SAT and SMT Solving

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Outline

- Summary of Last Week
- Bit Vectors

Summary of Last Week

Input to Satisfiability Problem for Equality Logic

conjunction φ of equality logic literals over set of variables V

Definitions

- $\varphi_=$ is set of positive literals (equality literals) in φ
- $\varphi_\neq$ is set of negative literals (inequality literals) in φ
- equality graph is undirected graph $G_=(\varphi) = (V, \varphi_=, \varphi_\neq)$

Definitions

equality graph $G_=(\varphi) = (V, \varphi_=, \varphi_\neq)$

- contradictory cycle is cycle with exactly one $\varphi_\neq$ edge
- contradictory cycle is simple if it contains no node twice

Lemma

φ is satisfiable iff $G_=(\varphi)$ contains no simple contradictory cycles

Idea (Branch and Bound)

- given $\mathbb{R}^2$ solution α, add constraints to exclude α but preserve $\mathbb{Z}^2$ solutions:
  if $a < \alpha(x) < a_1$, use Simplex on problems $C \wedge x \leqslant a$ and $C \wedge x \geqslant a + 1$
- might not terminate if solution space is unbounded

Algorithm BranchAndBound(φ)

Input: LIA constraint φ
Output: unsatisfiable, or satisfying assignment

```
let res be result of deciding φ over ℝ                ▷ e.g. by Simplex
if res is unsatisfiable then
    return unsatisfiable
else if res is solution over ℤ then
    return res
else
    let x be variable assigned non-integer value q in res
    res = BranchAndBound(φ ∧ x ⩽ ⌊q⌋)
    return res ≠ unsatisfiable ? res : BranchAndBound(φ ∧ x ⩾ ⌈q⌉)
```

Definition (Cut)

given solution α to problem over $\mathbb{R}^n$, cut is inequality $a_1x_1 + \cdots + a_nx_n \leqslant b$ which is not satisfied by α but by every $\mathbb{Z}^n$-solution

Gomory Cuts: Assumptions

- DPLL(T) Simplex returned solution α to

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

- for some $i \in B$ have $\alpha(x_i) \notin \mathbb{Z}$ and for all $j \in N$ value $\alpha(x_j)$ is l_j or u_j

Notation

- write $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$
- by assumption all nonbasic variables are assigned bounds, so can split

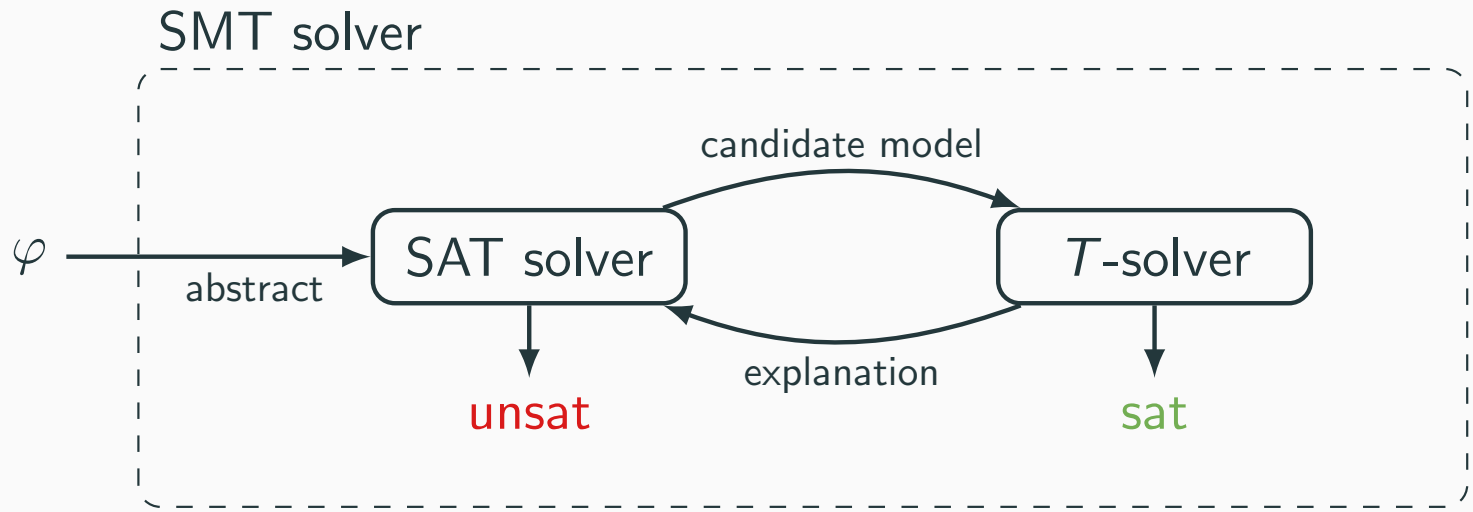
$$L^+ = \{\, j \in L \mid \alpha(x_j) = l_j \text{ and } A_{ij} \geqslant 0 \,\} \quad U^+ = \{\, j \in U \mid \alpha(x_j) = u_j \text{ and } A_{ij} \geqslant 0$$
$$L^- = \{\, j \in L \mid \alpha(x_j) = l_j \text{ and } A_{ij} < 0 \,\} \quad U^- = \{\, j \in U \mid \alpha(x_j) = u_j \text{ and } A_{ij} < 0$$

Lemma (Gomory Cut)

$$\sum_{j \in L^+} \frac{A_{ij}}{1-c}(x_j - l_j) - \sum_{j \in U^-} \frac{A_{ij}}{1-c}(u_j - x_j) - \sum_{j \in L^-} \frac{A_{ij}}{c}(x_j - l_j) + \sum_{j \in U^+} \frac{A_{ij}}{c}(u_j - x_j) \geqslant 1$$

Bit Vectors

How to Be Lazy



Theory T	T**-solving method**	
▶ equality logic	equality graphs	✓
▶ equality + uninterpreted functions (EUF)	congruence closure	✓
▶ linear real arithmetic (LRA)	DPLL(T) Simplex	✓
▶ linear integer arithmetic (LIA)	DPLL(T) Simplex + cuts	✓
▶ bitvectors (BV)	bit-blasting	
▶ arrays (A)		

Definition (Bit Vector Theory)

- variable $\mathbf{x}_k$ is list of length k of propositional variables $x_{k-1} \ldots x_2 x_1 x_0$
- constant n_k is bit list of length k
- formulas built according to grammar

$$
\begin{aligned}
\mathit{formula} &:= (\mathit{formula} \vee \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\neg \mathit{formula}) \mid \mathit{atom} \\
\mathit{atom} &:= \mathit{term}\ \mathit{rel}\ \mathit{term} \mid \mathit{true} \mid \mathit{false} \\
\mathit{rel} &:= \ = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\mathit{term} &:= (\mathit{term}\ \mathit{binop}\ \mathit{term}) \mid (\mathit{unop}\ \mathit{term}) \mid \mathit{var} \mid \mathit{constant} \mid \mathit{term}[i{:}j] \mid \\
&\qquad (\mathit{formula}\ ?\ \mathit{term} : \mathit{term}) \\
\mathit{binop} &:= + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid \mid \mid \hat{} \mid :: \\
\mathit{unop} &:= \sim \mid -
\end{aligned}
$$

- axioms are equality axioms plus rules for arithmetic/comparison/bitwise operations on bit vectors of length k
- solution assigns bit list of length k to variables $\mathbf{x}_k$

Examples

- $\mathbf{x}_4 + \mathbf{y}_4 = \mathbf{7}_4$
  satisfiable: $v(\mathbf{x}_4) = \mathbf{4}_4$ and $v(\mathbf{y}_4) = \mathbf{3}_4$
- $\mathbf{x}_4 + \mathbf{2}_4 <_u \mathbf{x}_4$
  satisfiable: $v(\mathbf{x}_4) = \mathbf{15}_4$

overflow semantics!

- $(\mathbf{x}_4 \times \mathbf{y}_4 = \mathbf{6}_4) \wedge (\mathbf{x}_4 \,\&\, \mathbf{y}_4 = \mathbf{2}_4)$
  satisfiable: $v(\mathbf{x}_4) = \mathbf{3}_4$, $v(\mathbf{y}_4) = \mathbf{2}_4$
- $(\mathbf{x}_4 \geqslant_u \mathbf{y}_4) \wedge \neg(\mathbf{x}_4 \geqslant_s \mathbf{y}_4)$
  satisfiable: $v(\mathbf{x}_4) = \mathbf{8}_4$, $v(\mathbf{y}_4) = \mathbf{0}_4$
- $(\mathbf{x}_4 \ll \mathbf{2}_4 = \mathbf{12}_4) \wedge (\mathbf{x}_4 + \mathbf{1}_4 = \mathbf{12}_4)$
  satisfiable: $v(\mathbf{x}_4) = \mathbf{11}_4$
- $(\mathbf{8}_4 \gg_u \mathbf{2}_4 = \mathbf{2}_4) \wedge (\mathbf{8}_4 \gg_s \mathbf{2}_4 = \mathbf{14}_4)$
  holds
- $(\mathbf{x}_4[1{:}0] :: \mathbf{x}_4[3{:}2] = \mathbf{2}_4) \wedge (\mathbf{y}_4[2{:}0] = \mathbf{7}_3)$
  satisfiable: $v(\mathbf{x}_4) = \mathbf{8}_4$ and $v(\mathbf{y}_4) = \mathbf{15}_4$

unsigned $\gg_u$ shifts in 0s
signed $\gg_s$ shifts in sign bits

$\mathbf{x}[i{:}j]$ denotes $x_i \ldots x_j$
and :: is concatenation

Notation for Constants

- $\mathbf{n}_k$ is binary representation of n in k bits
- $\mathbf{xn}_k$ is binary representation of hexadecimal n in k bits

Example

- $\mathbf{0}_1$, $\mathbf{3}_2$, $\mathbf{10}_4$, $\mathbf{1024}_{32}$,...
- $\mathbf{x0}_4$, $\mathbf{xa}_4$, $\mathbf{xb0}_8$, $\mathbf{x11cf}_{16}$,$\mathbf{xffffffff}_{32}$,...

More examples

negation uses two's complement

- $-\mathbf{a}_4 = \mathbf{a}_4$
  satisfiable: $v(\mathbf{a}_4) = -\mathbf{8}_4 = \mathbf{x8}_4$
- $\mathbf{a}_8 \div_u \mathbf{b}_8 = \mathbf{a}_8 \gg_u \mathbf{1}_8$
  satisfiable: $v(\mathbf{a}_8) = \mathbf{8}_8$ and $v(\mathbf{b}_8) = \mathbf{2}_8$

satisfied for powers of 2 (and 0)

- $\mathbf{a}_8 \,\&\, (\mathbf{a}_8 - \mathbf{1}_8) = \mathbf{0}_8$
  satisfiable: $v(\mathbf{a}_8) = \mathbf{8}_8$ or $\mathbf{x0}_8$, $\mathbf{x1}_8$, $\mathbf{x2}_8$, $\mathbf{x4}_8$, $\mathbf{x8}_8$, $\mathbf{x10}_8$, $\mathbf{x20}_8$, $\mathbf{x40}_8$, $\mathbf{x80}_8$

Remarks

- theory is decidable because carrier is finite
- common decision procedures use translation to SAT (bit blasting)
  - eager: no DPLL(T), bit-blast entire formula to SAT problem
  - lazy: second SAT solver as BV theory solver, bit-blast only BV atoms
- solvers heavily rely on preprocessing via rewriting

Example (Preprocessing)

$$\mathbf{x}_1 \neq \mathbf{0}_1 \wedge (\mathbf{y}_3 :: \mathbf{x}_1) \,\%_u\, \mathbf{2}_4 = \mathbf{0}_4 \rightarrow \mathbf{x}_1 = \mathbf{1}_1 \wedge (\mathbf{y}_3 :: \mathbf{x}_1) \,\%_u\, \mathbf{2}_4 = \mathbf{0}_4$$
$$\rightarrow (\mathbf{y}_3 :: \mathbf{1}_1) \,\%_u\, \mathbf{2}_4 = \mathbf{0}_4 \rightarrow \mathsf{F}$$

Definition (Bit Blasting: Formulas)

bit blasting transformation $\mathbf{B}$ transforms BV formula into propositional formula:

$$\mathbf{B}(\varphi \vee \psi) = \mathbf{B}(\varphi) \vee \mathbf{B}(\psi)$$
$$\mathbf{B}(\varphi \wedge \psi) = \mathbf{B}(\varphi) \wedge \mathbf{B}(\psi)$$
$$\mathbf{B}(\neg\varphi) = \neg\mathbf{B}(\varphi)$$
$$\mathbf{B}(t_1 \; \mathit{rel} \; t_2) = \mathbf{B}_r(u_1 \; \mathit{rel} \; u_2) \wedge \varphi_1 \wedge \varphi_2 \quad \text{if } \mathbf{B}_t(t_1) = (u_1, \varphi_1) \text{ and } \mathbf{B}_t(t_2) = (u_2, \varphi_2)$$

bit blasting $\mathbf{B}_t$ for term t returns (result u, side condition φ)

$\mathbf{B}_r$ transforms atom into propositional formula

Definition (Bit Blasting: Atoms)

for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ set

- equality

$$\mathbf{B}_r(\mathbf{x}_k = \mathbf{y}_k) = (x_k \leftrightarrow y_k) \wedge \cdots \wedge (x_1 \leftrightarrow y_1) \wedge (x_0 \leftrightarrow y_0)$$

- inequality

$$\mathbf{B}_r(\mathbf{x}_k \neq \mathbf{y}_k) = (x_k \oplus y_k) \vee \cdots \vee (x_1 \oplus y_1) \vee (x_0 \oplus y_0)$$

- unsigned greater-than or equal

$$\mathbf{B}_r(\mathbf{x}_1 \geqslant_u \mathbf{y}_1) = y_0 \rightarrow x_0$$

$$\mathbf{B}_r(\mathbf{x}_{k+1} \geqslant_u \mathbf{y}_{k+1}) = (x_k \wedge \neg y_k) \vee ((x_k \leftrightarrow y_k) \wedge \mathbf{B}(\mathbf{x}[k-1{:}0] \geqslant \mathbf{y}[k-1{:}0]))$$

- unsigned greater-than

$$\mathbf{B}(\mathbf{x}_k >_u \mathbf{y}_k) = \mathbf{B}(\mathbf{x}_k \geqslant \mathbf{y}_k) \wedge \mathbf{B}(\mathbf{x}_k \neq \mathbf{y}_k)$$

Definition (Bit Blasting: Bitwise Operations)

for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ use fresh variable $\mathbf{z}_k$ and set

- bitwise and

$$\mathbf{B}_t(\mathbf{x}_k \,\&\, \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \wedge y_i)$$

- bitwise or

$$\mathbf{B}_t(\mathbf{x}_k | \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \vee y_i)$$

- bitwise exclusive or

$$\mathbf{B}_t(\mathbf{x}_k \,\hat{}\, \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \oplus y_i)$$

- bitwise negation

$$\mathbf{B}_t(-\mathbf{x}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow \neg x_i$$

Definition (Bit Blasting: Concatenation, Extraction, If)

- concatenation

$$\mathbf{B}_t(\mathbf{x}_k :: \mathbf{y}_m) = (\mathbf{x}_k\mathbf{y}_m, \mathsf{T})$$

for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_m$

- extraction

$$\mathbf{B}_t(\mathbf{x}[n{:}m]) = (\mathbf{z}_{n-m+1}, \varphi) \quad \varphi = \bigwedge_{i=0}^{n-m} z_i \leftrightarrow x_{i+m}$$

for bit vector $\mathbf{x}_k$, $k > n \geqslant m \geqslant 0$ and fresh variable $\mathbf{z}_{n-m+1}$

- if-then-else

$$\mathbf{B}_t(p \ ? \ \mathbf{x}_k : \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} (p \rightarrow (z_i \leftrightarrow x_i)) \wedge (\neg p \rightarrow (z_i \leftrightarrow y_i))$$

for formula p and bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$

Definition (Bit Blasting: Addition and Subtraction)

- addition

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = (\mathbf{s}_k, \varphi)$$

where

$$\varphi = (c_0 \leftrightarrow x_0 \wedge y_0) \wedge (s_0 \leftrightarrow x_0 \oplus y_0) \wedge \bigwedge_{i=1}^{k-1} (c_i \leftrightarrow \mathrm{min2}(x_i, y_i, c_{i-1})) \wedge (s_i \leftrightarrow x_i \oplus y_i \oplus c_{i-1})$$

ripple-carry adder: $\mathbf{c}_k$ are carry bits

for fresh variables $\mathbf{s}_k$ and $\mathbf{c}_k$ and $\mathrm{min2}(a, b, d) = (a \wedge b) \vee (a \wedge d) \vee (b \wedge d)$

- unary minus

$$\mathbf{B}_t(-\mathbf{x}_k) = \mathbf{B}_t(\sim \mathbf{x}_k + \mathbf{1}_k)$$

- subtraction

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = \mathbf{B}_t(\mathbf{x}_k + (-\mathbf{y}_k)$$

Definition (Bit Blasting: Multiplication and Division)

for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ set

- multiplication

  $$\mathbf{B}_t(\mathbf{x}_k \times \mathbf{y}_k) = \mathbf{B}_t(\mathsf{mul}(\mathbf{x}_k, \mathbf{y}_k, 0))$$

  where

  $$\mathsf{mul}(\mathbf{x}_k, \mathbf{y}_k, k) = \mathbf{0}_k$$

  $$\mathsf{mul}(\mathbf{x}_k, \mathbf{y}_k, i) = \mathsf{mul}(\mathbf{x}_k \ll \mathbf{1}_k, \mathbf{y}_k, i+1) + (y_i \ ? \ \mathbf{x}_k : \mathbf{0}_k) \qquad \text{if } i < k$$

shift-and-add

- unsigned division

  $$\mathbf{B}_t(\mathbf{x}_k \div_u \mathbf{y}_k) = (\mathbf{q}_k, \varphi)$$

  $$\varphi = \mathbf{B}(\mathbf{y}_k \neq \mathbf{0}_k \rightarrow (\mathbf{q}_k \times \mathbf{y}_k + \mathbf{r}_k = \mathbf{x}_k \wedge \mathbf{r}_k < \mathbf{y}_k \wedge \mathbf{q}_k < \mathbf{x}_k))$$

  for fresh variables $\mathbf{q}_k$ and $\mathbf{r}_k$

Example (SMT-LIB 2 for BV)

$(\mathbf{a}_4 + \mathbf{b}_4 <_u \mathbf{b}_4) \wedge (\mathbf{a}_4 \neq \mathbf{10}_4) \wedge (\mathbf{a}_4 \,\&\, \mathbf{b}_4 = \mathbf{8}_4)$ is expressed as

```
(declare-const a (_ BitVec 4))
(declare-const b (_ BitVec 4))
(assert (bvult (bvadd a b) b))
(assert (not (= a #xa)))
(assert (= (bvand a b) #b1000))
(check-sat)
```

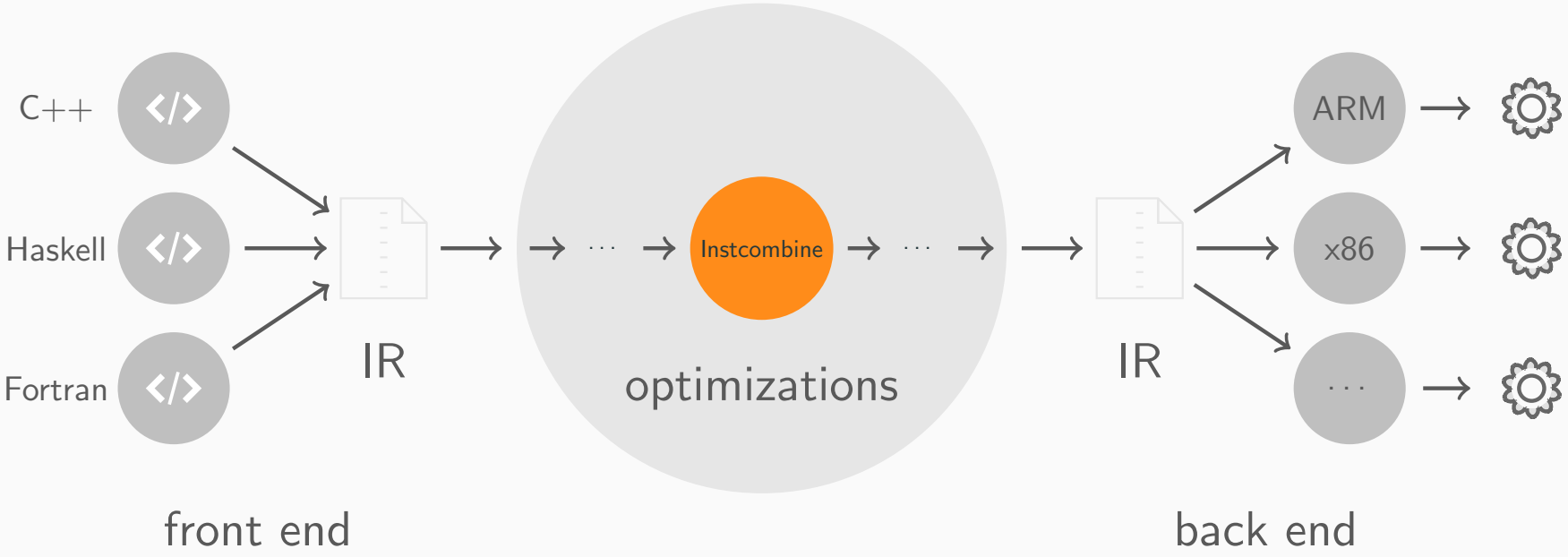
BV in SMT-LIB 2

- `(_ BitVec k)` is sort of bitvectors of length k
- `#xa` is constant in hexadecimal
- `#b1000` is constant in binary
- `bvadd`, `bvsub`, `bvmul` are arithmetic operations,
  `bvudiv` and `bvsdiv` are unsigned and signed division
- `bvult` and `bvule` are unsigned, `bvslt` and `bvsle` are signed $<$ and $\leqslant$
- `bvshl`, `bvlshr`, `bvashr` are shifts
- `bvand`, `bvor` are bitwise logical operations

Application: Verifying Compiler Optimizations (1)

LLVM

- open-source umbrella project: set of reusable toolchain components: libraries, assemblers, compilers, debuggers, ...
- compilation toolchain includes peephole optimizations in Instcombine pass



Application: Verifying Compiler Optimizations (2)

Instcombine Pass

- over 1000 algebraic simplifications of expressions
  - transform multiplies with constant power-of-two argument into shifts
  - bitwise operators with constant operands are always grouped so that shifts are performed first, then ors, then ands, then xors
  - changing bitwidth of variables
  - ...
- code is community maintained
- sometimes optimizations have errors—and compiler bugs are critical

Example

```
int foo(int z) {
int x = 4 * (z | 1001);
return -256 & x;
}
```

```
define i32 @foo(i32) #0 {
 %2 = or i32 %0, 1001
 %3 = mul nsw i32 4, %2
 %4 = xor i32 -256, %3
 ret i32 %4
}
```

Instcombine →

```
define i32 @foo(i32) #0 {
 %2 = shl i32 %0, 2
 %3 = or i32 %2, 4004
 %4 = xor i32 %3, -256
 ret i32 %4
}
```

Application: Verifying Compiler Optimizations (3)

Alive Project

- represent Instcombine optimizations in domain-specific language, e.g.

```
Name: PR20186
%a = sdiv %X, C
%r = sub 0, %a
=>
%r = sdiv %X, -C
```

- check correctness by means of SMT encoding

```
(declare-const x (_ BitVec 32))
(declare-const c (_ BitVec 32))
(declare-const before (_ BitVec 32))
(declare-const after (_ BitVec 32))
(assert (= before (bvsub #x00000000 (bvsdiv x c))))
(assert (= after (bvsdiv x (bvneg c))))
(assert (not (= before after)))
(assert (not (= c #x00000000)))
(check-sat)
```

- wrong for `c = x = #x80000000`

Bit Vectors in python/z3

```
from z3 import *
x = BitVec('x', 32) # create variable named x with 32 bits
c = BitVec('c', 32)

before = BitVecVal(0, 32) - (x / c)
after = x / - c

solver = Solver()
solver.add(c != BitVecVal(0, 32)) # exclude case where c=0
solver.add(after != before)

result = solver.check()
if result == z3.sat:
  m = solver.model()
  print m[x], m[c] # 2147483648 2147483648
  print m.eval(before), m.eval(after) # 4294967295 1
```

Application: Detecting Nontermination

```
int bsearch(int a[], int k, unsigned int lo, unsigned int hi) {
  unsigned int mid;
  while (lo < hi) {
    mid = (lo + hi)/2;
    if (a[mid] < k)
      lo = mid + 1;
    else if (a[mid] > k)
      hi = mid - 1;
    else
      return mid;
  }
  return -1;
}
```

- (former) implementation of binary search in Java library
- loops for inputs `lo=1` and `hi=UINT_MAX` if `a[0] < k`.
- SMT encoding can find values such that parameters stay the same in recursive call

Bibliography

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