BinarySearch({C_1, ..., C_m})
  φ := {C_1 ∨ b_1, ..., C_m ∨ b_m}
  return search(φ,0,m)
```

$b_1, \ldots, b_m$ are fresh variables

```
function search(φ, L, U)
  if L ⩾ U then
    return U
  mid := ⌊(U+L)/2⌋
  if SAT(φ ∧ CNF(∑_{i=1}^{m} b_i ⩽ mid)) then
    return search(φ, L, mid)
  else
    return search(φ, mid + 1, U)
```

Theorem

$\texttt{BinarySearch}(\varphi) = \mathrm{minUNSAT}(\varphi)$

Unsatisfiable Cores

Definitions

for unsatisfiable CNF formula φ given as set of clauses

- $\psi \subseteq \varphi$ such that $\bigwedge_{C \in \psi} C$ is unsatisfiable is unsatisfiable core (UC) of φ
- minimal unsatisfiable core ψ is UC such that every subset of ψ is satisfiable
- MUC (minimum unsatisfiable core) is UC such that $|\psi|$ is minimal

Example

$$\varphi = \{\neg x, \qquad x \vee z, \qquad \neg y \vee \neg z, \qquad x, \qquad y \vee \neg z\}$$

unsatisfiable cores are

- φ
- $\{\, \neg x,\ x \vee z,\ \neg y \vee \neg z,\ y \vee \neg z \,\}$ minimal
- $\{\, \neg x,\ x \,\}$ minimal and MUC

Remark

MUC is always minimal unsatisfiable core

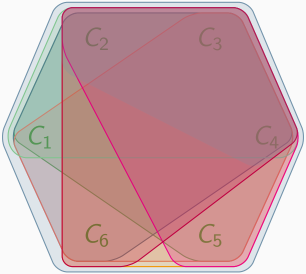
Example

$$\varphi = \{C_1, \ldots, C_6\}$$

$C_1: x_1 \vee \neg x_3$ $\qquad$ $C_2: x_2$ $\qquad$ $C_3: \neg x_2 \vee x_3$

$C_4: \neg x_2 \vee \neg x_3$ $\qquad$ $C_5: x_2 \vee x_3$ $\qquad$ $C_6: \neg x_1 \vee x_2 \vee \neg x_3$

φ has 9 unsatisfiable cores:



$UC_1 = \{C_1, C_2, C_3, C_4, C_5, C_6\}$
$UC_2 = \{C_1, C_2, C_3, C_4, C_5\}$
$UC_3 = \{C_1, C_2, C_3, C_4, C_6\}$
$UC_4 = \{C_1, C_3, C_4, C_5, C_6\}$
$UC_5 = \{C_2, C_3, C_4, C_5, C_6\}$
$UC_6 = \{C_1, C_2, C_3, C_4\}$
$UC_7 = \{C_2, C_3, C_4, C_5\}$
$UC_8 = \{C_2, C_3, C_4, C_6\}$
$UC_9 = \{C_2, C_3, C_4\}$ minimal and MUC

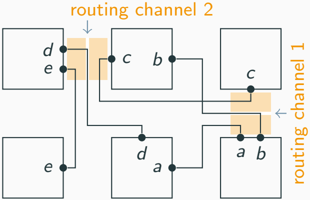
Application: FPGA Routing

Field Programmable Gate Arrays (FPGAs)

- can simulate microprocessors but faster for special tasks (from complex combinatorics to mere logic)
- logic blocks connected by "routing channels"
- "routing": determine which channels are used for what

Example (Encoding Routing Requirements)

- consider connections a, b, c, d, e of 2 bits each
- liveness: want to route $\geqslant 1$ bit of a, b, c, d, e
- 2 routing channels of 2 tracks each
- exclusivity: each channel has only 2 tracks
- unsatisfiable: UCs indicate problems



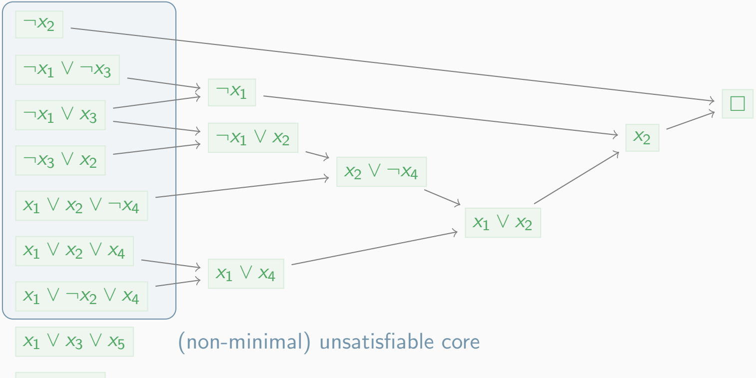
$a_0 \vee a_1$	$\neg a_0 \vee \neg b_0$	$\neg c_0 \vee \neg d_0$
$b_0 \vee b_1$	$\neg a_0 \vee \neg c_0$	$\neg c_0 \vee \neg e_0$
$c_0 \vee c_1$	$\neg b_0 \vee \neg c_0$	$\neg d_0 \vee \neg e_0$
$d_0 \vee d_1$	$\neg a_1 \vee \neg b_1$	$\neg c_1 \vee \neg d_1$
$e_0 \vee e_1$	$\neg a_1 \vee \neg c_1$	$\neg c_1 \vee \neg e_1$
	$\neg b_1 \vee \neg c_1$	$\neg d_1 \vee \neg e_1$

UC_1: channel 1 capacity exceeded
UC_2: channel 2 capacity exceeded
UC_3: c is overconstrained
UC_4: c is overconstrained

Idea

- repeatedly pick clause C from φ and check satisfiability:
  if $\varphi \setminus \{C\}$ is satisfiable, keep C, otherwise drop C
- SAT solvers can give resolution proof if conflict detected:
  use resolution graphs for more efficient implementation of this idea

Example (Resolution Graph)



Definition (Resolution Graph)

directed acyclic graph $G = (V, E)$ is resolution graph for set of clauses φ if

1. $V = V_i \uplus V_c$ is set of clauses and $V_i = \varphi$,
2. V_i nodes have no incoming edges, initial nodes
3. there is exactly one node $\square$ without outgoing edges,
4. $\forall C \in V_c \; \exists$ edges $D \to C$, $D' \to C$ such that C is resolvent of D and D', and
5. there are no other edges.

Notation

- $Reach_G(C)$ is set of nodes reachable from C in G
- $Reach_G^E(C)$ is set of edges reachable from C in G
- $BReach(C)$ is set of nodes backwards reachable from C in G
- $\overline{N}$ is $V \setminus N$ for any set of nodes N

Algorithm minUnsatCore(φ)

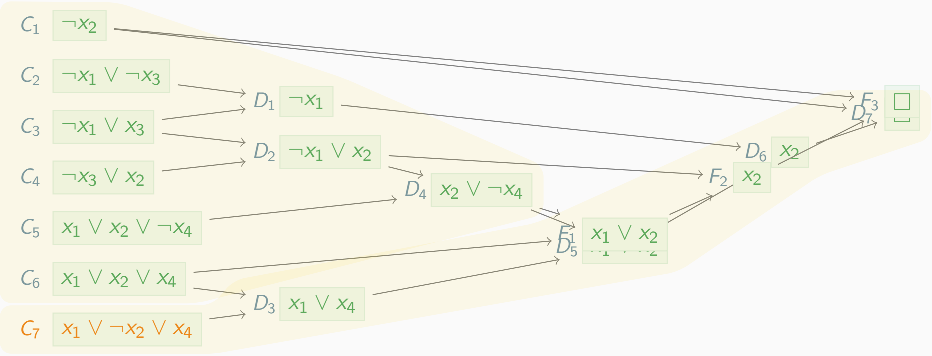
Input: unsatisfiable formula φ
Output: minimal unsatisfiable core of φ

build refutation graph $G = (V_i \uplus V_c, E)$ for φ
while $\exists$ unmarked clause in V_i **do**
  $C \leftarrow$ unmarked clause in V_i
  if SAT($\overline{Reach_G(C)}$) **then** ▷ subgraph without C satisfiable?
    mark C ▷ C is UC member
  else
    build refutation graph $G' = (V_i' \uplus V_c', E')$ for $\overline{Reach_G(C)}$
    $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
    $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
    $G \leftarrow (V_i \cup V_c, E)$
    $G \leftarrow G|_{BReach_G(\Box)}$ ▷ restrict to nodes with path to $\Box$
return V_i

Theorem

if φ unsatisfiable then minUnsatCore(φ) is minimal unsatisfiable core of φ

Example



minUnsatCore(φ)

re-use relevant resolvents: fewer steps to $\square$

- pick C_7
- $Reach_G(C_7) = \{C_7, D_3, D_5, D_6, D_7\}$, so $\overline{Re(C_7, G)} = \{C_1, \dots, C_6, D_1, D_2, D_4\}$
- check $\text{SAT}(\overline{Reach_G(C_7)})$
- unsatisfiable: get new resolution graph G_7 for $\varphi \cup \{D_1, D_2, D_4\}$
- construct resolution graph G' for φ by adding edges from G to G_7
- set G to G' restricted to $BReach_{G'}(\square)$
- after 5 more loop iterations: return $\{C_1, C_3, \dots, C_6\}$

Bounds for Maximum Satisfiability

Definition

if $\varphi = C_1 \wedge \cdots \wedge C_m$ is CNF formula then blocked formula

$$\varphi_B = (C_1 \vee b_1) \wedge \cdots \wedge (C_m \vee b_m)$$

for fresh variables $b_1, \ldots, b_m$

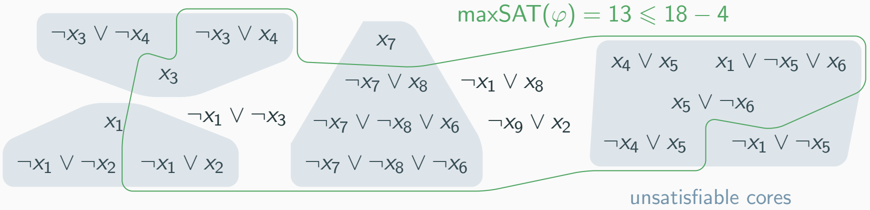
Lemma (Lower Bound)

Let v satisfy φ_B and $B_{\mathsf{T}} = \{b_i \mid v(b_i) = \mathsf{T}\}$. Then $\mathrm{maxSAT}(\varphi) \geqslant |\varphi| - |B_{\mathsf{T}}|$

Lemma (Upper Bound)

If φ contains k disjoint unsatisfiable cores then $\mathrm{maxSAT}(\varphi) \leqslant |\varphi| - k$

Example (Upper Bound)



Algorithm by Fu and Malik

Algorithm by Fu and Malik

Definition (Partial minUNSAT**)**

pminUNSAT(χ, φ) is minimal $|\psi|$ such that $\psi \subseteq \varphi$ and $\chi \wedge \bigwedge_{C \in \psi} \neg C$ satisfiable

Lemma

$$|\varphi| = |\text{pminUNSAT}(\chi, \varphi)| + |\text{pmaxSAT}(\chi, \varphi)|$$

Idea

- maximal satisfying valuation falsifies at least one clause in unsatisfiable core
- repeatedly call SAT solver on relaxed formula excluding unsatisfiable core until resulting formula is satisfiable
- returns pminUNSAT for hard clauses χ, soft clauses φ

Algorithm FuMalik(χ, φ)

Input: clause set φ and satisfiable clause set χ
Output: minUNSAT(χ, φ)

```
cost ← 0
while ¬SAT(χ ∪ φ) do
    UC ← unsatCore(χ ∪ φ)
    B ← ∅
    for C ∈ UC ∩ φ do                          ▷ loop over soft clauses in core
        b ← new blocking variable
        φ ← φ \ {C} ∪ {C ∨ b}
        B ← B ∪ {b}
    χ ← χ ∪ CNF(∑_{b∈B} b = 1)                 ▷ cardinality constraint is hard
    cost ← cost + 1
return cost
```

Theorem

FuMalik$(\chi, \varphi) =$ pminUNSAT(χ, φ)

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2 \vee b_1$	$\neg x_1 \vee x_2 \vee b_2$	$\neg x_1 \vee x_7$	$x_1 \vee b_3$
	$\neg x_3 \vee x_4 \vee c_1$	$x_3 \vee c_2$	$\neg x_3 \vee \neg x_4 \vee c_3$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	$x_7 \vee d_1$
	$\neg x_7 \vee x_8 \vee d_2$	$\neg x_7 \vee \neg x_8 \vee x_6 \vee d_3$	$\neg x_7 \vee \neg x_8 \vee \neg x_6 \vee d_4$	$\neg x_1 \vee \neg x_3 \vee e_1$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$
  $\chi = \chi \cup \text{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \vee x_3,\ \neg x_7 \vee x_2,\ x_7 \vee x_2,\ x_1 \vee \neg x_2,\ \neg x_1 \vee \neg x_3$
  $\chi = \chi \cup \text{CNF}(e_1 = 1)$
  $cost = 4$
- satisfiable: $v(x_1) = v(x_2) = v(x_3) = v(x_5) = v(x_7) = \text{T}$ and $v(x_i) = \text{F}$ otherwise
- $\text{pminUNSAT}(\chi, \varphi) = 4$ and $\text{pmaxSAT}(\chi, \varphi) = 12$

Unsatisfiable Cores in Practice

Unsatisfiable Cores in z3

```
from z3 import *

x1,x2,x3 = Bool("x1"), Bool("x2"), Bool("x3")
phi = [ Or(Not(x1), Not(x2)), Or(Not(x1), x2),\
  Or(Not(x1), x3), x1, Or(Not(x3), x2)]

solver = Solver()
solver.set(unsat_core=True)

# assert clauses in phi with names phi0 ... phi4
for i,c in enumerate(phi):
  solver.assert_and_track(c, "phi" + str(i))

if solver.check() == z3.unsat:
  uc = solver.unsat_core()
  print(uc) # [phi0, phi1, phi3]
```

Literature

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