Induced Operations

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  - Correctness

Induced Operations

Key Idea

Use Galois connections to transform computations into more approximate computations with better time-, space-, or termination behavior.

Induced Operations

Two possible ways:

Inducing along the Abstraction function

- Replace a computation using L by a computation using M:

  Analysis using M is an upper approximation to the analysis *induced* by L (loss of precision).

Inducing along the Concretisation function

- Use M for approximating the fixed point computations in L:

  Ensure convergence of fixed points by using the more approximate complete lattice M while maintaining precision of the analysis.

Inducing along the Abstraction Function

Assumptions

- Galois connections $(L_1, \alpha_1, \gamma_1, M_1)$ and $(L_2, \alpha_2, \gamma_2, M_2)$
- Analysis $f_p \; : \; L_1 \to L_2$

Goal

Replace f_p by new more approximate analysis $g_p \; : \; M_1 \to M_2$.

- Candidate for g_p: $\alpha_2 \circ f_p \circ \gamma_1$

Inducing along the Abstraction Function

Example

Consider analysis $f_{plus}(\mathbb{Z}\mathbb{Z}) = \{z_1 + z_2 \mid (z_1, z_2) \in \mathbb{Z}\mathbb{Z}\}$ using complete lattices $(\mathcal{P}(\mathbb{Z}), \subseteq)$ and $(\mathcal{P}(\mathbb{Z} \times \mathbb{Z}), \subseteq)$.

Galois Connections

- $(\mathcal{P}(\mathbb{Z}), \alpha_{sign}, \gamma_{sign}, \mathcal{P}(\mathbf{Sign}))$
  - $\alpha_{sign}(Z) = \{sign(z) \mid z \in Z\}$
  - $\gamma_{sign}(S) = \{z \in Z \mid sign(z) \in S\}$
- $(\mathcal{P}(\mathbb{Z} \times \mathbb{Z}), \alpha_{SS'}, \gamma_{SS'}, \mathcal{P}(\mathbf{Sign} \times \mathbf{Sign}))$
  - $\alpha_{SS'}(ZZ) = \{(sign(z_1), sign(z_2)) \mid (z_1, z_2) \in ZZ\}$
  - $\gamma_{SS'}(SS) = \{(z_1, z_2) \mid (sign(z_1), sign(z_2)) \in SS\}$

Inducing along the Abstraction Function

Example (cont'd)

Construct analysis $g_{plus} : \mathcal{P}(\mathbf{Sign} \times \mathbf{Sign}) \to \mathcal{P}(\mathbf{Sign})$ from f_{plus}, using candidate $g_{plus} = \alpha_{sign} \circ f_{plus} \circ \gamma_{SS'}$.

$$
\begin{aligned}
g_{plus}(SS) &= \alpha_{sign}(f_{plus}(\gamma_{SS'}(SS))) \\
&= \alpha_{sign}(f_{plus}(\{(z_1, z_2) \mid (sign(z_1), sign(z_2)) \in SS\})) \\
&= \alpha_{sign}(\{z_1 + z_2 \mid (sign(z_1), sign(z_2)) \in SS\}) \\
&= \{sign(z_1 + z_2) \mid (sign(z_1), sign(z_2)) \in SS\} \\
&= \bigcup \{s_1 \oplus s_2 \mid (s_1, s_2) \in SS\}
\end{aligned}
$$

where $\oplus$ is the "addition" on signs.

Recap

- Galois connections $(L_i, \alpha_i, \gamma_i, M_i),\ i \in \{1, 2\}$
- Analysis $f_p : L_1 \rightarrow L_2$
- Analysis $g_p : M_1 \rightarrow M_2$

Correctness relations

- Representation functions $\beta_i : V_i \rightarrow L_i$
- Correctness relations $R_i : V_i \times L_i \rightarrow \{true, false\}$ generated by $\beta_i : V_i \rightarrow L_i$
- Correctness relations $S_i : V_i \times M_i \rightarrow \{true, false\}$ generated by $\alpha_i \circ \beta_i : V_i \rightarrow M_i$

Lemma 4.41

If $(L_i, \alpha_i, \gamma_i, M_i)$ are Galois connections and $\beta_i : V_i \rightarrow L_i$ are representation functions, then

$$((\alpha_1 \circ \beta_1) \twoheadrightarrow (\alpha_2 \circ \beta_2))(\leadsto) = \alpha_2 \circ ((\beta_1 \twoheadrightarrow \beta_2)(\leadsto)) \circ \gamma_1$$

holds for all $\leadsto$.

Proof (Lemma 4.41)

Simply calculate:

$$
\begin{aligned}
((\alpha_1 \circ \beta_1) \twoheadrightarrow (\alpha_2 \circ \beta_2))(\rightsquigarrow)(m_1) \\
&= \bigsqcup \{\alpha_2(\beta_2(v_2)) \mid \alpha_1(\beta_1(v_1)) \sqsubseteq m_1 \wedge v_1 \rightsquigarrow v_2\} \\
&= \alpha_2 \left(\bigsqcup \{\beta_2(v_2)) \mid \beta_1(v_1) \sqsubseteq \gamma_1(m_1) \wedge v_1 \rightsquigarrow v_2\} \right) \\
&= \alpha_2((\beta_1 \twoheadrightarrow \beta_2)(\rightsquigarrow)(\gamma_1(m_1)) \\
&= (\alpha_2 \circ ((\beta_1 \twoheadrightarrow \beta_2)(\rightsquigarrow)) \circ \gamma_1)(m_1)
\end{aligned}
$$

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Inducing along the Abstraction Function

Correctness cont'd

Lemma 4.41 yields:

$$(p \vdash \cdot \rightsquigarrow \cdot)(R_1 \twoheadrightarrow R_2)f_p \wedge \alpha_2 \circ f_p \circ \gamma_1 \sqsubseteq g_p$$
$$\Rightarrow (p \vdash \cdot \rightsquigarrow \cdot)(S_1 \twoheadrightarrow S_2)g_p$$

In words: if f_p is correct and g_p is an upper approximation to the induced analysis $\alpha_2 \circ f_p \circ \gamma_1$ then also g_p is correct.

Inducing along the Abstraction Function

Correctness cont'd

Proof

1. Suppose $(p \vdash \cdot \rightsquigarrow \cdot)(R_1 \twoheadrightarrow R_2)f_p$ and $\alpha_2 \circ f_p \circ \gamma_1 \sqsubseteq g_p$.
2. Since $(L_i, \alpha_i, \gamma_i, M_i)$ are Galois connections and f_p and g_p are monotone we get $f_p \sqsubseteq \gamma_2 \circ g_p \circ \alpha_1$.
3. Using the first assumption and Lemma 4.8:

$$
\begin{aligned}
&(p \vdash \cdot \rightsquigarrow \cdot)(R_1 \twoheadrightarrow R_2)f_p \wedge f_p \sqsubseteq \gamma_2 \circ g_p \circ \alpha_1 \\
&\quad \Rightarrow (\beta_1 \twoheadrightarrow \beta_2)(p \vdash \cdot \rightsquigarrow \cdot) \sqsubseteq f_p \wedge f_p \sqsubseteq \gamma_2 \circ g_p \circ \alpha_1 \\
&\quad \Rightarrow (\beta_1 \twoheadrightarrow \beta_2)(p \vdash \cdot \rightsquigarrow \cdot) \sqsubseteq \gamma_2 \circ g_p \circ \alpha_1 \\
&\quad \Rightarrow \alpha_2 \circ (\beta_1 \twoheadrightarrow \beta_2)(p \vdash \cdot \rightsquigarrow \cdot) \circ \gamma_1 \sqsubseteq g_p \\
&\quad \Rightarrow (\alpha_1 \circ \beta_1 \twoheadrightarrow \alpha_2 \circ \beta_2)(p \vdash \cdot \rightsquigarrow \cdot) \sqsubseteq g_p \\
&\quad \Rightarrow (p \vdash \cdot \rightsquigarrow \cdot)(S_1 \twoheadrightarrow S_2)g_p
\end{aligned}
$$

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Definition

A function $f_p : L_1 \to L_2$ is *optimal* for the program p if and only if correctness of a function $f' : L_1 \to L_2$ amounts to $f_p \sqsubseteq f'$

Equivalently, f_p is *optimal* if and only if $(\beta_1 \twoheadrightarrow \beta_2)(p \vdash \cdot \rightsquigarrow \cdot) = f_p$

Lemma 4.41 may then be read as saying that if $f_p : L_1 \to L_2$ is optimal then so is $\alpha_2 \circ f_p \circ \gamma_1 : M_1 \to M_2$.

Inducing along the Abstraction Function

Fixed Points

Consider analysis $f_p : L_1 \to L_2$ requires computation of the least fixed point of a monotone function $F : (L_1 \to L_2) \to (L_1 \to L_2)$ so that $f_p = lfp(F)$.

- $(L_i, \alpha_i, \gamma_i, M_i)$ give rise to $(L_1 \to L_2, \alpha, \gamma, M_1 \to M_2)$
- Let $G : (M_1 \to M_2) \to (M_1 \to M_2)$ be an upper approximation to $\alpha \circ F \circ \gamma$
- Take $g_p : M_1 \to M_2$ to be $g_p = lfp(G)$

Fact

Correctness of f_p carries over to g_p.

Lemma 4.42

Assume

- (L, α, γ, M) is a Galois connection
- $f : L \to L$ and $g : M \to M$ are monotone functions
- g is an upper approximation to f (i.e. $\alpha \circ f \circ \gamma \sqsubseteq g$)

Then follows

- $\forall m \in M : g(m) \sqsubseteq m \Rightarrow f(\gamma(m)) \sqsubseteq \gamma(m)$
- and furthermore $lfp(f) \sqsubseteq \gamma(lfp(g))$ and $\alpha(lfp(f)) \sqsubseteq lfp(g)$

Inducing along the Abstraction Function

Fixed Points - Correctness

Proof

Show $\forall m \in M : g(m) \sqsubseteq m \Rightarrow f(\gamma(m)) \sqsubseteq \gamma(m)$

$$\begin{aligned} g(m) \sqsubseteq m \wedge \alpha(f(\gamma(m))) \sqsubseteq g(m) \\ \Rightarrow \alpha(f(\gamma(m))) \sqsubseteq m \\ \Rightarrow f(\gamma(m)) \sqsubseteq \gamma(m) \end{aligned}$$

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Proof cont'd

From the previous result follows $\{\gamma(m) \mid g(m) \sqsubseteq m\} \subseteq \{l \mid f(l) \sqsubseteq l\}$ and hence (using Lemma 4.22)

$$\gamma\left(\bigsqcap\{m \mid g(m) \sqsubseteq m\}\right) = \bigsqcap\{\gamma(m) \mid g(m) \sqsubseteq m\} \sqsupseteq \bigsqcap\{l \mid f(l) \sqsubseteq l\}$$

Using Tarski's theorem and that a Galois connection is an adjunction:

$$\begin{aligned} &\gamma(\bigsqcap\{m \mid g(m) \sqsubseteq m\}) \sqsupseteq \bigsqcap\{l \mid f(l) \sqsubseteq l\} \\ &\Rightarrow \gamma(Red(g)) \sqsupseteq Red(f) \\ &\Rightarrow lfp(f) \sqsubseteq \gamma(lfp(g)) \\ &\Rightarrow \alpha(lfp(f)) \sqsubseteq lfp(g) \end{aligned}$$

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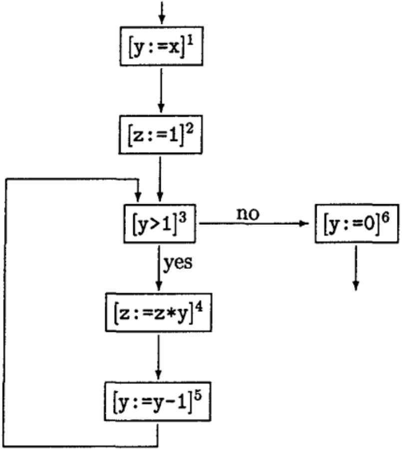
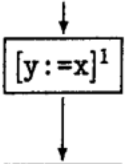
Excursion - Data Flow Analysis

- Model how the data flows through a program
- Construct Constraints describing the program
- Analysis describes state-changes with transfer functions



Excursion - Data Flow Analysis

- Statement with label $l = 1$
- Assigns x to y
- For some analysis A we write
  - $A_\circ(l)$ for the **entry**-state
  - $A_\bullet(l)$ for the **exit**-state
  - $f_l^A(\textbf{State})$ for the transfer function
- constraints are modeled for example with
  - $A_\bullet(1) \sqsupseteq f_1^A(A_\circ(1))$
  - $A_\circ(2) \sqsupseteq A_\bullet(1)$



Generalized Monotone Framework A

- complete lattice L
- finite flow $F \subseteq \textbf{Lab} \times \textbf{Lab}$
- finite set of extremal labels $E \subseteq \textbf{Lab}$
- extremal value $i \in L$
- a mapping f from the labels of F and E to monotone transfer functions $L \to L$
- Constraints $A^{\sqsupseteq}$

$$A_\circ(l) \sqsupseteq \bigsqcup\{A_\bullet(l') \mid (l', l) \in F\} \sqcup i_E^l \quad \text{where } i_E^l = \begin{cases} i & l \in E \\ \bot & l \notin E \end{cases}$$

$$A_\bullet(l) \sqsupseteq f_l(A_\circ(l))$$

Generalized Monotone Framework A

- $(A_\circ, A_\bullet) \models A^{\sqsupseteq}$ whenever $A_\circ, A_\bullet$ is a solution to the constraints $A^{\sqsupseteq}$
- consider the associated monotone function
  $\vec{f}(A_\circ, A_\bullet) = (\lambda l.A_\circ(l)\ ,\ \lambda l.A_\bullet(l))$
- $(A_\circ, A_\bullet) \sqsupseteq \vec{f}(A_\circ, A_\bullet)$ is equivalent to $(A_\circ, A_\bullet) \models A^{\sqsupseteq}$

Generalized Monotone Framework B

- let (L, α, γ, M) be a Galois Connection
- B is as A, but has
  - the mapping g from labels of F and E to monotone transfer functions $M \to M$, that satisfies $g_l \sqsupseteq \alpha \circ f_l \circ \gamma$
  - the extremal value $j \sqsupseteq \alpha(i)$
- As in A we get the constraints $B^{\sqsupseteq}$ for B and the associated monotone function $\vec{g}$

Fact

$$(B_\circ, B_\bullet) \models B^{\sqsupseteq} \implies (\gamma \circ B_\circ, \gamma \circ B_\bullet) \models A^{\sqsupseteq}$$

Sets of States Analysis SS

- complete lattice $(\mathcal{P}(\textbf{State}), \subseteq)$
- flow $F = flow(S_*)$
- set $E = \{init(S_*)\}$ of extremal labels
- extremal value $i = \textbf{State}$
- transfer functions given by $f_\cdot^{SS}$:

$$f_l^{SS}(\Sigma) = \begin{cases} \{\sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma] \mid \sigma \in \Sigma\} & \text{if } [x := a]^l \text{ is in } S_* \\ \Sigma & \text{if } [skip]^l \text{ is in } S_* \\ \Sigma & \text{if } [b]^l \text{ is in } S_* \end{cases}$$

where $\Sigma \subseteq \textbf{State}$

Fact

The *SS* analysis is correct

Inducing along the Abstraction Function

A Worked Example

Constant Propagation Analysis

- complete lattice $\widehat{\textbf{State}}_{\textbf{CP}} = ((\textbf{Var} \to \textbf{Z}^{\top})_{\bot}, \sqsubseteq)$
- flow $F = \mathit{flow}(S_*)$
- extremal labels $E = \{\mathit{init}(S_*)\}$
- extremal value $i = \lambda x.\top$
- transfer functions of the constant propagation analysis [a] given by $f_{\cdot}^{CP}$

[a] Principles of Program Analysis, page 71, Table 2.7

Inducing along the Abstraction Function

A Worked Example

The relationship between the two analyses is established by the representation function

$$\beta_{CP} : \textbf{State} \rightarrow \widehat{\textbf{State}}_{\textbf{CP}}$$
$$\beta_{CP}(\sigma) = \sigma$$

Galois Connection

β_{CP} gives rise to a Galois connection $(\mathcal{P}(\textbf{State}), \alpha_{CP}, \gamma_{CP}, \widehat{\textbf{State}}_{\textbf{CP}})$

$$\alpha_{CP}(\Sigma) = \bigsqcup\{\beta_{CP}(\sigma) \mid \sigma \in \Sigma\}$$
$$\gamma_{CP}(\hat{\sigma}) = \{\sigma \mid \beta_{CP}(\sigma) \sqsubseteq \hat{\sigma}\}$$

Conclusion

one can now show

$$\forall l \in \textbf{Lab} : f_l^{CP} \sqsupseteq \alpha_{CP} \circ f_l^{SS} \circ \gamma_{CP}$$
$$\gamma_{CP}(\lambda x.\top) = \textbf{State}$$

and hence CP is an upper approximation to the analysis induced from SS by the Galois connection and therefore correct.

Inducing along the Concretisation Function

Why?

Inducing by abstraction function has a critical disadvantage. It lose precision along the analysis.

Inducing by Concretisation Function

instead of replacing the analysis using L with analysis using M;

- We perform normally on L (to not lose precision).
- but we only use M to approximate the fixed point computations done in L (to ensure convergence of the fixed points).

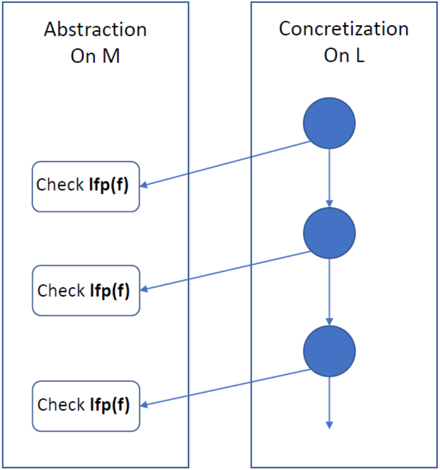
Inducing by Concretisation Function

Inducing by Concretisation Function

Using widening operator $\nabla_M : M \times M \to M$

- to define $\nabla_L : L \times L \to L$
  by using the formula $l_1 \nabla_L l_2 = \gamma(\alpha(l_1) \nabla_M \alpha(l_2))$
- we can approximate $\mathbf{lfp(f)}$ over L.

Concretisation process



Inducing along the concretization function

Widening Operator

Why Widening Operator?

- We can't guarantee reaching stability eventually.
- or reaching least upper bound that equals **lfp(f)**.

Widening Operator

- used to obtain approximations of the least fixed points.
- used to limit the number of computation steps needed.

lemma 4.45

If $(\mathbf{L}, \alpha, \gamma, \mathbf{M})$ is a Galois insertion such that
$\gamma(\bot_{\mathbf{M}}) = \bot_{\mathbf{L}}$ and if $\nabla_{\mathbf{M}} : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$ is a widening operator.
Then $\nabla_{\mathbf{L}} : \mathbf{L} \times \mathbf{L} \rightarrow \mathbf{L}$ is a widening operator defined by the formula
$\mathbf{l_1} \nabla_{\mathbf{L}} \mathbf{l_2} = \gamma(\alpha(\mathbf{l_1} \nabla_{\mathbf{M}} \alpha(\mathbf{l_2}))$.
this satisfies $\mathbf{lfp}_{\nabla_{\mathbf{L}}}(\mathbf{f}) = \gamma(\mathbf{lfp}_{\nabla_{\mathbf{M}}}(\alpha \circ \mathbf{f} \circ \gamma))$ for all monotone functions
$\mathbf{f} : \mathbf{L} \rightarrow \mathbf{L}$.

Proof

- given $\nabla_{\mathbf{L}}$ is a widening operator, $\exists n_f \geq 0, \mathbf{lfp}_{\nabla_{\mathbf{L}}}(\mathbf{f}) = \mathbf{f}^{\mathbf{n_f}}_{\nabla_{\mathbf{L}}} = \mathbf{f}^{\mathbf{n}}_{\nabla_{\mathbf{L}}}$
- given $\nabla_{\mathbf{M}}$ is a widening operator, $\exists n_g \geq 0, \mathbf{lfp}_{\nabla_{\mathbf{M}}}(\mathbf{g}) = \mathbf{g}^{\mathbf{n_g}}_{\nabla_{\mathbf{M}}} = \mathbf{g}^{\mathbf{n}}_{\nabla_{\mathbf{M}}}$
- if we can prove that: $\mathbf{f}^{\mathbf{n}}_{\nabla_{\mathbf{L}}} = \gamma(\mathbf{g}^{\mathbf{n}}_{\nabla_{\mathbf{M}}})$
- we can obtain that: $\mathbf{lfp}_{\nabla_{\mathbf{L}}}(\mathbf{f}) = \gamma(\mathbf{lfp}_{\nabla_{\mathbf{M}}}(\mathbf{g}))$

The Proof

by induction on **n:**

- base case: $n = 0$.

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \perp_L$ and $g^0_{\nabla_M} = \perp_M$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \perp_L$ and $g^0_{\nabla_M} = \perp_M$

  assume $\perp_L = \gamma(\perp_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Rightarrow \alpha(f(f^n_{\nabla_L})) \sqsubseteq \alpha(f^n_{\nabla_L})$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Rightarrow \alpha(f(f^n_{\nabla_L})) \sqsubseteq \alpha(f^n_{\nabla_L})$

  $\Rightarrow \alpha(f(\gamma(g^n_{\nabla_M}))) \sqsubseteq \alpha(\gamma(g^n_{\nabla_M}))$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $$f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$$

  $$\begin{aligned} f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} &\Rightarrow \alpha(f(f^n_{\nabla_L})) \sqsubseteq \alpha(f^n_{\nabla_L}) \\ &\Rightarrow \alpha(f(\gamma(g^n_{\nabla_M}))) \sqsubseteq \alpha(\gamma(g^n_{\nabla_M}) \\ &\Rightarrow g(g^n_{\nabla_M}) \sqsubseteq \alpha(\gamma(g^n_{\nabla_M}) \end{aligned}$$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $$f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$$

  $$\begin{aligned} f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} &\Rightarrow \alpha(f(f^n_{\nabla_L})) \sqsubseteq \alpha(f^n_{\nabla_L}) \\ &\Rightarrow \alpha(f(\gamma(g^n_{\nabla_M}))) \sqsubseteq \alpha(\gamma(g^n_{\nabla_M}) \\ &\Rightarrow g(g^n_{\nabla_M}) \sqsubseteq \alpha(\gamma(g^n_{\nabla_M}) \\ &\Rightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M} \end{aligned}$$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$

  $g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M} \Rightarrow \gamma(g(g^n_{\nabla_M})) \sqsubseteq \gamma(g^n_{\nabla_M})$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$

  $$g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M} \Rightarrow \gamma(g(g^n_{\nabla_M})) \sqsubseteq \gamma(g^n_{\nabla_M})$$
  $$\Rightarrow \gamma(\alpha(f(\gamma(g^n_{\nabla_M})))) \sqsubseteq \gamma(g^n_{\nabla_M})$$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$

  $$\begin{aligned} g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M} &\Rightarrow \gamma(g(g^n_{\nabla_M})) \sqsubseteq \gamma(g^n_{\nabla_M}) \\ &\Rightarrow \gamma(\alpha(f(\gamma(g^n_{\nabla_M})))) \sqsubseteq \gamma(g^n_{\nabla_M}) \\ &\Rightarrow \gamma(\alpha(f(f^n_{\nabla_L}))) \sqsubseteq f^n_{\nabla_L} \end{aligned}$$

The Proof

by induction on **n:**

- base case: n = 0.

  $f^0_{\nabla_L} = \bot_L$ and $g^0_{\nabla_M} = \bot_M$

  assume $\bot_L = \gamma(\bot_M)$

  $\Rightarrow f^0_{\nabla_L} = \gamma(g^0_{\nabla_M})$

- for induction step over n.

  $f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \Leftrightarrow g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M}$

  $$\begin{aligned} g(g^n_{\nabla_M}) \sqsubseteq g^n_{\nabla_M} &\Rightarrow \gamma(g(g^n_{\nabla_M})) \sqsubseteq \gamma(g^n_{\nabla_M}) \\ &\Rightarrow \gamma(\alpha(f(\gamma(g^n_{\nabla_M})))) \sqsubseteq \gamma(g^n_{\nabla_M}) \\ &\Rightarrow \gamma(\alpha(f(f^n_{\nabla_L}))) \sqsubseteq f^n_{\nabla_L} \\ &\Rightarrow f(f^n_{\nabla_L}) \sqsubseteq f^n_{\nabla_L} \end{aligned}$$

The Proof

- induction step: $n > 0$.

The Proof

- induction step: n $>$ 0.

$$f^{n}_{\nabla_L} = \begin{cases} f^{n-1}_{\nabla_L} & \textit{if} \quad f(f^{n-1}_{\nabla_L}) \sqsubseteq f^{n-1}_{\nabla_L} \\ f^{n-1}_{\nabla_L} \nabla_L f(f^{n-1}_{\nabla_L}) & \textit{otherwise} \end{cases}$$

The Proof

- induction step: n > 0.

$$f^n_{\nabla_L} = \begin{cases} f^{n-1}_{\nabla_L} & \textit{if} \quad f(f^{n-1}_{\nabla_L}) \sqsubseteq f^{n-1}_{\nabla_L} \\ f^{n-1}_{\nabla_L} \nabla_L f(f^{n-1}_{\nabla_L}) & \textit{otherwise} \end{cases}$$

$$= \begin{cases} f^{n-1}_{\nabla_L} & \textit{if} \quad g(g^{n-1}_{\nabla_M}) \sqsubseteq g^{n-1}_{\nabla_M} \\ f^{n-1}_{\nabla_L} \nabla_L f(f^{n-1}_{\nabla_L}) & \textit{otherwise} \end{cases}$$

The Proof

- induction step: n > 0.

$$f_{\nabla_L}^{n} = \begin{cases} f_{\nabla_L}^{n-1} & \mathit{if} \quad f(f_{\nabla_L}^{n-1}) \sqsubseteq f_{\nabla_L}^{n-1} \\ f_{\nabla_L}^{n-1} \nabla_L f(f_{\nabla_L}^{n-1}) & \mathit{otherwise} \end{cases}$$

$$= \begin{cases} f_{\nabla_L}^{n-1} & \mathit{if} \quad g(g_{\nabla_M}^{n-1}) \sqsubseteq g_{\nabla_M}^{n-1} \\ f_{\nabla_L}^{n-1} \nabla_L f(f_{\nabla_L}^{n-1}) & \mathit{otherwise} \end{cases}$$

$$= \begin{cases} \gamma(g_{\nabla_M}^{n-1}) & \mathit{if} \quad g(g_{\nabla_M}^{n-1}) \sqsubseteq g_{\nabla_M}^{n-1} \\ \gamma(\alpha\boldsymbol{(}\gamma(g_{\nabla_M}^{n-1}) \nabla_M f(\gamma(g_{\nabla_M}^{n-1}))\boldsymbol{)}) & \mathit{otherwise} \end{cases}$$

The Proof

- induction step: n > 0.

$$f^n_{\nabla_L} = \begin{cases} f^{n-1}_{\nabla_L} & \textit{if} \quad f(f^{n-1}_{\nabla_L}) \sqsubseteq f^{n-1}_{\nabla_L} \\ f^{n-1}_{\nabla_L} \nabla_L f(f^{n-1}_{\nabla_L}) & \textit{otherwise} \end{cases}$$

$$= \begin{cases} f^{n-1}_{\nabla_L} & \textit{if} \quad g(g^{n-1}_{\nabla_M}) \sqsubseteq g^{n-1}_{\nabla_M} \\ f^{n-1}_{\nabla_L} \nabla_L f(f^{n-1}_{\nabla_L}) & \textit{otherwise} \end{cases}$$

$$= \begin{cases} \gamma(g^{n-1}_{\nabla_M}) & \textit{if} \quad g(g^{n-1}_{\nabla_M}) \sqsubseteq g^{n-1}_{\nabla_M} \\ \gamma(\alpha\boldsymbol{(}\gamma(g^{n-1}_{\nabla_M}) \nabla_M f(\gamma(g^{n-1}_{\nabla_M})))\boldsymbol{)} & \textit{otherwise} \end{cases}$$

$$= \gamma\left(\begin{cases} (g^{n-1}_{\nabla_M}) & \textit{if} \quad g(g^{n-1}_{\nabla_M}) \sqsubseteq g^{n-1}_{\nabla_M} \\ g^{n-1}_{\nabla_M} \nabla_M g(g^{n-1}_{\nabla_M}) & \textit{otherwise} \end{cases}\right)$$

The Proof

- induction step: n > 0.

$$f^n_{\nabla_L} = \begin{cases} f^{n-1}_{\nabla_L} & \textit{if} \quad f(f^{n-1}_{\nabla_L}) \sqsubseteq f^{n-1}_{\nabla_L} \\ f^{n-1}_{\nabla_L} \nabla_L f(f^{n-1}_{\nabla_L}) & \textit{otherwise} \end{cases}$$

$$= \begin{cases} f^{n-1}_{\nabla_L} & \textit{if} \quad g(g^{n-1}_{\nabla_M}) \sqsubseteq g^{n-1}_{\nabla_M} \\ f^{n-1}_{\nabla_L} \nabla_L f(f^{n-1}_{\nabla_L}) & \textit{otherwise} \end{cases}$$

$$= \begin{cases} \gamma(g^{n-1}_{\nabla_M}) & \textit{if} \quad g(g^{n-1}_{\nabla_M}) \sqsubseteq g^{n-1}_{\nabla_M} \\ \gamma(\alpha\boldsymbol{(}\gamma(g^{n-1}_{\nabla_M}) \nabla_M f(\gamma(g^{n-1}_{\nabla_M}))\boldsymbol{)}) & \textit{otherwise} \end{cases}$$

$$= \gamma\left(\begin{cases} (g^{n-1}_{\nabla_M}) & \textit{if} \quad g(g^{n-1}_{\nabla_M}) \sqsubseteq g^{n-1}_{\nabla_M} \\ g^{n-1}_{\nabla_M} \nabla_M g(g^{n-1}_{\nabla_M}) & \textit{otherwise} \end{cases}\right)$$

$$= \gamma(g^n_{\nabla_M})$$

Proof

- given $\nabla_{\mathbf{L}}$ is a widening operator, $\exists n_f \geq 0, \mathbf{lfp}_{\nabla_\mathbf{L}}(\mathbf{f}) = \mathbf{f}^{\mathbf{n_f}}_{\nabla_\mathbf{L}} = \mathbf{f}^{\mathbf{n}}_{\nabla_\mathbf{L}}$
- given $\nabla_{\mathbf{M}}$ is a widening operator, $\exists n_g \geq 0, \mathbf{lfp}_{\nabla_\mathbf{M}}(\mathbf{g}) = \mathbf{g}^{\mathbf{n_g}}_{\nabla_\mathbf{M}} = \mathbf{g}^{\mathbf{n}}_{\nabla_\mathbf{M}}$
- We have proven that: $\mathbf{f}^{\mathbf{n}}_{\nabla_\mathbf{L}} = \gamma(\mathbf{g}^{\mathbf{n}}_{\nabla_\mathbf{M}})$
- which prove that: $\mathbf{lfp}_{\nabla_\mathbf{L}}(\mathbf{f}) = \gamma(\mathbf{lfp}_{\nabla_\mathbf{M}}(\mathbf{g}))$

So, we can perform our analysis over **L** without lossing precision.

Thank you for your attention,
Questions?