Simply typed lambda calculus is strongly normalizing

The following proof of strong normalization of $\Lambda^{\rightarrow}$, using induction loading on the predicate SN to deal with creation of redexes and substitutions to deal with contraction of existing ones, is a standard strong computibility-style proof. The predicate SC on $\Lambda^{\rightarrow}$ is defined by:

$$M \in SC =^{\text{def}} \forall \vec{R} \in SC . M\vec{R} \in \Lambda^{\rightarrow} \text{ is SN}$$

From the definition, $SC \implies \text{SN}$, SC is closed under β, and SC holds for every variable x, since $\vec{R} \in SC \implies \vec{R} \in \text{SN} \implies x\vec{R} \in \text{SN}$.

Theorem 1. $\Lambda^{\rightarrow}$ *is* SN.

Proof We prove *for all terms M and type preserving substitutions σ mapping free variables of M to terms in SC, $M^\sigma \in SC$*, by induction on the derivation of $M \in \Lambda^{\rightarrow}$, from which the theorem follows by setting σ to the identity.

(var) $x^\sigma = \sigma(x) \in SC$, by assumption,

(app) Let $\vec{R} \in SC$, then $(MN)^\sigma \vec{R} = (M^\sigma N^\sigma)\vec{R} = M^\sigma N^\sigma \vec{R} \in \text{SN}$, by ih for M (and N),

(abs) Let $\vec{R} \in SC$. $(\lambda x.M)^\sigma \vec{R} = (\lambda x.M^\sigma)\vec{R}$. By ih and assumption, M^σ and $\vec{R}$ are SN, so an infinite reduction must be of the form $(\lambda x.M^\sigma)P\vec{Q} \twoheadrightarrow_\beta (\lambda x.M')P'\vec{Q} \rightarrow_\beta M'^{[x:=P']}\vec{Q'} \rightarrow_\beta^\infty \ldots$ with $\vec{R} = P\vec{Q}$, but $M'^{[x:=P']}\vec{Q'} \twoheadleftarrow_\beta M^{\sigma[x:=P]}\vec{Q} \in \text{SN}$ by ih. $\odot$