SAT and SMT Solving

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Outline

- Summary of Last Week
- Simplex Algorithm

Summary of Last Week

Definition (DPLL(T) systems)

- basic system $\mathcal{B}$: unit propagate, decide, fail, T-backjump, T-propagate
- full system $\mathcal{F}$: $\mathcal{B}$ plus T-learn, T-forget, and restart

Theorem (Correctness)

For derivation with final state S_n:

$$\parallel F \quad \Longrightarrow_{\mathcal{F}} \quad S_1 \quad \Longrightarrow_{\mathcal{F}} \quad S_2 \quad \Longrightarrow_{\mathcal{F}} \quad \dots \quad \Longrightarrow_{\mathcal{F}} \quad S_n$$

- *if* $S_n = \text{FailState}$ *then* F *is T-unsatisfiable*
- *if* $S_n = M \parallel F'$ *and* M *is T-consistent then* F *is T-satisfiable and* $M \vDash_T F$

Theorem (Termination)

Γ: $\parallel F \Longrightarrow^*_{\mathcal{F}} S_0 \Longrightarrow^*_{\mathcal{F}} S_1 \Longrightarrow^*_{\mathcal{F}} \dots$ *is finite if*

- *there is no infinite sub-derivation of only T-learn and T-forget steps, and*
- *for every sub-derivation* $S_i \overset{\text{restart}}{\Longrightarrow}_{\mathcal{F}} S_{i+1} \Longrightarrow^*_{\mathcal{F}} S_j \overset{\text{restart}}{\Longrightarrow}_{\mathcal{F}} S_{j+1} \Longrightarrow^*_{\mathcal{F}} S_k$ *with no restart steps in* $S_{i+1} \Longrightarrow^*_{\mathcal{F}} S_j$ *and* $S_{j+1} \Longrightarrow^*_{\mathcal{F}} S_k$*:*
  - *there are more $\mathcal{B}$-steps in* $S_j \Longrightarrow^*_{\mathcal{F}} S_k$ *than in* $S_i \Longrightarrow^*_{\mathcal{F}} S_j$*, or*
  - *a clause is learned in* $S_j \Longrightarrow^*_{\mathcal{F}} S_k$ *that is never forgotten in* Γ

Congruence Closure

Input: set of equations E and equation $s \approx t$, both without variables

Output: *valid* ($E \vDash_T s \approx t$) or *invalid* ($E \nvDash_T s \approx t$)

1. build congruence classes
   (a) put different subterms of terms in $E \cup \{s \approx t\}$ in separate sets
   (b) merge sets $\{\dots, t_1, \dots\}$ and $\{\dots, t_2, \dots\}$ for all $t_1 \approx t_2$ in E
   (c) merge sets $\{\dots, f(t_1, \dots, t_n), \dots\}$ and $\{\dots, f(u_1, \dots, u_n), \dots\}$ if t_i and u_i belong to same set for all $1 \leqslant i \leqslant n$, repeatedly
1. if s and t belong to same set then return *valid* else return *invalid*

Deciding Satisfiability of EUF Conjunctions

- consider EUF conjunction φ with free variables $x_1, \ldots, x_n$
- split φ into positive and negative literals

$$\varphi = \left(\bigwedge P\right) \wedge \left(\bigwedge \neg N\right)$$

- determine satisfiability

$\varphi = (\bigwedge P) \wedge (\bigwedge \neg N)$	unsatisfiable	
$\iff \exists x_1 \ldots x_n . (\bigwedge P) \wedge (\bigwedge \neg N)$	unsatisfiable	
$\iff (\bigwedge \widehat{P}) \wedge (\bigwedge \neg \widehat{N})$	unsatisfiable	skolemization
$\iff \neg \left((\bigwedge \widehat{P}) \wedge (\bigwedge \neg \widehat{N})\right)$	valid	φ unsat iff $\neg\varphi$ valid
$\iff \bigwedge \widehat{P} \to \bigvee \widehat{N}$	valid	deMorgan
$\iff \exists\, s \approx t$ in $\widehat{N}$ such that $\bigwedge \widehat{P} \to s \approx t$ valid		semantics of $\vee$
$\iff \exists\, s \approx t$ in $\widehat{N}$ such that $\bigwedge \widehat{P} \vDash_T s \approx t$		semantics of $\vDash$

Simplex Algorithm

Effects and Side Effects

- guaranteed to solve all your real arithmetic problems
- encountering Simplex can cause initial dizzyness
- in rare cases solving systems of linear inequalities can become addictive

Definition (Theory of Linear Arithmetic over C)

- for variables $x_1, \dots, x_n$, formulas built according to grammar

$$\varphi ::= \varphi \land \varphi \mid t = t \mid t < t \mid t \leqslant t$$

$$t ::= a_1 x_1 + \dots + a_n x_n + b \qquad \text{for } a_1, \dots, a_n, b \in \text{in carrier } C$$

- axioms are equality axioms plus calculation rules of arithmetic over C
- solution assigns values in C to $x_1, \dots, x_n$

Definitions

- Linear Real Arithmetic (LRA) uses carrier $C = \mathbb{R}$
- Linear Integer Arithmetic (LIA) uses carrier $C = \mathbb{Z}$

Example

- $x + y + z = 2 \land z > y \land y > -1$
  is satisfiable in LRA and LIA, e.g. with $v(x) = v(y) = 0$ and $v(z) = 2$
- $x < 3 \land 2x > 4$
  is unsatisfiable in LIA but satisfiable in LRA, e.g. with $v(x) = 2.5$

Satisfiability Problem for Linear Arithmetic

- integers (LIA): NP-complete
- reals (LRA) or rationals: polynomial Simplex algorithm

Some History

exponential worst-case complexity

1947 Danzig proposed Simplex algorithm to solve optimization problem:

$$\text{maximize } c(\vec{x}) \qquad \text{such that} \qquad A\vec{x} \leqslant b \text{ and } \vec{x} \geqslant 0$$

for linear objective function c, matrix A, vector b, and vector of variables $\vec{x}$

- also known as linear programming

1979 Khachiyan proposed polynomial version based on ellipsoid method

1984 Karmakar proposed polynomial version based on interior points method

2000- SMT solvers use DPLL(T) version to solve satisfiability problem

$$A\vec{x} \leqslant b$$

Problem Input (General Form)

- m equalities

$$a_1x_1 + \ldots a_nx_n = 0$$

- (optional) lower and upper bounds on variables

$$l_i \leqslant x_i \leqslant u_i$$

Lemma

any LRA problem without $<$ can be turned into equisatisfiable general form

Example

$$\begin{aligned} x - y &\geqslant -1 \\ y &\leqslant 4 \\ x + y &\geqslant 6 \\ 3x - y &\leqslant 7 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} -x + y - s_1 &= 0 & s_1 &\leqslant 1 \\ y - s_2 &= 0 & s_2 &\leqslant 4 \\ -x - y - s_3 &= 0 & s_3 &\leqslant -6 \\ 3x - y - s_4 &= 0 & s_4 &\leqslant 7 \end{aligned} \qquad \text{slack variables}$$

- s_1, s_2, s_3, s_4 are slack variables
- x, y are problem variables

Representation

- represent equalities using $m \times (n+m)$ matrix A

$$\begin{aligned} -x + y - s_1 &= 0 & s_1 &\leqslant 1 \\ y - s_2 &= 0 & s_2 &\leqslant 4 \\ -x - y - s_3 &= 0 & s_3 &\leqslant -6 \\ 3x - y - s_4 &= 0 & s_4 &\leqslant 7 \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 3 & -1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{array}{l} s_1 \leqslant 1 \\ s_2 \leqslant 4 \\ s_3 \leqslant -6 \\ s_4 \leqslant 7 \end{array}$$

- simplified matrix presentation

$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

$\leftarrow$ nonbasic variables

basic variables $\rightarrow$

Notation

- simplified matrix is tableau
- B is set of basic variables (in tableau listed vertically)
- N is set of non-basic variables (in tableau listed horizontally)

DPLL(T) Simplex Algorithm

Input: conjunction of LRA literals φ without $<$

Output: satisfiable or unsatisfiable

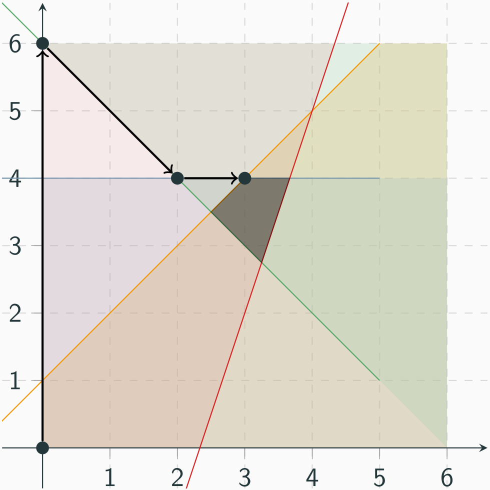
1. transform φ into general form and construct tableau
2. fix order on variables and assign 0 to each variable
3. if all basic variables satisfy their bounds then return satisfiable
4. let $x_i \in B$ be variable that violates its bounds
5. search for suitable variable $x_j \in N$ for pivoting with x_i
6. return unsatisfiable if search unsuccessful
7. perform pivot operation on x_i and x_j
9. update assignment
10. go to step 3

Simplex, Visually

- constraints

$$x - y \geqslant -1$$
$$y \leqslant 4$$
$$x + y \geqslant 6$$
$$3x - y \leqslant 7$$

- solution space
- Simplex solution search



Example

tableau

$$\begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 1 & -1 \\ 1 & 0 \\ 2 & -1 \end{pmatrix} \end{array}$$

constraints

$s_1 \leqslant 1$
$s_2 \leqslant 4$
$s_3 \leqslant -6$
$s_4 \leqslant 7$

assignment

x	y	s_1	s_2	s_3	s_4
3	4	1	4	-7	7

1. Iteration 1
   - s_3 violates its bounds
   - decreasing s_3 requires to increase x or y (both suitable since they have no upper bound)
   - pivot s_3 with y:

     $$y = -x - s_3 \qquad s_1 = -2x - s_3$$
     $$s_2 = -x - s_3 \qquad s_4 = 4x + s_3$$

   - update assignment

     $$s_3 = s_3 - 6 = -6 \qquad y = 6$$
     $$s_1 = 6 \qquad s_2 = 6 \qquad s_4 = -6$$

DPLL(T) Simplex Algorithm

non-basic $\vec{x}_N$

basic $\vec{x}_B$

$$\begin{array}{cc} & x_j \\ x_i & \begin{pmatrix} \cdots & & \cdots \\ & A_{ij} & \\ \cdots & & \cdots \end{pmatrix} \end{array}$$

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Invariant

- (1) is satisfied and (2) holds for all nonbasic variables

Pivoting

- swap basic x_i and non-basic x_j, so $i \in B$ and $j \in N$

$$x_i = \sum_{k \in N} A_{ik} x_k \quad \Longrightarrow \quad x_j = \frac{1}{A_{ij}} \left(x_i - \sum_{k \in N - \{j\}} A_{ik} x_k \right) \tag{$\star$}$$

- new tableau A' consists of ($\star$) and $A_{B-\{i\}}\vec{x}_N = \vec{x}_{B-\{i\}}$ with ($\star$) substituted

Update

- assignment of x_i is updated to previously violated bound l_i or u_i,
- assignment of x_k is recomputed using ($\star$) and A' for all $k \in B - \{i\} \cup \{j\}$

DPLL(T) Simplex Algorithm

$$A\vec{x}_N = \vec{x}_B \qquad (1)$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \qquad (2)$$

Suitability

- basic variable x_i violates lower and/or upper bound
- pick nonbasic variable x_j such that
  - if $x_i < l_i$: $A_{ij} > 0$ and $x_j < u_j$ or $A_{ij} < 0$ and $x_j > l_j$
  - if $x_i > u_i$: $A_{ij} > 0$ and $x_j > l_j$ or $A_{ij} < 0$ and $x_j < u_j$

Observation

- problem is unsatisfiable if no suitable pivot exists

Bland's Rule

- pick lexicographically smallest (i, j) that is suitable pivot
- guarantees termination

How to Treat Strict Inequalities

replace in LRA formula φ every strict inequality

$$a_1x_1 + \cdots + a_nx_n < b$$

by non-strict inequality

$$a_1x_1 + \cdots + a_nx_n \leqslant b - \delta$$

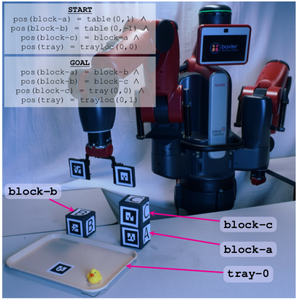
to obtain formula φ_δ in LRA without $<$, and treat δ symbolically during Simplex algorithm

Lemma

$$\varphi \textit{ is satisfiable} \quad\iff\quad \exists \textit{ rational number } \delta > 0 \textit{ such that } \varphi_\delta \textit{ is satisfiable}$$

Application: Motion Planning for Robots

- robots needs to plan motions
  to place objects correctly
- instance of *constraint based planning*
- encoding
  - fix number of time slots $t_1, \ldots, t_n$
  - action variable a_i for time t_i encodes which action performed at time t_i (one action per time)
  - actions require precondition and imply postcondition
  - use arithmetic to minimize path



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(Almost) Everything is Better With Arithmetic

LRA and LIA admit more efficient encodings of

- n-queens
- Sudoku
- graph coloring
- Minesweeper
- travelling salesperson
- rabbit problem
- planning problems
- scheduling problems
- component configuration problems
- everything with cardinality constraints
- ...

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