SAT and SMT Solving

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SS 2018

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Outline

- Summary of Last Week
- Deciding Equality Logic
- Branch and Bound
- Cutting Planes

Summary of Last Week

Definition (Theory of Linear Arithmetic over C)

- for variables $x_1, \dots, x_n$, formulas built according to grammar

$$\varphi ::= \varphi \wedge \varphi \mid t = t \mid t < t \mid t \leqslant t$$

$$t ::= a_1 x_1 + \cdots + a_n x_n + b \qquad \text{for } a_1, \dots, a_n, b \in \text{in carrier } C$$

- axioms are equality axioms plus calculation rules of arithmetic over C
- solution assigns values in C to $x_1, \dots, x_n$

Definitions

- carrier $\mathbb{R}$: linear real arithmetic (LRA),
  DPLL(T) simplex algorithm is decision procedure
- carrier $\mathbb{Z}$: linear integer arithmetic (LIA)

DPLL(T) Simplex Algorithm (1)

- linear arithmetic constraint solving over real or rational variables
- $x_1, \ldots, x_n$ are split into basic variables $\vec{x}_B$ and nonbasic variables $\vec{x}_N$

Input

constraints plus upper and lower bounds for $x_1, \ldots, x_n$:

$$A\,\vec{x}_N = \vec{x}_B \qquad \text{with tableau } A \in \mathbb{R}^{|B|\times|N|} \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Output

satisfying assignment or “unsatisfiable”

Invariant

(1) is satisfied and (2) holds for all nonbasic variables x_i

DPLL(T) Simplex Algorithm (2)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Method

- if (2) holds for all basic variables, return current assignment
- otherwise select basic variable x_i (so $i \in B$) which violates (2)
- select suitable nonbasic variable x_j (so $j \in N$) such that x_i and x_j can be swapped in a pivoting step, resulting in new tableau

  $$A' \, x_{N'} = x_{B'}$$

  with $N' = N \cup \{i\} - \{j\}$ and $B' = B \cup \{j\} - \{i\}$
- change value of x_i to l_i or u_i and update values of basic variables accordingly

DPLL(T) Simplex Algorithm (3)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Pivoting

- swap basic x_i and non-basic x_j

$$x_i = \sum_{k \in N} A_{ik} x_k \qquad \Longrightarrow \qquad x_j = \frac{1}{A_{ij}} \left(x_i - \sum_{k \in N - \{j\}} A_{ik} x_k \right) \tag{$\star$}$$

- new tableau A' consists of ($\star$) and $A_{B-\{i\}}\vec{x}_N = \vec{x}_{B-\{i\}}$ with ($\star$) substituted

Update

- assignment of x_i is updated to previously violated bound l_i or u_i,
- assignment of x_k is recomputed using ($\star$) and A' for all $k \in B - \{i\} \cup \{j\}$

DPLL(T) Simplex Algorithm (4)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Suitability

- basic variable x_i violates lower and/or upper bound
- pick nonbasic variable x_j such that
  - if $x_i < l_i$: $A_{ij} > 0$ and $x_j < u_j$ or $A_{ij} < 0$ and $x_j > l_j$
  - if $x_i > u_i$: $A_{ij} > 0$ and $x_j > l_j$ or $A_{ij} < 0$ and $x_j < u_j$
- problem is unsatisfiable if no suitable pivot exists

Bland's Rule

- pick lexicographically smallest (i, j) that is suitable pivot
- guarantees termination

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Example (due to B. Felgenhauer)

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_3 \\ x_4 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array}$$

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x_1	x_2	x_3	x_4
0	0	0	0

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$$\downarrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \end{array} & \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -4 & 0 & -4 & -8 \end{array}$$

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x_1	x_2	x_3	x_4
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$$\downarrow$$

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$$\downarrow$$

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$$\begin{array}{c|cc} & x_3 & x_4 \\ x_1 & -\frac{1}{3} & \frac{2}{3} \\ x_2 & \frac{2}{3} & -\frac{1}{3} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -\frac{10}{3} & -\frac{1}{3} & -4 & -7 \end{array}$$

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$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

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\begin{array}{c@{}c} & \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \end{array} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -\frac{10}{3} & -\frac{1}{3} & -4 & -7 \end{array} \arrow[d] & \begin{array}{c@{}c} & \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \end{array} & \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 3 & -4 & -5 & 2 \end{array} \\
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$$\downarrow$$

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$$\longrightarrow$$

$$\begin{array}{cc} & \begin{matrix} x_1 & x_2 \end{matrix} \\ \begin{matrix} x_3 \\ x_4 \end{matrix} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -4 & -9 & -6 \end{array}$$

$$\uparrow$$

$$\begin{array}{cc} & \begin{matrix} x_3 & x_2 \end{matrix} \\ \begin{matrix} x_1 \\ x_4 \end{matrix} & \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 3 & -4 & -5 & 2 \end{array}$$

$$\uparrow$$

$$\begin{array}{cc} & \begin{matrix} x_3 & x_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \end{matrix} & \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline \frac{7}{3} & -\frac{11}{3} & -5 & 1 \end{array}$$

Example (due to B. Felgenhauer)

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

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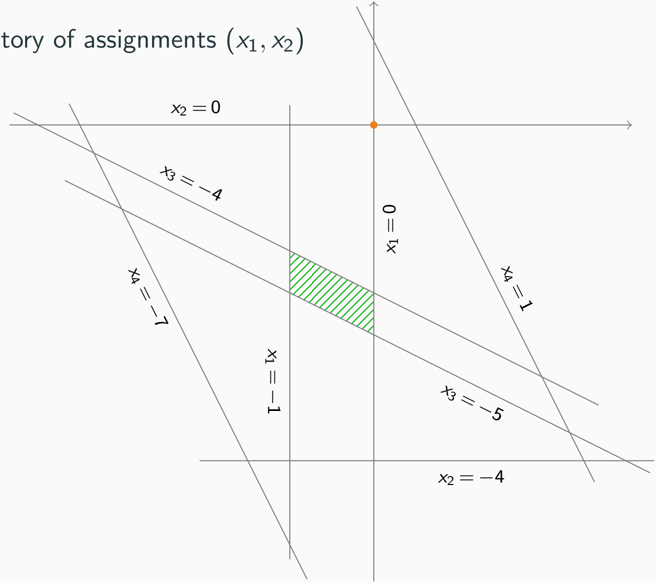
\end{tikzcd}$$

trajectory of assignments (x_1, x_2)



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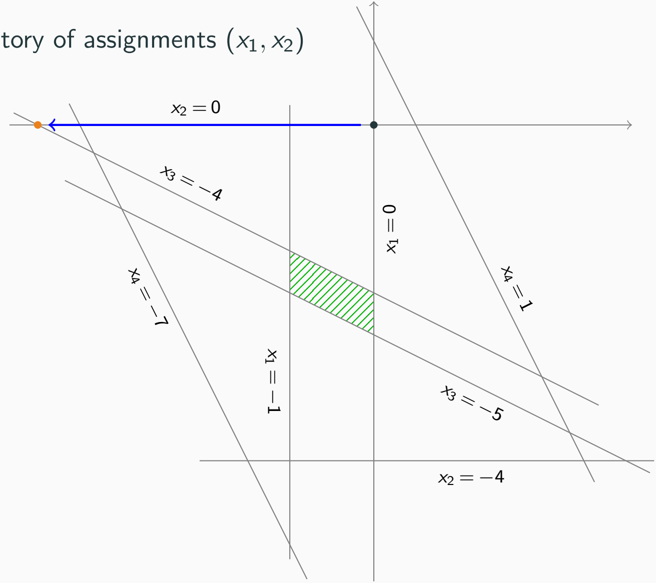

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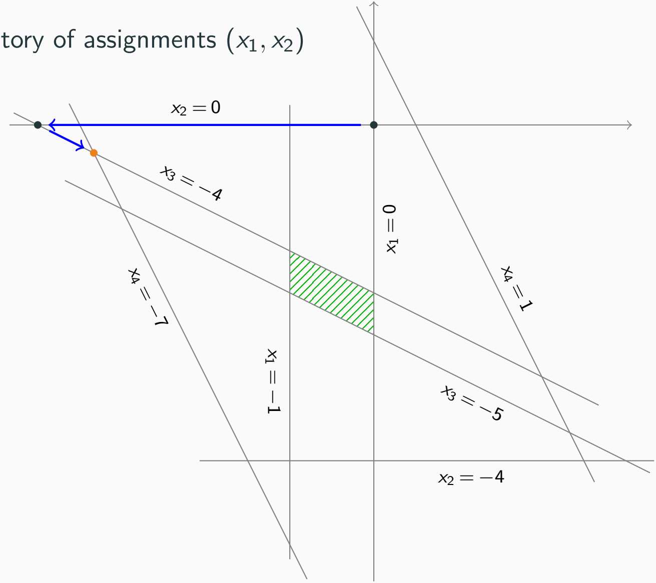

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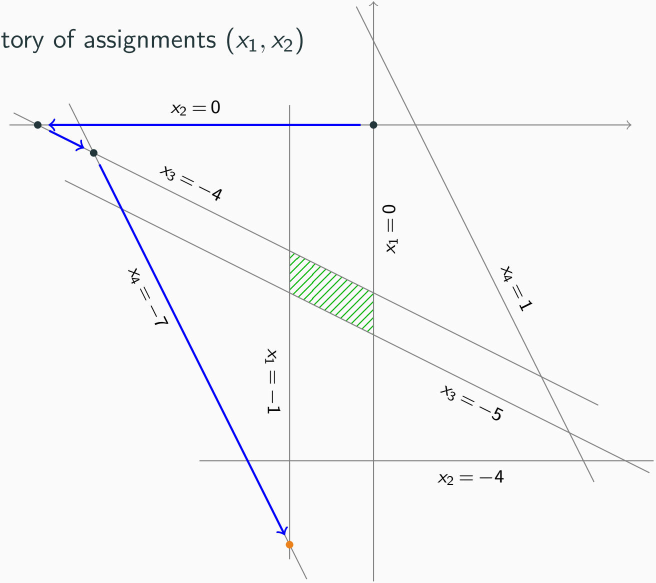

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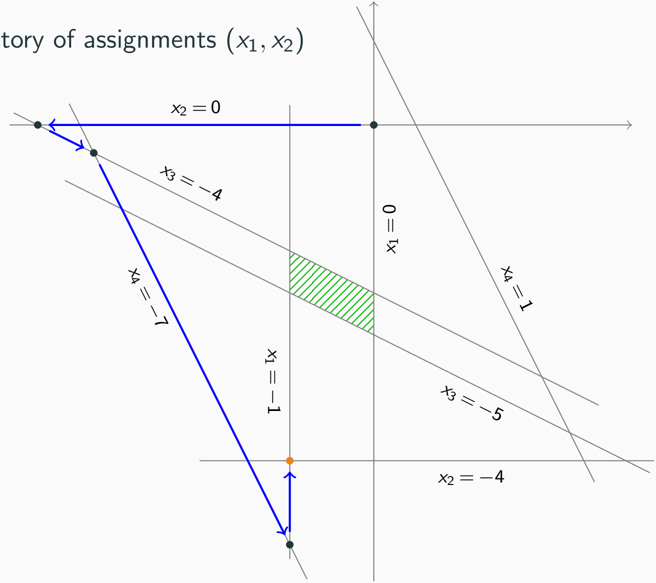

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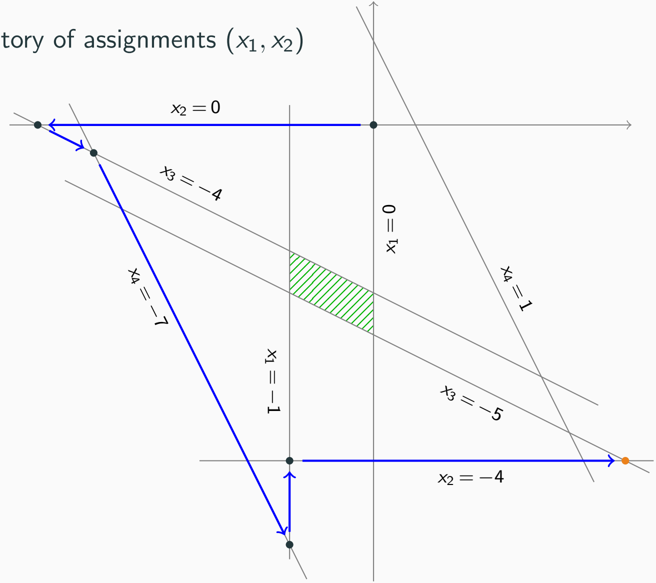

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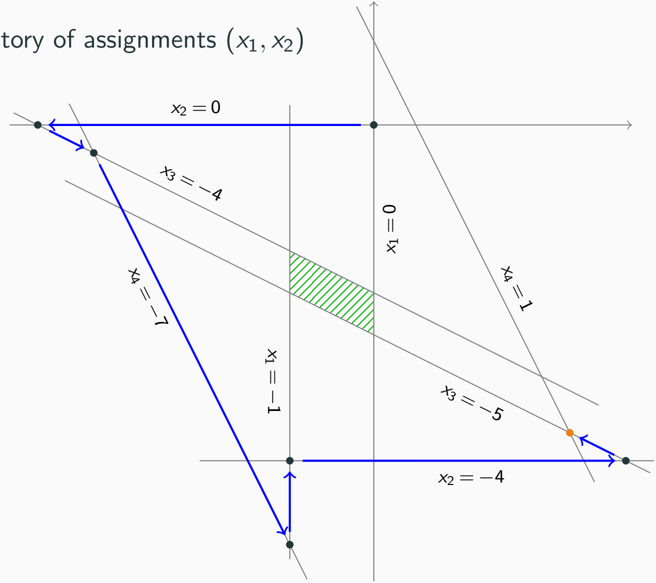

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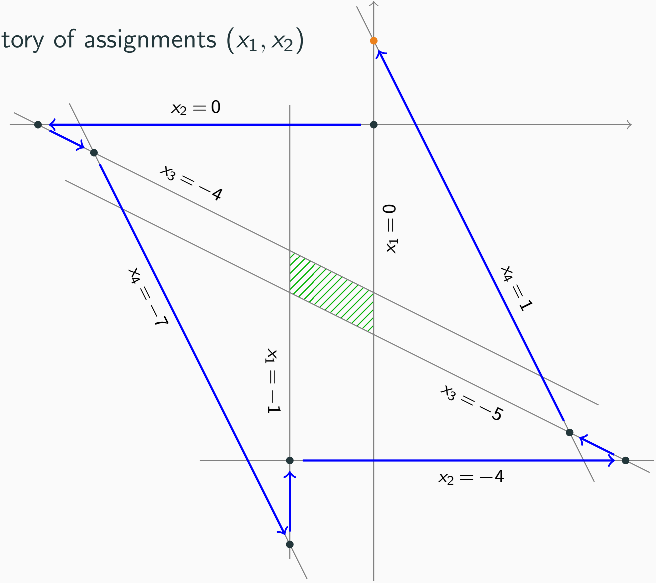

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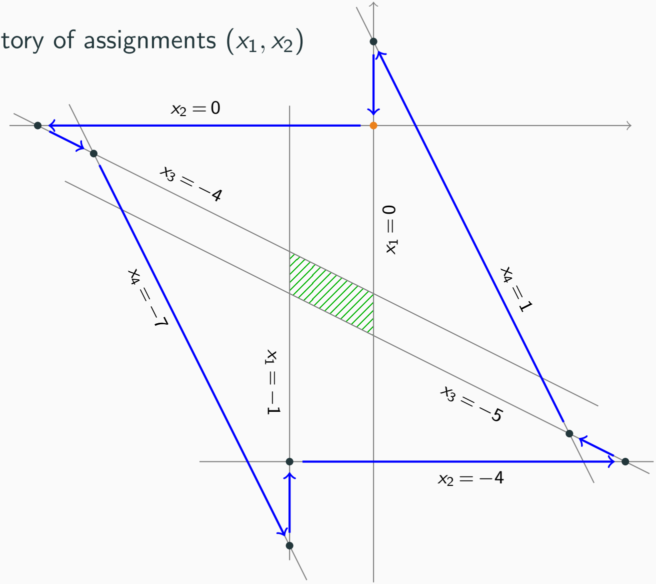

Example (due to B. Felgenhauer)

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_3 \\ x_4 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \end{array} & \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -4 & 0 & -4 & -8 \end{array}$$

violation of Bland's rule

$$\downarrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -\frac{10}{3} & -\frac{1}{3} & -4 & -7 \end{array}$$

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$$\begin{array}{ccc} & x_1 & x_2 \\ x_3 & 1 & 2 \\ x_4 & 2 & 1 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{ccc} & x_3 & x_2 \\ x_1 & 1 & -2 \\ x_4 & 2 & -3 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -4 & 0 & -4 & -8 \end{array}$$

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$$\downarrow$$

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$$\downarrow$$

$$\begin{array}{c|cc} & x_3 & x_1 \\ \hline x_2 & \frac{1}{2} & -\frac{1}{2} \\ x_4 & \frac{1}{2} & \frac{3}{2} \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -\frac{3}{2} & -4 & -\frac{7}{2} \end{array}$$

Example (due to B. Felgenhauer)

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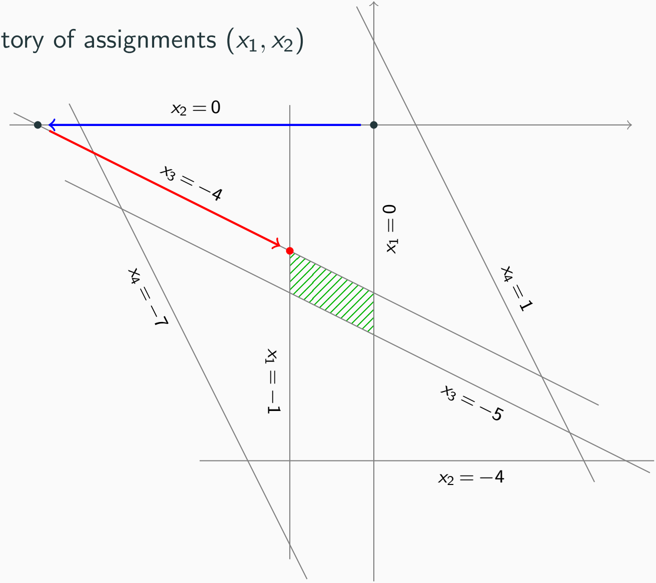
$$\downarrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_3 & x_1 \end{array} \\ \begin{array}{c} x_2 \\ x_4 \end{array} & \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -\frac{3}{2} & -4 & -\frac{7}{2} \end{array}$$

satisfying assignment

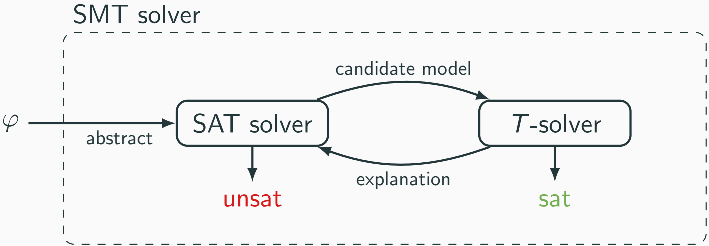
trajectory of assignments (x_1, x_2)



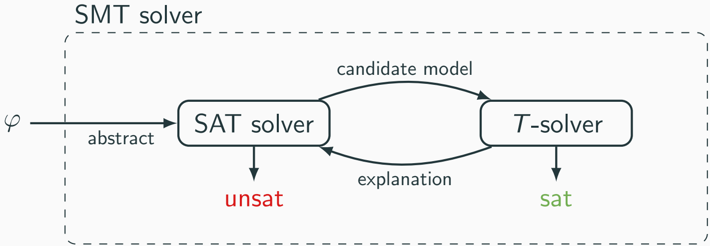
Outline

- Summary of Last Week
- Deciding Equality Logic
- Branch and Bound
- Cutting Planes

How to Be Lazy



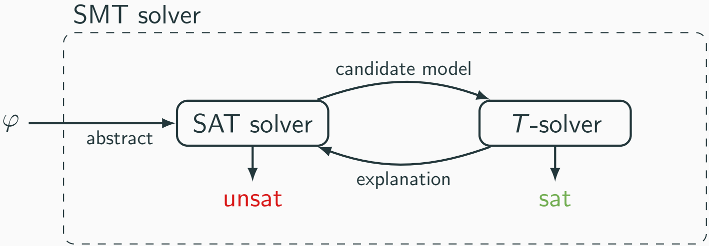
How to Be Lazy



Theory T

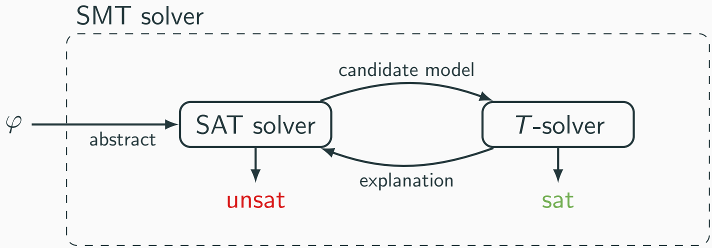
- equality logic
- equality + uninterpreted functions (EUF)
- linear real arithmetic (LRA)
- linear integer arithmetic (LIA)
- bitvectors (BV)
- arrays (A)

How to Be Lazy



Theory T	T**-solving method**	
equality logic		
equality + uninterpreted functions (EUF)	congruence closure	✓
linear real arithmetic (LRA)	DPLL(T) Simplex	✓
linear integer arithmetic (LIA)		
bitvectors (BV)		
arrays (A)		

How to Be Lazy



SMT solver

φ

abstract

SAT solver

candidate model

T-solver

explanation

unsat

sat

Theory T	T**-solving method**	
equality logic	equality graphs	
equality + uninterpreted functions (EUF)	congruence closure	✓
linear real arithmetic (LRA)	DPLL(T) Simplex	✓
linear integer arithmetic (LIA)	DPLL(T) Simplex + cuts	
bitvectors (BV)		
arrays (A)		

Deciding Equality Logic

Input to Satisfiability Problem for Equality Logic

conjunction φ of equality logic literals over set of variables V

Definitions

- $\varphi_{=}$ is set of positive literals (equality literals) in φ
- $\varphi_{\neq}$ is set of negative literals (inequality literals) in φ

Input to Satisfiability Problem for Equality Logic

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Example

conjunction of equality literals

$$\varphi = x_1 = x_2 \wedge x_1 \neq x_3 \wedge x_3 = x_5 \wedge x_4 \neq x_6 \wedge x_6 \neq x_7 \wedge x_5 = x_9 \wedge$$
$$x_5 = x_7 \wedge x_8 \neq x_9 \wedge x_9 = x_{10} \wedge x_7 = x_9 \wedge x_5 \neq x_{10}$$

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- $\varphi_=$: $\quad x_1 = x_2,\ x_3 = x_5,\ x_5 = x_7,\ x_9 = x_{10},\ x_7 = x_9,\ x_5 = x_9$

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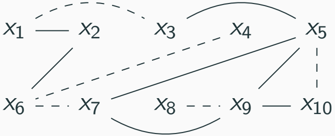
- equality graph



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Definitions

equality graph $G_=(\varphi) = (V, \varphi_=, \varphi_{\neq})$

- contradictory cycle is cycle with exactly one $\varphi_{\neq}$ edge

Input to Satisfiability Problem for Equality Logic

conjunction φ of equality logic literals over set of variables V

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- contradictory cycle is simple if it contains no node twice

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conjunction φ of equality logic literals over set of variables V

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Definitions

equality graph $G_=(\varphi) = (V, \varphi_=, \varphi_{\neq})$

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- contradictory cycle is simple if it contains no node twice

Lemma

φ is satisfiable iff $G_=(\varphi)$ contains no simple contradictory cycles

Example

conjunction of equality literals

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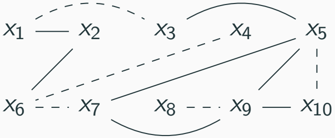

- contradictory cycles

Example

conjunction of equality literals

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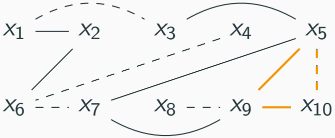

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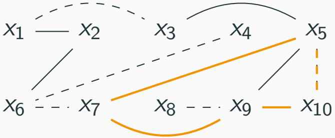
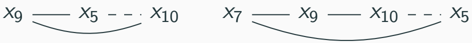


- contradictory cycles



Example

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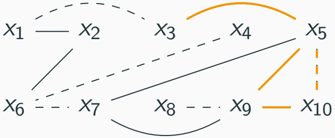

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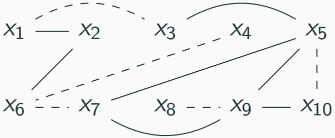

- contradictory cycles

simple simple not simple

Example

conjunction of equality literals

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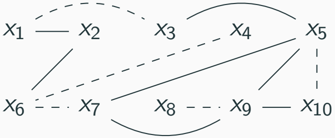
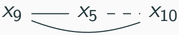
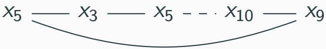


- contradictory cycles



simple

simple



not simple

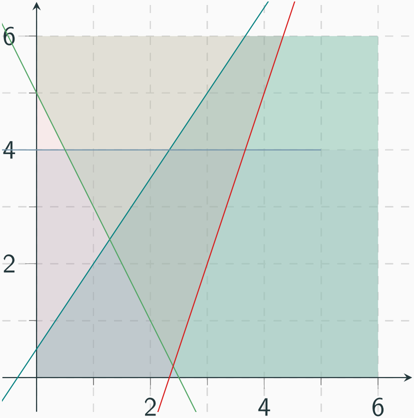
- unsatisfiable

Branch and Bound

Example

$$\begin{aligned} 3x - 2y &\geqslant -1 \\ y &\leqslant 4 \\ 2x + y &\geqslant 5 \\ 3x - y &\leqslant 7 \end{aligned}$$

- looking for solution in $\mathbb{Z}^2$



Example

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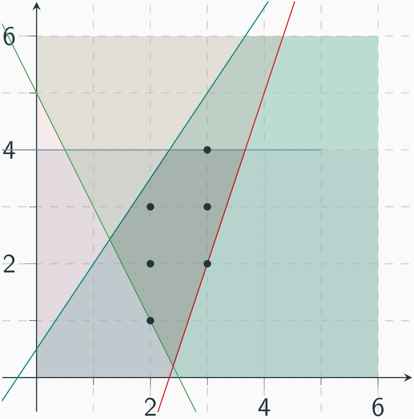
- looking for solution in $\mathbb{Z}^2$
- infinite $\mathbb{R}^2$ solution space, six solutions in $\mathbb{Z}^2$



Example

$$
\begin{aligned}
3x - 2y &\geqslant -1 \\
y &\leqslant 4 \\
2x + y &\geqslant 5 \\
3x - y &\leqslant 7
\end{aligned}
$$

- looking for solution in $\mathbb{Z}^2$
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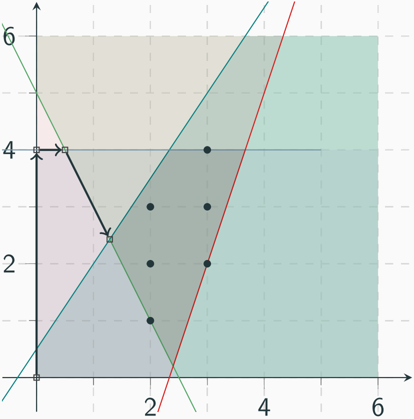
- Simplex returns $(\frac{9}{7}, \frac{17}{7})$



Example

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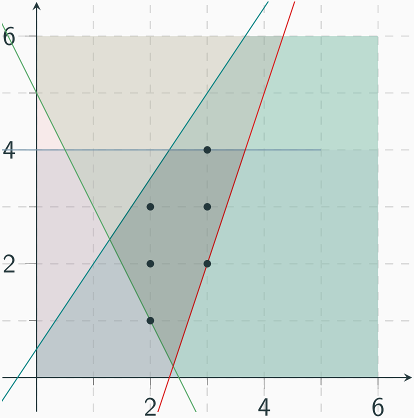

Idea (Branch and Bound)

- add constraints that exclude solution in $\mathbb{R}^2$ but do not change solutions in $\mathbb{Z}^2$

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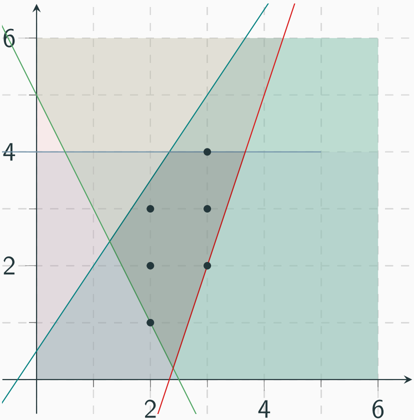

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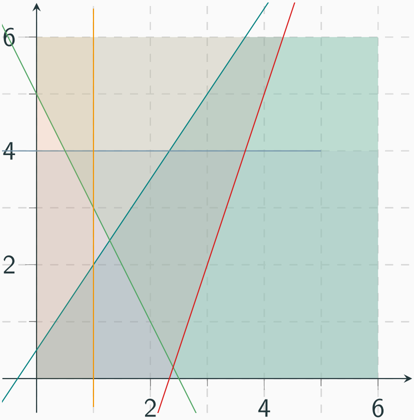

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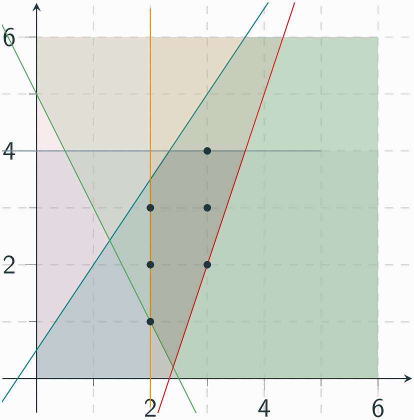

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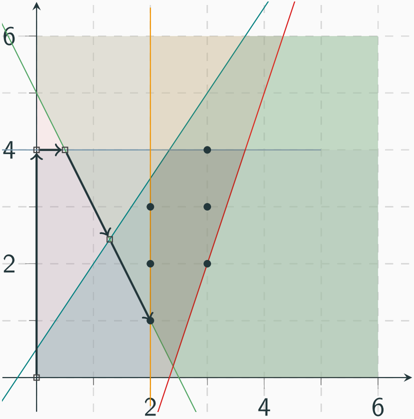

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  - $C \land x \leqslant 1$ unsatisfiable
  - $C \land x \geqslant 2$ satisfiable, Simplex can return $(2, 1)$

```
Algorithm BranchAndBound($\varphi$)
Input:  LIA constraint $\varphi$
Output: unsatisfiable, or satisfying assignment
  let res be result of deciding $\varphi$ over $\mathbb{R}$        ▷ e.g. by Simplex
  if res is unsatisfiable then
      return unsatisfiable
  else if res is solution over $\mathbb{Z}$ then
      return res
  else
      let $x$ be variable assigned non-integer value $q$ in res
      res = BranchAndBound($\varphi \wedge x \leqslant \lfloor q \rfloor$)
      return res $\neq$ unsatisfiable ? res : BranchAndBound($\varphi \wedge x \geqslant \lceil q \rceil$)
```

Remarks

- BranchAndBound might not terminate if solution space is unbounded

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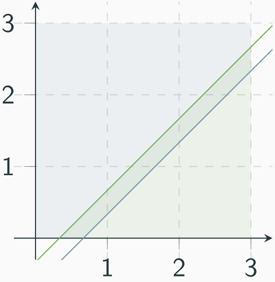

- $3x - 3y \geqslant 1 \wedge 3x - 3y \leqslant 2$ has no solution in $\mathbb{Z}^2$
- BranchAndBound keeps adding $x \geqslant n$, $y \geqslant m$

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- bounds for solution can be derived from tableau, but are often too high for efficient practical procedures

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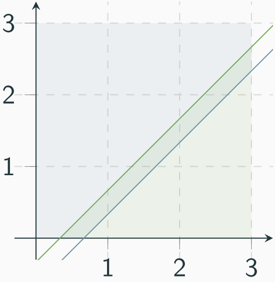

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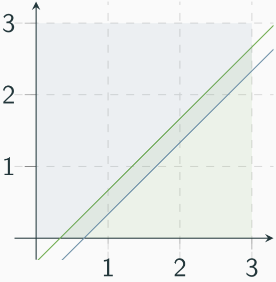
- use cutting planes to restrict solution space more efficiently

Example



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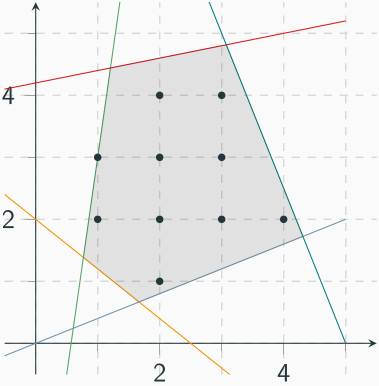
Cutting Planes

Example



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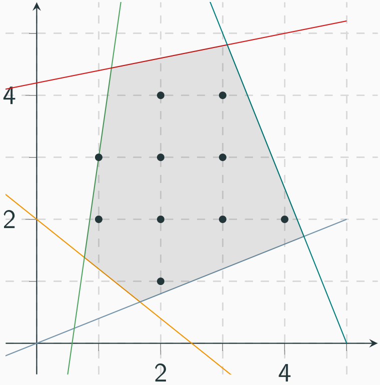

Definition (Cut)

given solution α to problem over $\mathbb{R}^n$, cut is inequality $a_1x_1 + \cdots + a_nx_n \leqslant b$ which is not satisfied by α but by every $\mathbb{Z}^n$-solution

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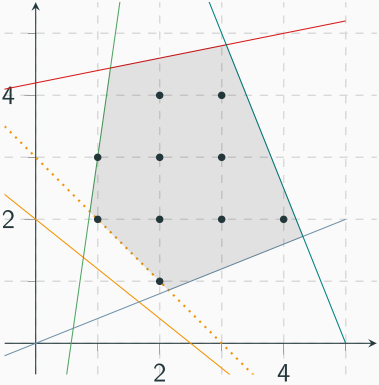

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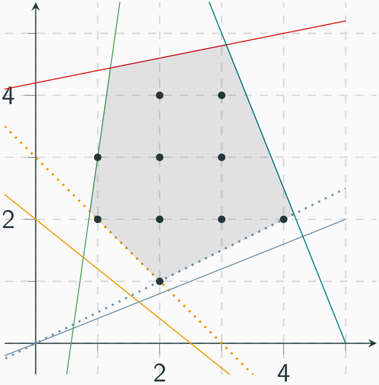

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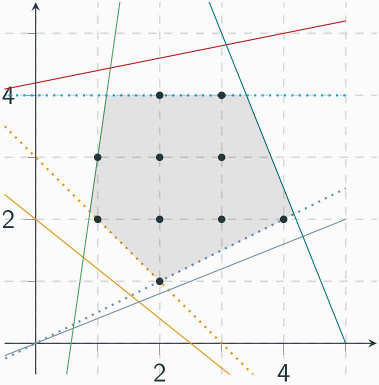

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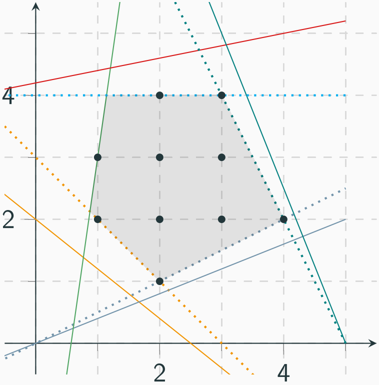

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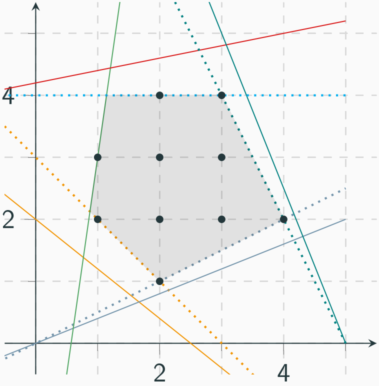

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Method

like in BranchAndBound, keep adding cuts until integer solution found

Gomory Cuts: Assumptions

- DPLL(T) Simplex returned solution α to

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

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Lemma (Gomory Cut)

cut is given by inequality

$$\sum_{j \in L^+} \frac{A_{ij}}{1-c}(x_j - l_j) - \sum_{j \in U^-} \frac{A_{ij}}{1-c}(u_j - x_j) - \sum_{j \in L^-} \frac{A_{ij}}{c}(x_j - l_j) + \sum_{j \in U^+} \frac{A_{ij}}{c}(u_j - x_j) \geqslant 1$$

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

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Proof (1)

- consider potential integer solution $\vec{x}$ to (1) and (2)

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Proof (1)

- consider potential integer solution $\vec{x}$ to (1) and (2)
- $\vec{x}$ satisfies i-th row of (1):

$$x_i = \sum_{j \in N} A_{ij} x_j \tag{3}$$

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$$x_i - \alpha(x_i) = \sum_{j \in N} A_{ij} (x_j - \alpha(x_j)) \tag{5}$$

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- subtract (4) from (3):

$$\begin{aligned} x_i - \alpha(x_i) &= \sum_{j \in N} A_{ij}(x_j - \alpha(x_j)) \\ &= \sum_{j \in L} A_{ij}(x_j - l_j) - \sum_{j \in U} A_{ij}(u_j - x_j) \end{aligned} \tag{5}$$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

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- abbreviate

$$\mathcal{L}^+ = \sum_{j\in L^+} A_{ij}(x_j - l_j)$$
$$\mathcal{L}^- = \sum_{j\in L^-} A_{ij}(x_j - l_j)$$

so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$

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  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$, can write $\alpha(x_i) = \lfloor \alpha(x_i) \rfloor + c$, so

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- for integer solution $\vec{x}$ left-hand side must be integer, so also right-hand side
- abbreviate

$$\mathcal{L}^+ = \sum_{j \in L^+} A_{ij}(x_j - l_j) \qquad \mathcal{U}^+ = \sum_{j \in U^+} A_{ij}(u_j - x_j)$$
$$\mathcal{L}^- = \sum_{j \in L^-} A_{ij}(x_j - l_j) \qquad \mathcal{U}^- = \sum_{j \in U^-} A_{ij}(u_j - x_j)$$

  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$, can write $\alpha(x_i) = \lfloor \alpha(x_i) \rfloor + c$, so

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- for integer solution $\vec{x}$ left-hand side must be integer, so also right-hand side
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$$\mathcal{L}^+ = \sum_{j \in L^+} A_{ij}(x_j - l_j) \qquad \mathcal{U}^+ = \sum_{j \in U^+} A_{ij}(u_j - x_j)$$
$$\mathcal{L}^- = \sum_{j \in L^-} A_{ij}(x_j - l_j) \qquad \mathcal{U}^- = \sum_{j \in U^-} A_{ij}(u_j - x_j)$$

  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$ and $\mathcal{L}^- \leqslant 0$,

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$, can write $\alpha(x_i) = \lfloor \alpha(x_i) \rfloor + c$, so

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- for integer solution $\vec{x}$ left-hand side must be integer, so also right-hand side
- abbreviate

$$\mathcal{L}^+ = \sum_{j \in L^+} A_{ij}(x_j - l_j) \qquad \mathcal{U}^+ = \sum_{j \in U^+} A_{ij}(u_j - x_j)$$
$$\mathcal{L}^- = \sum_{j \in L^-} A_{ij}(x_j - l_j) \qquad \mathcal{U}^- = \sum_{j \in U^-} A_{ij}(u_j - x_j)$$

  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$ and $\mathcal{L}^- \leqslant 0$, $\mathcal{U}^- \leqslant 0$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$, can write $\alpha(x_i) = \lfloor \alpha(x_i) \rfloor + c$, so

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- for integer solution $\vec{x}$ left-hand side must be integer, so also right-hand side
- abbreviate

$$\begin{aligned} \mathcal{L}^+ &= \sum_{j \in L^+} A_{ij}(x_j - l_j) & \qquad \mathcal{U}^+ &= \sum_{j \in U^+} A_{ij}(u_j - x_j) \\ \mathcal{L}^- &= \sum_{j \in L^-} A_{ij}(x_j - l_j) & \qquad \mathcal{U}^- &= \sum_{j \in U^-} A_{ij}(u_j - x_j) \end{aligned}$$

  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$ and $\mathcal{L}^- \leqslant 0$, $\mathcal{U}^- \leqslant 0$
- distinguish $\mathcal{L} \geqslant \mathcal{U}$ or $\mathcal{L} < \mathcal{U}$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
    - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
    - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
    - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$

since $\mathcal{L}^+ \geqslant \mathcal{L}$
and $\mathcal{U}^- \leqslant \mathcal{U}$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
    - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
    - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
    - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- if $\mathcal{L} < \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- if $\mathcal{L} < \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1$$)

since $\mathcal{U}^+ \geqslant \mathcal{U}$
and $\mathcal{L}^- \leqslant \mathcal{L}$

- if $\mathcal{L} < \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- if $\mathcal{L} < \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- if $\mathcal{L} < \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$
- terms $\mathcal{L}^+$, $\mathcal{U}^+$, $-\mathcal{L}^-$ and $-\mathcal{U}^-$ always non-negative, as well as c and $1 - c$

Proof (3)

- both sides are integer in equation
$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$
- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- if $\mathcal{L} < \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$
- terms $\mathcal{L}^+$, $\mathcal{U}^+$, $-\mathcal{L}^-$ and $-\mathcal{U}^-$ always non-negative, as well as c and $1 - c$
- add (7) and (8) to obtain cut
$$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) + \frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \qquad \blacksquare$$

Proof (3)

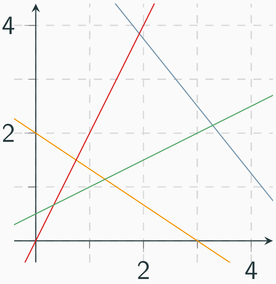
- both sides are integer in equation
$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$
- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- if $\mathcal{L} < \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$
- terms $\mathcal{L}^+$, $\mathcal{U}^+$, $-\mathcal{L}^-$ and $-\mathcal{U}^-$ always non-negative, as
- add (7) and (8) to obtain cut

$$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) + \frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1$$

the desired monster inequality!

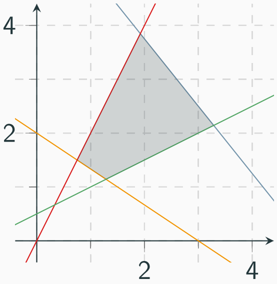
∎

Example



$$\begin{aligned} -2x - 3y &\leqslant -6 \\ -2x + y &\leqslant 0 \\ x - 2y &\leqslant -1 \\ 5x + 4y &\leqslant 25 \end{aligned}$$

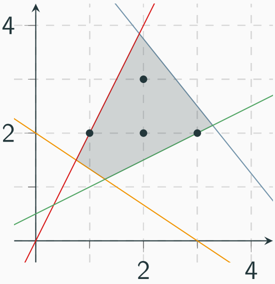
Example



$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
$$5x + 4y \leqslant 25$$

- infinite $\mathbb{R}^2$-solution space

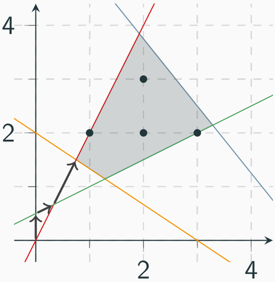
Example



$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
$$5x + 4y \leqslant 25$$

- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$

Example



$$-2x - 3y \leqslant -6$$

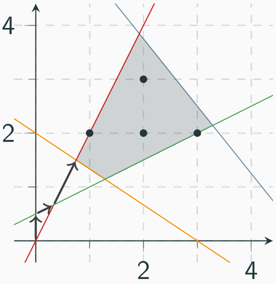
$$-2x + y \leqslant 0$$

$$x - 2y \leqslant -1$$

$$5x + 4y \leqslant 25$$

- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

Example



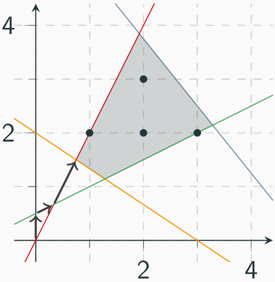
$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
$$5x + 4y \leqslant 25$$

- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

$$\begin{array}{ccc}
 & \begin{matrix} x & \quad y \end{matrix} & \\
\begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{pmatrix} -2 & -3 \\ -2 & 1 \\ 1 & -2 \\ 5 & 4 \end{pmatrix} & \begin{matrix} s_1 \leqslant -6 \\ s_2 \leqslant 0 \\ s_3 \leqslant -1 \\ s_4 \leqslant 25 \end{matrix}
\end{array}$$

intial tableau

Example



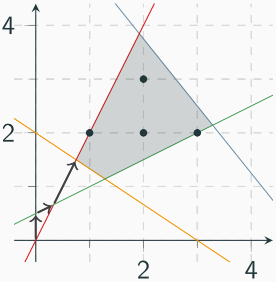
$$-2x-3y\leqslant -6$$
$$-2x+y\leqslant 0$$
$$x-2y\leqslant -1$$
$$5x+4y\leqslant 25$$

- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

$$\begin{array}{c}\begin{array}{c|cc} & x & y \\ s_1 & -2 & -3 \\ s_2 & -2 & 1 \\ s_3 & 1 & -2 \\ s_4 & 5 & 4 \end{array}\end{array} \quad \begin{array}{l} s_1\leqslant -6 \\ s_2\leqslant 0 \\ s_3\leqslant -1 \\ s_4\leqslant 25\end{array} \quad \longrightarrow \quad \begin{array}{c|cc} & s_2 & s_1 \\ s_3 & -\frac{7}{8} & \frac{3}{8} \\ x & -\frac{3}{8} & -\frac{1}{8} \\ y & \frac{1}{4} & -\frac{1}{4} \\ s_4 & -\frac{7}{8} & -\frac{13}{8} \end{array} \quad \begin{array}{ll} x=\frac{3}{4} & s_1=-6 \\ y=\frac{3}{2} & s_2=0 \\ & s_3=-2\frac{1}{4} \\ & s_4=9\frac{3}{4} \end{array}$$

intial tableau — final tableau — solution

Example



$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
$$5x + 4y \leqslant 25$$

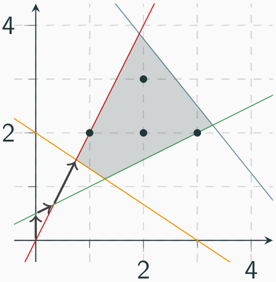
- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -2 & -3 \\ -2 & 1 \\ 1 & -2 \\ 5 & 4 \end{pmatrix} \end{array} \begin{array}{l} \\ s_1 \leqslant -6 \\ s_2 \leqslant 0 \\ s_3 \leqslant -1 \\ s_4 \leqslant 25 \end{array} \longrightarrow \begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} \\ -\frac{7}{8} & -\frac{13}{8} \end{pmatrix} \end{array} \quad \begin{array}{ll} x = \frac{3}{4} & s_1 = -6 \\ y = \frac{3}{2} & s_2 = 0 \\ & s_3 = -2\frac{1}{4} \\ & s_4 = 9\frac{3}{4} \end{array}$$

intial tableau — final tableau — solution

- nonbasic variables $s_2 = 0$ and $s_1 = -6$ at bounds

Example



$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
$$5x + 4y \leqslant 25$$

- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

$$\begin{array}{cc} & \begin{matrix} x & \quad y \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{pmatrix} -2 & -3 \\ -2 & 1 \\ 1 & -2 \\ 5 & 4 \end{pmatrix} \end{array} \begin{array}{l} s_1 \leqslant -6 \\ s_2 \leqslant 0 \\ s_3 \leqslant -1 \\ s_4 \leqslant 25 \end{array} \longrightarrow \begin{array}{cc} & \begin{matrix} s_2 & \quad s_1 \end{matrix} \\ \begin{matrix} s_3 \\ x \\ y \\ s_4 \end{matrix} & \begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} \\ -\frac{7}{8} & -\frac{13}{8} \end{pmatrix} \end{array} \quad \begin{array}{ll} x = \frac{3}{4} & s_1 = -6 \\ y = \frac{3}{2} & s_2 = 0 \\ & s_3 = -2\frac{1}{4} \\ & s_4 = 9\frac{3}{4} \end{array}$$

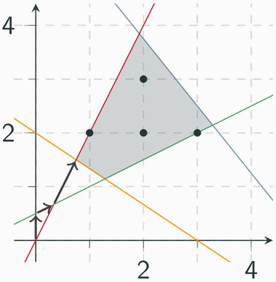
intial tableau

final tableau

solution

- nonbasic variables $s_2 = 0$ and $s_1 = -6$ at bounds, basic x is assigned $\frac{3}{4} \notin \mathbb{Z}$

Example



$$
\begin{aligned}
-2x - 3y &\leqslant -6\\
-2x + y &\leqslant 0\\
x - 2y &\leqslant -1\\
5x + 4y &\leqslant 25
\end{aligned}
$$

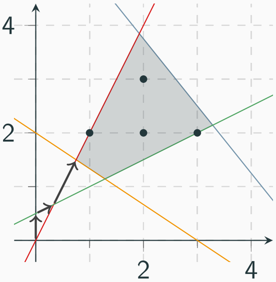
- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

$$
\begin{array}{cc}
 & \begin{array}{cc} x & y \end{array} \\
\begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} &
\begin{pmatrix} -2 & -3 \\ -2 & 1 \\ 1 & -2 \\ 5 & 4 \end{pmatrix}
\end{array}
\quad
\begin{array}{l}
s_1 \leqslant -6\\
s_2 \leqslant 0\\
s_3 \leqslant -1\\
s_4 \leqslant 25
\end{array}
\quad \longrightarrow \quad
\begin{array}{cc}
 & \begin{array}{cc} s_2 & s_1 \end{array} \\
\begin{array}{c} s_3 \\ x \\ y \\ s_4 \end{array} &
\begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} \\ -\frac{7}{8} & -\frac{13}{8} \end{pmatrix}
\end{array}
\quad
\begin{array}{l}
x = \frac{3}{4}\\
y = \frac{3}{2}
\end{array}
\quad
\begin{array}{l}
s_1 = -6\\
s_2 = 0\\
s_3 = -2\frac{1}{4}\\
s_4 = 9\frac{3}{4}
\end{array}
$$

intial tableau — final tableau — solution

- nonbasic variables $s_2 = 0$ and $s_1 = -6$ at bounds, basic x is assigned $\frac{3}{4} \notin \mathbb{Z}$
- from $c = \frac{3}{4}$

Example



$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
$$5x + 4y \leqslant 25$$

- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{pmatrix} -2 & -3 \\ -2 & 1 \\ 1 & -2 \\ 5 & 4 \end{pmatrix} \end{array} \quad \begin{array}{l} s_1 \leqslant -6 \\ s_2 \leqslant 0 \\ s_3 \leqslant -1 \\ s_4 \leqslant 25 \end{array} \quad \longrightarrow \quad \begin{array}{cc} & \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{array}{c} s_3 \\ x \\ y \\ s_4 \end{array} & \begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} \\ -\frac{7}{8} & -\frac{13}{8} \end{pmatrix} \end{array} \quad \begin{array}{ll} x = \frac{3}{4} & s_1 = -6 \\ y = \frac{3}{2} & s_2 = 0 \\ & s_3 = -2\frac{1}{4} \\ & s_4 = 9\frac{3}{4} \end{array}$$

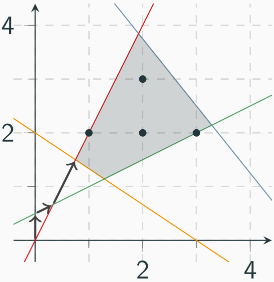
intial tableau

final tableau

solution

- nonbasic variables $s_2 = 0$ and $s_1 = -6$ at bounds, basic x is assigned $\frac{3}{4} \notin \mathbb{Z}$
- from $c = \frac{3}{4}$ obtain Gomory cut $\frac{1}{4}(\frac{3}{8}(0 - s_2) + \frac{1}{8}(-6 - s_1)) \geqslant 1$

Example



$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
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- infinite $\mathbb{R}^2$-solution space
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intial tableau

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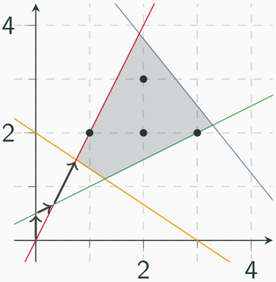
Example



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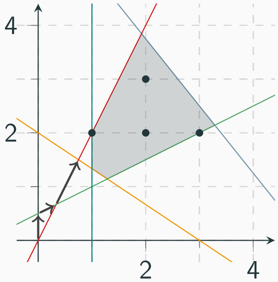

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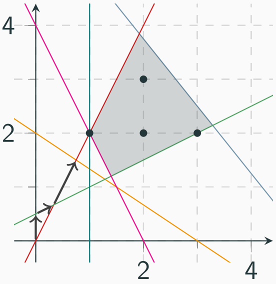

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intial tableau — final tableau — solution

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LIA Application: Finding Work Schedules

Example (Scheduling Problem)

Is there a six-week cyclic work schedule for 22 employees who work 8-hour shifts, 24/7, 5 working days per week such that the following holds: In morning and afternoon shifts 6 employees are present, in night shifts 3. Joe does only morning shifts, Sally does not work on Sundays. Nobody works more than 6 days in a row.

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- number of employees n
- set of shifts A (activities to be distributed)
- length of schedule (e.g. one week) and cyclicity
- requirement matrix R: R_{ij} is # employees required in shift i of day j
- prohibited shift sequences, maximal length of work blocks, ...

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LIA Encoding

- integer variable corresponding to employee for each activity
- cardinality constraints for requirement matrix
- ...

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