

SAT and SMT Solving

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Outline

- Summary of Last Week
- Bit Vectors

Summary of Last Week

Input to Satisfiability Problem for Equality Logic

conjunction φ of equality logic literals over set of variables V

Definitions

- ▶ $\varphi_ =$ is set of positive literals (equality literals) in φ
- ▶ φ_{\neq} is set of negative literals (inequality literals) in φ
- ▶ equality graph is undirected graph $G_=(\varphi) = (V, \varphi_ =, \varphi_{\neq})$

Definitions

equality graph $G_=(\varphi) = (V, \varphi_ =, \varphi_{\neq})$

- ▶ contradictory cycle is cycle with exactly one φ_{\neq} edge
- ▶ contradictory cycle is simple if it contains no node twice

Lemma

φ is satisfiable iff $G_=(\varphi)$ contains no simple contradictory cycles

Idea (Branch and Bound)

- ▶ given \mathbb{R}^2 solution α , add constraints to exclude α but preserve \mathbb{Z}^2 solutions:
if $a < \alpha(x) < a_1$, use Simplex on problems $C \wedge x \leq a$ and $C \wedge x \geq a + 1$
- ▶ might not terminate if solution space is unbounded

Algorithm BranchAndBound(φ)

Input: LIA constraint φ

Output: unsatisfiable, or satisfying assignment

let res be result of deciding φ over \mathbb{R}

▷ e.g. by Simplex

if res is unsatisfiable **then**

return unsatisfiable

else if res is solution over \mathbb{Z} **then**

return res

else

let x be variable assigned non-integer value q in res

$res = \text{BranchAndBound}(\varphi \wedge x \leq \lfloor q \rfloor)$

return $res \neq \text{unsatisfiable} ? res : \text{BranchAndBound}(\varphi \wedge x \geq \lceil q \rceil)$

Definition (Cut)

given solution α to problem over \mathbb{R}^n , cut is inequality $a_1x_1 + \dots + a_nx_n \leq b$ which is not satisfied by α but by every \mathbb{Z}^n -solution

Gomory Cuts: Assumptions

- ▶ DPLL(T) Simplex returned solution α to

$$A\vec{x}_N = \vec{x}_B \quad (1)$$

$$-\infty \leq l_i \leq x_i \leq u_i \leq +\infty \quad (2)$$

- ▶ for some $i \in B$ have $\alpha(x_i) \notin \mathbb{Z}$ and for all $j \in N$ value $\alpha(x_j)$ is l_j or u_j

Notation

- ▶ write $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$
- ▶ by assumption all nonbasic variables are assigned bounds, so can split

$$L^+ = \{j \in L \mid \alpha(x_j) = l_j \text{ and } A_{ij} \geq 0\} \quad U^+ = \{j \in U \mid \alpha(x_j) = u_j \text{ and } A_{ij} \geq 0\}$$

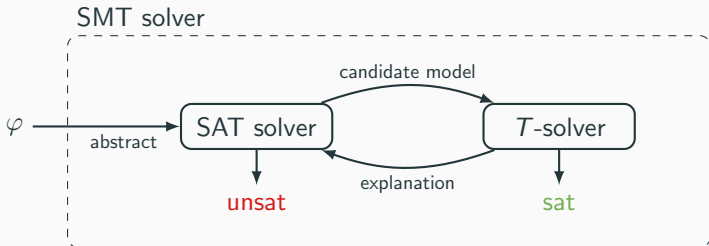
$$L^- = \{j \in L \mid \alpha(x_j) = l_j \text{ and } A_{ij} < 0\} \quad U^- = \{j \in U \mid \alpha(x_j) = u_j \text{ and } A_{ij} < 0\}$$

Lemma (Gomory Cut)

$$\sum_{j \in L^+} \frac{A_{ij}}{1-c} (x_j - l_j) - \sum_{j \in U^-} \frac{A_{ij}}{1-c} (u_j - x_j) - \sum_{j \in L^-} \frac{A_{ij}}{c} (x_j - l_j) + \sum_{j \in U^+} \frac{A_{ij}}{c} (u_j - x_j) \geq 1$$

Bit Vectors

How to Be Lazy



Theory T

- ▶ equality logic
- ▶ equality + uninterpreted functions (EUF)
- ▶ linear real arithmetic (LRA)
- ▶ linear integer arithmetic (LIA)
- ▶ bitvectors (BV)
- ▶ arrays (A)

T -solving method

- equality graphs ✓
- congruence closure ✓
- DPLL(T) Simplex ✓
- DPLL(T) Simplex + cuts ✓
- bit-blasting

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$formula := (formula \vee formula) \mid (formula \wedge formula) \mid (\neg formula) \mid atom$

$atom := term \ rel \ term \mid true \mid false$

$rel := = \mid \neq \mid \geq_u \mid \geq_s \mid >_u \mid >_s$

$term := (term \ binop \ term) \mid (unop \ term) \mid var \mid constant \mid term[i:j] \mid$
 $(formula \ ? \ term : \ term)$

$binop := + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid \mid \mid \wedge \mid ::$

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- ▶ axioms are equality axioms plus rules for arithmetic/comparison/bitwise operations on bit vectors of length k
- ▶ **solution** assigns bit list of length k to variables \mathbf{x}_k

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$\mathbf{x}[i:j]$ denotes $x_i \dots x_j$
and $::$ is concatenation

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- ▶ \mathbf{xn}_k is binary representation of hexadecimal n in k bits

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- ▶ $\mathbf{0}_1, \mathbf{3}_2, \mathbf{10}_4, \mathbf{1024}_{32}, \dots$
- ▶ $\mathbf{x0}_4, \mathbf{xa}_4, \mathbf{xb0}_8, \mathbf{x11cf}_{16}, \mathbf{xffffffff}_{32}, \dots$

More examples

negation uses two's complement

- ▶ $-\mathbf{a}_4 = \mathbf{a}_4$
satisfiable: $v(\mathbf{a}_4) = -\mathbf{8}_4 = \mathbf{x8}_4$
- ▶ $\mathbf{a}_8 \div_u \mathbf{b}_8 = \mathbf{a}_8 \ggg_u \mathbf{1}_8$
satisfiable: $v(\mathbf{a}_8) = \mathbf{8}_8$ and $v(\mathbf{b}_8) = \mathbf{2}_8$
- ▶ $\mathbf{a}_8 \& (\mathbf{a}_8 - \mathbf{1}_8) = \mathbf{0}_8$
satisfiable: $v(\mathbf{a}_8) = \mathbf{8}_8$ or $\mathbf{x0}_8, \mathbf{x1}_8, \mathbf{x2}_8, \mathbf{x4}_8, \mathbf{x8}_8, \mathbf{x10}_8, \mathbf{x20}_8, \mathbf{x40}_8, \mathbf{x80}_8$

Notation for Constants

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satisfied for powers of 2 (and 0)

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bit blasting transformation \mathbf{B} transforms BV formula into propositional formula:

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\mathbf{B}_r transforms atom into propositional formula

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bit blasting \mathbf{B}_t for term t
returns (result u , side condition φ)

Definition (Bit Blasting: Atoms)

for bit vectors \mathbf{x}_k and \mathbf{y}_k set

► equality

$$\mathbf{B}_r(\mathbf{x}_k = \mathbf{y}_k) = (x_k \leftrightarrow y_k) \wedge \cdots \wedge (x_1 \leftrightarrow y_1) \wedge (x_0 \leftrightarrow y_0)$$

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- ▶ inequality

$$\mathbf{B}_r(\mathbf{x}_k \neq \mathbf{y}_k) = (x_k \oplus y_k) \vee \cdots \vee (x_1 \oplus y_1) \vee (x_0 \oplus y_0)$$

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$$\mathbf{B}_r(\mathbf{x}_1 \geq_u \mathbf{y}_1) = y_0 \rightarrow x_0$$

$$\mathbf{B}_r(\mathbf{x}_{k+1} \geq_u \mathbf{y}_{k+1}) = (x_k \wedge \neg y_k) \vee ((x_k \leftrightarrow y_k) \wedge \mathbf{B}(\mathbf{x}[k-1:0] \geq \mathbf{y}[k-1:0]))$$

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$$\mathbf{B}(\mathbf{x}_k >_u \mathbf{y}_k) = \mathbf{B}(\mathbf{x}_k \geq \mathbf{y}_k) \wedge \mathbf{B}(\mathbf{x}_k \neq \mathbf{y}_k)$$

Definition (Bit Blasting: Bitwise Operations)

for bit vectors \mathbf{x}_k and \mathbf{y}_k use fresh variable \mathbf{z}_k and set

- ▶ bitwise and

$$\mathbf{B}_t(\mathbf{x}_k \& \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \quad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \wedge y_i)$$

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- ▶ bitwise negation

$$\mathbf{B}_t(-\mathbf{x}_k) = (\mathbf{z}_k, \varphi) \quad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow \neg x_i$$

Definition (Bit Blasting: Concatenation, Extraction, If)

► concatenation

$$\mathbf{B}_t(\mathbf{x}_k :: \mathbf{y}_m) = (\mathbf{x}_k \mathbf{y}_m, \top)$$

for bit vectors \mathbf{x}_k and \mathbf{y}_m

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$$\mathbf{B}_t(\mathbf{x}_k :: \mathbf{y}_m) = (\mathbf{x}_k \mathbf{y}_m, \top)$$

for bit vectors \mathbf{x}_k and \mathbf{y}_m

- ▶ extraction

$$\mathbf{B}_t(\mathbf{x}[n:m]) = (\mathbf{z}_{n-m+1}, \varphi) \quad \varphi = \bigwedge_{i=0}^{n-m} z_i \leftrightarrow x_{i+m}$$

for bit vector \mathbf{x}_k , $k > n \geq m \geq 0$ and fresh variable \mathbf{z}_{n-m+1}

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- ▶ if-then-else

$$\mathbf{B}_t(p ? \mathbf{x}_k : \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \quad \varphi = \bigwedge_{i=0}^{k-1} (p \rightarrow (z_i \leftrightarrow x_i)) \wedge (\neg p \rightarrow (z_i \leftrightarrow y_i))$$

for formula p and bit vectors \mathbf{x}_k and \mathbf{y}_k

Definition (Bit Blasting: Addition and Subtraction)

► addition

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ripple-carry adder:
 c_k are carry bits

for fresh variables \mathbf{s}_k and \mathbf{c}_k and $\text{min2}(a, b, d) = (a \wedge b) \vee (a \wedge d) \vee (b \wedge d)$

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- ▶ unary minus

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- ▶ subtraction

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = \mathbf{B}_t(\mathbf{x}_k + (-\mathbf{y}_k))$$

Definition (Bit Blasting: Multiplication and Division)

for bit vectors \mathbf{x}_k and \mathbf{y}_k set

- ▶ multiplication

$$\mathbf{B}_t(\mathbf{x}_k \times \mathbf{y}_k) = \mathbf{B}_t(\text{mul}(\mathbf{x}_k, \mathbf{y}_k, 0))$$

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$$\text{mul}(\mathbf{x}_k, \mathbf{y}_k, i) = \text{mul}(\mathbf{x}_k \ll \mathbf{1}_k, \mathbf{y}_k, i + 1) + (y_i ? \mathbf{x}_k : \mathbf{0}_k) \quad \text{if } i < k$$

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shift-and-add

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$$\text{mul}(\mathbf{x}_k, \mathbf{y}_k, i) = \text{mul}(\mathbf{x}_k \ll \mathbf{1}_k, \mathbf{y}_k, i + 1) + (y_i ? \mathbf{x}_k : \mathbf{0}_k) \quad \text{if } i < k$$

shift-and-add

- ▶ unsigned division

$$\mathbf{B}_t(\mathbf{x}_k \div_u \mathbf{y}_k) = (\mathbf{q}_k, \varphi)$$

$$\varphi = \mathbf{B}(\mathbf{y}_k \neq \mathbf{0}_k \rightarrow (\mathbf{q}_k \times \mathbf{y}_k + \mathbf{r}_k = \mathbf{x}_k \wedge \mathbf{r}_k < \mathbf{y}_k \wedge \mathbf{q}_k < \mathbf{x}_k))$$

for fresh variables \mathbf{q}_k and \mathbf{r}_k

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$(a_4 + b_4 <_u b_4) \wedge (a_4 \neq 10_4) \wedge (a_4 \& b_4 = 8_4)$ is expressed as

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- ▶ `bvand`, `bvor` are bitwise logical operations

Application: Verifying Compiler Optimizations (1)

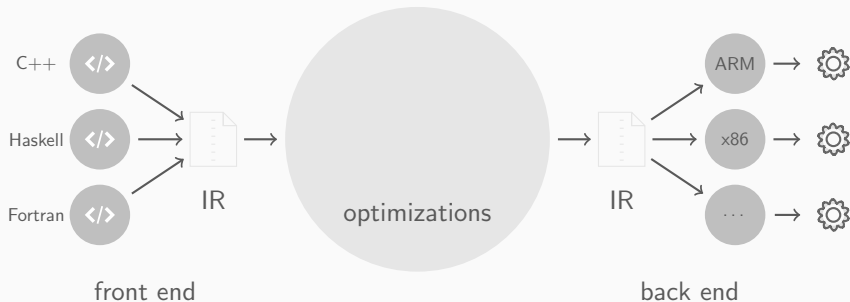
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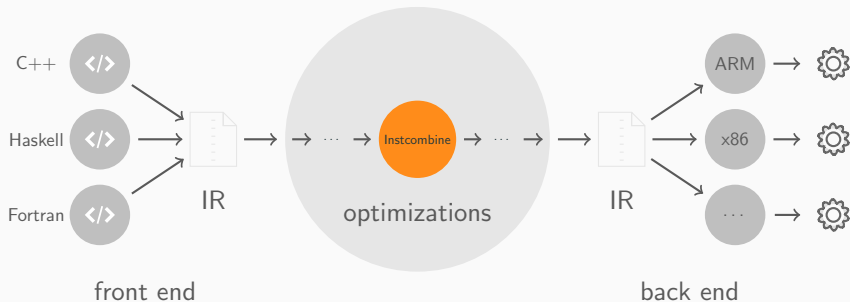
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Application: Verifying Compiler Optimizations (1)

LLVM

- ▶ open-source umbrella project: set of reusable toolchain components: libraries, assemblers, compilers, debuggers, ...
- ▶ compilation toolchain includes peephole optimizations in **Instcombine** pass



Application: Verifying Compiler Optimizations (2)

Instcombine Pass

- ▶ over 1000 algebraic simplifications of expressions
 - ▶ transform multiplies with constant power-of-two argument into shifts
 - ▶ bitwise operators with constant operands are always grouped so that shifts are performed first, then ors, then ands, then xors
 - ▶ changing bitwidth of variables
 - ▶ ...

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Example

```
int foo(int z) {  
  int x = 4 * (z | 1001);  
  return -256 & x;  
}
```

.....→

```
define i32 @foo(i32) #0 {  
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```

Instcombine →

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 - ▶ ...
- ▶ code is community maintained
- ▶ sometimes optimizations have **errors**—and compiler bugs are critical

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Application: Verifying Compiler Optimizations (3)

Alive Project

- ▶ represent `Instcombine` optimizations in domain-specific language, e.g.

```
Name: PR20186
```

```
%a = sdiv %X, C
```

```
%r = sub 0, %a
```

```
=>
```

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%r = sdiv %X, -C
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```

- ▶ check correctness by means of SMT encoding

```
(declare-const x (_ BitVec 32))
(declare-const c (_ BitVec 32))
(declare-const before (_ BitVec 32))
(declare-const after (_ BitVec 32))
(assert (= before (bvsdiv #x00000000 (bvsdiv x c))))
(assert (= after (bvsdiv x (bvneg c))))
(assert (not (= before after)))
(assert (not (= c #x00000000)))
(check-sat)
```



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(assert (not (= c #x00000000)))
(check-sat)
```



- ▶ **wrong** for `c = x = #x80000000`

Bit Vectors in python/z3

```
from z3 import *
x = BitVec('x', 32) # create variable named x with 32 bits
c = BitVec('c', 32)

before = BitVecVal(0, 32) - (x / c)
after = x / - c

solver = Solver()
solver.add(c != BitVecVal(0, 32)) # exclude case where c=0
solver.add(after != before)

result = solver.check()
if result == z3.sat:
    m = solver.model()
    print m[x], m[c] # 2147483648 2147483648
    print m.eval(before), m.eval(after) # 4294967295 1
```

Application: Detecting Nontermination

```
int bsearch(int a[], int k, unsigned int lo, unsigned int hi) {
    unsigned int mid;
    while (lo < hi) {
        mid = (lo + hi)/2;
        if (a[mid] < k)
            lo = mid + 1;
        else if (a[mid] > k)
            hi = mid - 1;
        else
            return mid;
    }
    return -1;
}
```

- ▶ (former) implementation of binary search in Java library
- ▶ loops for inputs $lo=1$ and $hi=UINT_MAX$ if $a[0] < k$.
- ▶ SMT encoding can find values such that parameters stay the same in recursive call



Daniel Kroening and Ofer Strichman

Bit Vectors

Chapter 6 of Decision Procedures — An Algorithmic Point of View
Springer, 2008



Nuno Lopes, David Menendez, Sarantosh Nagarakatte, and John Regehr.

Provably Correct Peephole Optimizations with Alive.

Proc. 36th PLDI, pp. 22–32, 2013.