SAT and SMT Solving

Sarah Winkler

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Department of Computer Science
University of Innsbruck

Outline

- Summary of Last Week
- Deciding Equality Logic
- Branch and Bound
- Cutting Planes

Summary of Last Week

Definition (Theory of Linear Arithmetic over C)

- for variables $x_1, \ldots, x_n$, formulas built according to grammar

$$\varphi ::= \varphi \wedge \varphi \mid t = t \mid t < t \mid t \leqslant t$$
$$t ::= a_1 x_1 + \cdots + a_n x_n + b \qquad \text{for } a_1, \ldots, a_n, b \in \text{in carrier } C$$

- axioms are equality axioms plus calculation rules of arithmetic over C
- solution assigns values in C to $x_1, \ldots, x_n$

Definitions

- carrier $\mathbb{R}$: linear real arithmetic (LRA),
  DPLL(T) simplex algorithm is decision procedure
- carrier $\mathbb{Z}$: linear integer arithmetic (LIA)

DPLL(T) Simplex Algorithm (1)

- linear arithmetic constraint solving over real or rational variables
- $x_1, \ldots, x_n$ are split into basic variables $\vec{x}_B$ and nonbasic variables $\vec{x}_N$

Input

constraints plus upper and lower bounds for $x_1, \ldots, x_n$:

$$A\,\vec{x}_N = \vec{x}_B \qquad \text{with tableau } A \in \mathbb{R}^{|B|\times|N|} \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Output

satisfying assignment or "unsatisfiable"

Invariant

(1) is satisfied and (2) holds for all nonbasic variables x_i

DPLL(T) Simplex Algorithm (2)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Method

- if (2) holds for all basic variables, return current assignment
- otherwise select basic variable x_i (so $i \in B$) which violates (2)
- select suitable nonbasic variable x_j (so $j \in N$) such that x_i and x_j can be swapped in a pivoting step, resulting in new tableau

  $$A' \, x_{N'} = x_{B'}$$

  with $N' = N \cup \{i\} - \{j\}$ and $B' = B \cup \{j\} - \{i\}$
- change value of x_i to l_i or u_i and update values of basic variables accordingly

DPLL(T) Simplex Algorithm (3)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Pivoting

- swap basic x_i and non-basic x_j

$$x_i = \sum_{k \in N} A_{ik} x_k \qquad \Longrightarrow \qquad x_j = \frac{1}{A_{ij}} \left(x_i - \sum_{k \in N - \{j\}} A_{ik} x_k \right) \tag{$\star$}$$

- new tableau A' consists of ($\star$) and $A_{B-\{i\}}\vec{x}_N = \vec{x}_{B-\{i\}}$ with ($\star$) substituted

Update

- assignment of x_i is updated to previously violated bound l_i or u_i,
- assignment of x_k is recomputed using ($\star$) and A' for all $k \in B - \{i\} \cup \{j\}$

DPLL(T) Simplex Algorithm (4)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Suitability

- basic variable x_i violates lower and/or upper bound
- pick nonbasic variable x_j such that
  - if $x_i < l_i$: $A_{ij} > 0$ and $x_j < u_j$ or $A_{ij} < 0$ and $x_j > l_j$
  - if $x_i > u_i$: $A_{ij} > 0$ and $x_j > l_j$ or $A_{ij} < 0$ and $x_j < u_j$
- problem is unsatisfiable if no suitable pivot exists

Bland's Rule

- pick lexicographically smallest (i, j) that is suitable pivot
- guarantees termination

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Example (due to B. Felgenhauer)

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

$$\begin{array}{ccc} & x_1 & x_2 \\ x_3 & 1 & 2 \\ x_4 & 2 & 1 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$\downarrow$

$$\begin{array}{ccc} & x_3 & x_2 \\ x_1 & 1 & -2 \\ x_4 & 2 & -3 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -4 & 0 & -4 & -8 \end{array}$$

$\downarrow$

$$\begin{array}{ccc} & x_3 & x_4 \\ x_1 & -\frac{1}{3} & \frac{2}{3} \\ x_2 & \frac{2}{3} & -\frac{1}{3} \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -\frac{10}{3} & -\frac{1}{3} & -4 & -7 \end{array}$$

$\downarrow$

$$\begin{array}{ccc} & x_1 & x_4 \\ x_3 & -3 & 2 \\ x_2 & -2 & 1 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -5 & -11 & -7 \end{array}$$

$\longrightarrow$

$$\begin{array}{ccc} & x_1 & x_2 \\ x_3 & 1 & 2 \\ x_4 & 2 & 1 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -4 & -9 & -6 \end{array}$$

$\uparrow$

$$\begin{array}{ccc} & x_3 & x_2 \\ x_1 & 1 & -2 \\ x_4 & 2 & -3 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 3 & -4 & -5 & 2 \end{array}$$

$\uparrow$

$$\begin{array}{ccc} & x_3 & x_4 \\ x_1 & -\frac{1}{3} & \frac{2}{3} \\ x_2 & \frac{2}{3} & -\frac{1}{3} \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline \frac{7}{3} & -\frac{11}{3} & -5 & 1 \end{array}$$

$\uparrow$

$$\begin{array}{ccc} & x_1 & x_4 \\ x_3 & -3 & 2 \\ x_2 & -2 & 1 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 1 & 2 & 1 \end{array}$$

$\longleftarrow$ (back to the first state)

Example (due to B. Felgenhauer)

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

$$\begin{array}{c} \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{c} \\ x_1 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -4 & 0 & -4 & -8 \end{array}$$

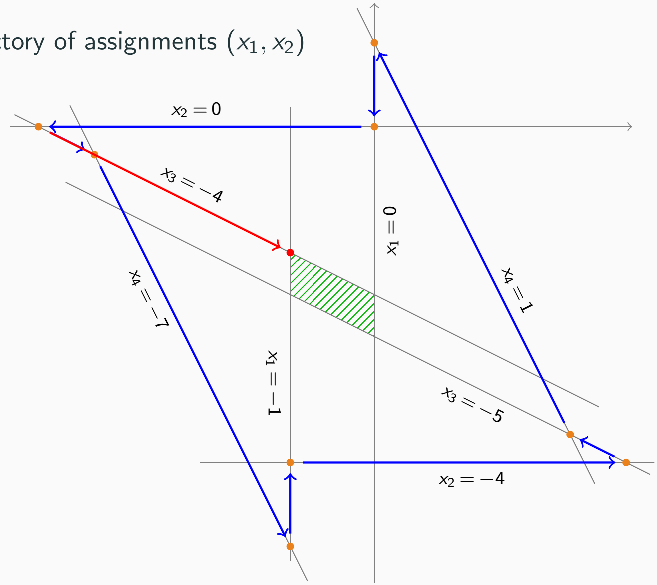
violation of Bland's rule

$$\downarrow$$

$$\begin{array}{c} \\ x_2 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} x_3 & x_1 \end{array} \\ \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -\frac{3}{2} & -4 & -\frac{7}{2} \end{array}$$

satisfying assignment

trajectory of assignments (x_1, x_2)



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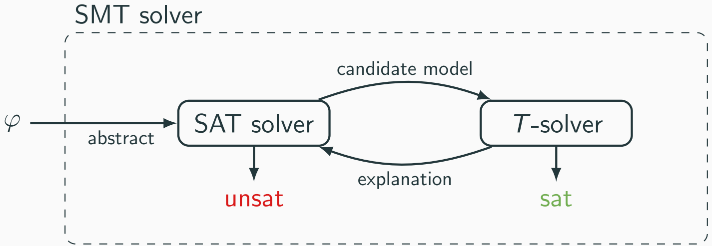
How to Be Lazy



Theory T	**T-solving method**	
equality logic	equality graphs	
equality + uninterpreted functions (EUF)	congruence closure	✓
linear real arithmetic (LRA)	DPLL(T) Simplex	✓
linear integer arithmetic (LIA)	DPLL(T) Simplex + cuts	
bitvectors (BV)		
arrays (A)		

Deciding Equality Logic

Input to Satisfiability Problem for Equality Logic

conjunction φ of equality logic literals over set of variables V

Definitions

- $\varphi_=$ is set of positive literals (equality literals) in φ
- $\varphi_{\neq}$ is set of negative literals (inequality literals) in φ
- equality graph is undirected graph $G_=(\varphi) = (V, \varphi_=, \varphi_{\neq})$

Definitions

equality graph $G_=(\varphi) = (V, \varphi_=, \varphi_{\neq})$

- contradictory cycle is cycle with exactly one $\varphi_{\neq}$ edge
- contradictory cycle is simple if it contains no node twice

Lemma

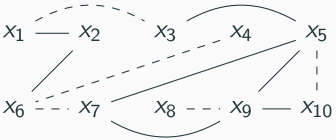
φ is satisfiable iff $G_=(\varphi)$ contains no simple contradictory cycles

Example

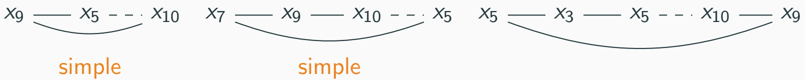
conjunction of equality literals

$$\varphi = x_1 = x_2 \wedge x_1 \neq x_3 \wedge x_3 = x_5 \wedge x_4 \neq x_6 \wedge x_6 \neq x_7 \wedge x_5 = x_9 \wedge$$
$$x_5 = x_7 \wedge x_8 \neq x_9 \wedge x_9 = x_{10} \wedge x_7 = x_9 \wedge x_5 \neq x_{10}$$

- $\varphi_=$: $\quad x_1 = x_2,\ x_3 = x_5,\ x_5 = x_7,\ x_9 = x_{10},\ x_7 = x_9,\ x_5 = x_9$
- $\varphi_{\neq}$: $\quad x_1 \neq x_3,\ x_4 \neq x_6,\ x_6 \neq x_7,\ x_8 \neq x_9,\ x_5 \neq x_{10}$
- equality graph



- contradictory cycles



- unsatisfiable

Input to Satisfiability Problem for Equality Logic

conjunction φ of equality logic literals over set of variables V

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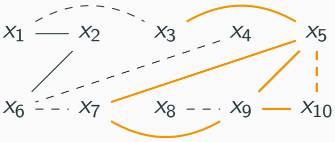
φ is satisfiable iff $G_=(\varphi)$ contains no simple contradictory cycles

Example

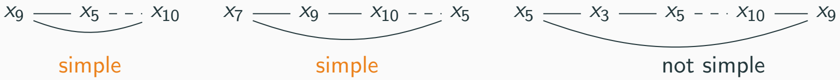
conjunction of equality literals

$$\varphi = x_1 = x_2 \land x_1 \neq x_3 \land x_3 = x_5 \land x_4 \neq x_6 \land x_6 \neq x_7 \land x_5 = x_9 \land$$
$$x_5 = x_7 \land x_8 \neq x_9 \land x_9 = x_{10} \land x_7 = x_9 \land x_5 \neq x_{10}$$

- $\varphi_=$: $x_1 = x_2,\ x_3 = x_5,\ x_5 = x_7,\ x_9 = x_{10},\ x_7 = x_9,\ x_5 = x_9$
- $\varphi_{\neq}$: $x_1 \neq x_3,\ x_4 \neq x_6,\ x_6 \neq x_7,\ x_8 \neq x_9,\ x_5 \neq x_{10}$
- equality graph



- contradictory cycles



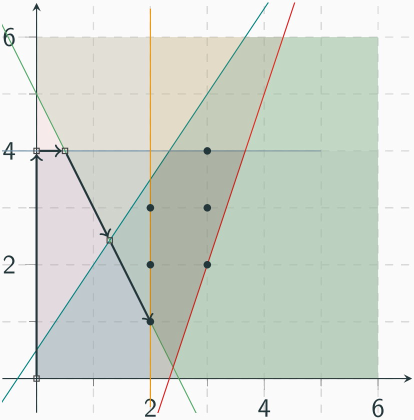
- unsatisfiable

Branch and Bound

Example

$3x - 2y \geqslant -1$
$y \leqslant 4$
$2x + y \geqslant 5$
$3x - y \leqslant 7$

- looking for solution in $\mathbb{Z}^2$
- infinite $\mathbb{R}^2$ solution space, six solutions in $\mathbb{Z}^2$
- Simplex returns $(\frac{9}{7}, \frac{17}{7})$



Idea (Branch and Bound)

- add constraints that exclude solution in $\mathbb{R}^2$ but do not change solutions in $\mathbb{Z}^2$
- in current solution $1 < x < 2$, so use Simplex on two augmented problems:
  - $C \wedge x \leqslant 1$ unsatisfiable
  - $C \wedge x \geqslant 2$ satisfiable, Simplex can return $(2, 1)$

Algorithm BranchAndBound(φ)

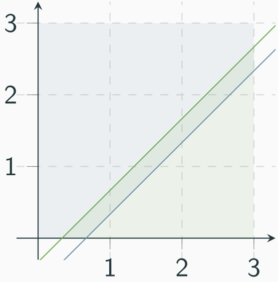
Input: LIA constraint φ
Output: unsatisfiable, or satisfying assignment

```
let res be result of deciding φ over ℝ                ▷ e.g. by Simplex
if res is unsatisfiable then
    return unsatisfiable
else if res is solution over ℤ then
    return res
else
    let x be variable assigned non-integer value q in res
    res = BranchAndBound(φ ∧ x ⩽ ⌊q⌋)
    return res ≠ unsatisfiable ? res : BranchAndBound(φ ∧ x ⩾ ⌈q⌉)
```

Remarks

- BranchAndBound might not terminate if solution space is unbounded
- bounds for solution can be derived from tableau, but are often too high for efficient practical procedures
- use cutting planes to restrict solution space more efficiently

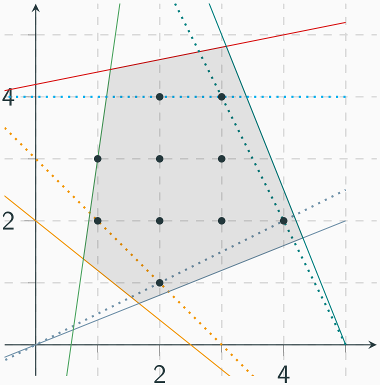
Example



- $3x - 3y \geqslant 1 \wedge 3x - 3y \leqslant 2$ has no solution in $\mathbb{Z}^2$
- BranchAndBound keeps adding $x \geqslant n$, $y \geqslant m$

Cutting Planes

Example



Definition (Cut)

given solution α to problem over $\mathbb{R}^n$, cut is inequality $a_1x_1 + \cdots + a_nx_n \leqslant b$ which is not satisfied by α but by every $\mathbb{Z}^n$-solution

Method

like in BranchAndBound, keep adding cuts until integer solution found

Gomory Cuts: Assumptions

- DPLL(T) Simplex returned solution α to

$$A\vec{x}_N = \vec{x}_B \tag{1}$$
$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

- for some $i \in B$ variable x_i is assigned $\alpha(x_i) \notin \mathbb{Z}$
- for all $j \in N$ value $\alpha(x_j)$ is l_j or u_j

Notation

- write $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$
- by assumption all nonbasic variables are assigned bounds, so can split

$$L = \{\, j \in N \mid \alpha(x_j) = l_j \,\} \qquad U = \{\, j \in N \mid \alpha(x_j) = u_j \,\}$$
$$L^+ = \{\, j \in L \mid A_{ij} \geqslant 0 \,\} \qquad U^+ = \{\, j \in U \mid A_{ij} \geqslant 0 \,\}$$
$$L^- = \{\, j \in L \mid A_{ij} < 0 \,\} \qquad U^- = \{\, j \in U \mid A_{ij} < 0 \,\}$$

Lemma (Gomory Cut)

cut is given by inequality

$$\sum_{j \in L^+} \frac{A_{ij}}{1-c}(x_j - l_j) - \sum_{j \in U^-} \frac{A_{ij}}{1-c}(u_j - x_j) - \sum_{j \in L^-} \frac{A_{ij}}{c}(x_j - l_j) + \sum_{j \in U^+} \frac{A_{ij}}{c}(u_j - x_j) \geqslant 1$$

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Proof (1)

- consider potential integer solution $\vec{x}$ to (1) and (2)
- $\vec{x}$ satisfies i-th row of (1):

$$x_i = \sum_{j \in N} A_{ij} x_j \tag{3}$$

- because α is solution have

$$\alpha(x_i) = \sum_{j \in N} A_{ij} \alpha(x_j) \tag{4}$$

- subtract (4) from (3):

$$\begin{aligned} x_i - \alpha(x_i) &= \sum_{j \in N} A_{ij}(x_j - \alpha(x_j)) \\ &= \sum_{j \in L} A_{ij}(x_j - l_j) - \sum_{j \in U} A_{ij}(u_j - x_j) \end{aligned} \tag{5}$$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$, can write $\alpha(x_i) = \lfloor \alpha(x_i) \rfloor + c$, so

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- for integer solution $\vec{x}$ left-hand side must be integer, so also right-hand side
- abbreviate

$$\mathcal{L}^+ = \sum_{j \in L^+} A_{ij}(x_j - l_j) \qquad \mathcal{U}^+ = \sum_{j \in U^+} A_{ij}(u_j - x_j)$$
$$\mathcal{L}^- = \sum_{j \in L^-} A_{ij}(x_j - l_j) \qquad \mathcal{U}^- = \sum_{j \in U^-} A_{ij}(u_j - x_j)$$

  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$ and $\mathcal{L}^- \leqslant 0$, $\mathcal{U}^- \leqslant 0$
- distinguish $\mathcal{L} \geqslant \mathcal{U}$ or $\mathcal{L} < \mathcal{U}$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1$$)

since $\mathcal{L}^+ \geqslant \mathcal{L}$
and $\mathcal{U}^- \leqslant \mathcal{U}$

- if $\mathcal{L} < \mathcal{U}$
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$

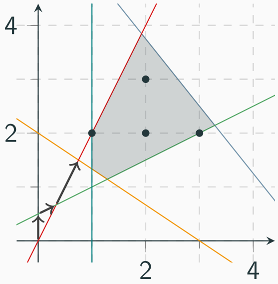
since $\mathcal{U}^+ \geqslant \mathcal{U}$
and $\mathcal{L}^- \leqslant \mathcal{L}$

- terms $\mathcal{L}^+$, $\mathcal{U}^+$, $-\mathcal{L}^-$ and $-\mathcal{U}^-$ always non-negative, as
- add (7) and (8) to obtain cut

$$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) + \frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1$$ ■

the desired
monster inequality!

Example



$$\begin{aligned} -2x - 3y &\leqslant -6 \\ -2x + y &\leqslant 0 \\ x - 2y &\leqslant -1 \\ 5x + 4y &\leqslant 25 \end{aligned}$$

- infinite $\mathbb{R}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -2 & -3 \\ -2 & 1 \\ 1 & -2 \\ 5 & 4 \end{pmatrix} \end{array} \begin{array}{l} s_1 \leqslant -6 \\ s_2 \leqslant 0 \\ s_3 \leqslant -1 \\ s_4 \leqslant 25 \end{array} \longrightarrow \begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} \\ -\frac{7}{8} & -\frac{13}{8} \end{pmatrix} \end{array} \quad \begin{array}{ll} x = \frac{3}{4} & s_1 = -6 \\ y = \frac{3}{2} & s_2 = 0 \\ & s_3 = -2\frac{1}{4} \\ & s_4 = 9\frac{3}{4} \end{array}$$

intial tableau — final tableau — solution

- nonbasic variables $s_2 = 0$ and $s_1 = -6$ at bounds, basic x is assigned $\frac{3}{4} \notin \mathbb{Z}$
- from $c = \frac{3}{4}$ obtain Gomory cut $\frac{1}{4}(\frac{3}{8}(0 - s_2) + \frac{1}{8}(-6 - s_1)) \geqslant 1$
- corresponds to $-\frac{3}{2}(-2x + y) - \frac{1}{2}(-2x - 3y) \geqslant 4$, simplified $x \geqslant 1$

LIA Application: Finding Work Schedules

Example (Scheduling Problem)

Is there a six-week cyclic work schedule for 22 employees who work 8-hour shifts, 24/7, 5 working days per week such that the following holds: In morning and afternoon shifts 6 employees are present, in night shifts 3. Joe does only morning shifts, Sally does not work on Sundays. Nobody works more than 6 days in a row.

Shift Schedule Requirements

- number of employees n
- set of shifts A (activities to be distributed)
- length of schedule (e.g. one week) and cyclicity
- requirement matrix R: R_{ij} is # employees required in shift i of day j
- prohibited shift sequences, maximal length of work blocks, ...

LIA Encoding

- integer variable corresponding to employee for each activity
- cardinality constraints for requirement matrix
- ...

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