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Approximation of Fixed Points

Chapter 4.2 of: Principles of Program Analysis

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Lattice

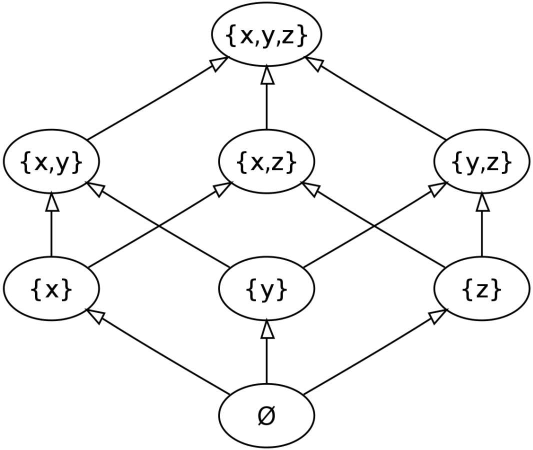

Figure: Hasse diagram of the "subset" lattice of xyz

Fixed Points

$$fix(f) : f(x) = x \tag{1}$$

$$Red(f) = \{l | f(l) \sqsubseteq l\} \tag{2}$$

$$Ext(f) = \{l | f(l) \sqsupseteq l\} \tag{3}$$

$$lfp(f) = \sqcap Fix(f) = \sqcap Red(f) \in Fix(f) \subseteq Red(f) \tag{4}$$

$$gfp(f) = \sqcup Fix(f) = \sqcup Ext(f) \in Fix(f) \subseteq Ext(f) \tag{5}$$

Knaster Tarski's lemma

The lets:

- Let $(L \sqsubseteq)$ be a complete lattice
- Let $f : L \Rightarrow L$ be increasing
- Let $P = \{x \in L | x \leq f(x)\}$
- Let $p = \vee P \Leftarrow$ join
- Let $x \in P$

Knaster Tarski's lemma

Proof:

- Then $x \subseteq f(x)$
- But also $x \subseteq p$
- and so $f(x) \subseteq f(p)$
- overall: $x \subseteq f(p)$
- so $f(p)$ is an upper bound
- so $f(p) \in P$
- therefor $f(p) \subseteq p$ is an upper bound

Result

f has a greatest and least fixed point

Knaster Tarski's lemma

Requirements

- Let $(L \sqsubseteq)$ be a complete lattice
- Let $f : L \Rightarrow L$ be increasing

Result

f has a greatest and least fixed point

Widening Operators

Upper Bound Operators

$$l_1 \sqsubseteq (l_1 \sqcup^{ubo} l_2) \sqsupseteq l_2 \tag{6}$$

applied to a sequence $(l_n)_n$

$$l_n^{\sqcup^{ubo}} = \begin{cases} l_n & \text{if } n = 0 \\ l_{n-1}^{\sqcup^{ubo}} \sqcup^{ubo} l_n & \text{otherwise} \end{cases} \tag{7}$$

Widening Operators

Upper Bound Operators

So $l_n^{\sqcup^{ubo}}$ does:

- beeing in a complete lattice
- guaranteeing ascending monotony

If only it also stabilised...

Widening Operators

Example with intervals

$$int = [0, \infty] \tag{8}$$

$$I = [0,0],[1,1],[2,2],[3,3],[4,4],[5,5],[6,6]... \tag{9}$$

$$I_n^{\sqcup^{int}} = [0,0],[0,1],[0,2],[0,3],[0,4].... \tag{10}$$

$$int = [0, 2] \tag{11}$$

$$I_n^{\sqcup^{int}} = [0,0],[0,1],[0,2],[0,3],[-\infty,\infty],[-\infty,\infty].... \tag{12}$$

Widening Operators

∇ is $\sqcup^{ubo}$ only stabilising.

So ∇ applied to a function:

$$f_{\nabla}^{n} = \begin{cases} \bot & \text{if } n = 0 \\ f_{\nabla}^{n-1} & \text{if } n > 0 \wedge f(f_{\nabla}^{n-1}) \sqsubseteq f_{\nabla}^{n-1} \\ f_{\nabla}^{n-1} \nabla f(f_{\nabla}^{n-1}) & \text{otherwise} \end{cases} \tag{13}$$

Widening Operators

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2nd case is the $lfp(f_{\nabla})$

Narrowing Operators

To improve on approximation by widening operator, whilst also guaranteeing termination

Thank you for your attention!

Thöni Michael