universität innsbruck

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# Interactive Theorem Proving using Isabelle/HOL

## Session 9

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## Topics

calculational reasoning, case analysis, code generation, computation induction, data type invariants, document preparation, finding theorems, first steps, functional programming in HOL, higher-order logic, history and motivation, induction, inductive definitions, Isabelle basics, Isabelle/Isar, Isabelle/ML, IsaFoR/CeTA, locales, manual termination proofs, multisets, natural deduction, notation, proof methods, PSL: a high-level proof strategy language, rule induction, rule inversion, session management, sets, simplification, sledgehammer, structural induction, structured proof, The Archive of Formal Proofs, the certification approach, total recursive functions, type classes, type definitions, well-foundedness

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# Overview

- Proof Methods
- Well-Foundedness
- Manual Termination Proofs
- Exercises

# Proof Methods

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  `⟨?A ∈ Pow ?B ⟹ ?c ∈ ?A ⟹ ?c ∈ ?B⟩`
- `simp`, `intro`, `elim`, `dest` – declare fact simplification/introduction/elimination/destruction rule

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- `have "A ∨ B ⟹ C" apply (erule disjE)` ⇝
  1. $A \Longrightarrow C$
  2. $B \Longrightarrow C$

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## Combined Proof Methods

- `force` (`fastforce`, `bestsimp`) – solve first subgoal by combination of equational and classical reasoning
- `auto` – apply combination of equational and classical reasoning to all subgoals and leave result as new subgoals

## Modifiers for Classical Methods

classical methods (like `blast` and `auto`) take following modifiers:

- `intro:` $fact^+$ – add additional intro rules
- `dest:` $fact^+$ – add additional dest rules
- `elim:` $fact^+$ – add additional elim rules
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## Note

when used with combined methods (like `force` and `auto`), modifiers for simplifier use prefix `simp` (like `simp add:`, `simp del:`, ...)

# Well-Foundedness

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## `Demo09.thy` – Equivalence Proof

prove that (I) implies (A) and (N) implies (C)

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## Lexicographic Product of Relations

- lexicographic product of relations $R$ and $S$, written $R$ `<*lex*>` $S$, give by $((x_1, x_2), (y_1, y_2)) \in R$ `<*lex*>` $S$ iff $(x_1, y_1) \in R$ or both, $x_1 = y_1$ and $(x_2, y_2) \in S$

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- lexicographic product of well-founded relations is well-founded

# Manual Termination Proofs

## Demo09.thy – Alternative Implementation for Grouping Repeated List Elements

```
fun split_same :: "'a ⇒ 'a list ⇒ ('a list × 'a list)"
  where
    "split_same x [] = ([], [])"
  | "split_same x (y # ys) =
    (if x = y then let (us, vs) = split_same x ys in (y # us, vs)
    else ([], y # ys))"

fun group :: "'a list ⇒ 'a list list"
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### Additional Simp Rules for Termination

- sometimes sufficient to add simp rules (only for termination proofs)
- use attribute `termination_simp`

## Demo09.thy – An Odd Even/Odd Function

```
fun evenodd :: "nat ⇒ bool ⇒ bool"
  where
    "evenodd 0 True ⟷ True"
  | "evenodd x False ⟷ ¬ (evenodd x True)"
  | "evenodd (Suc x) True ⟷ evenodd x False"
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### Explicit Termination Proofs

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  `function (sequential) f :: "T" ... by (pat_completeness) auto`

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- replace `fun f :: "T" ...` by
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- and add `termination by (relation "R") auto` for some well-founded relation `R` that covers recursive calls
- note: by default `fun` uses method `lexicographic_order` for termination proofs

## Demo09.thy – Higher-Order Recursion

```
datatype tree = Tree nat "tree list"

fun map :: "('a ⇒ 'b) ⇒ 'a list ⇒ 'b list"
  where
    "map f [] = []"
  | "map f (x#xs) = f x # map f xs"

fun mirror :: "tree ⇒ tree"
  where
    "mirror (Tree n ts) = Tree n (rev (map mirror ts))"
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### Congruence Rules

- express on which values higher-order arguments have to agree to yield same result
- `fundef_cong` – declares congruence rule for function definitions

# Exercises

## Exercises (start from `Exercises09.thy`)

**URL**

`http://cl-informatik.uibk.ac.at/teaching/ss19/itp/thys/Exercises09.thy`

## Further Reading

Alexander Krauss.
Defining Recursive Functions in Isabelle/HOL.
Isabelle documentation, 2018.

Alexander Krauss.
*Automating Recursive Definitions and Termination Proofs in Higher-Order Logic*.
PhD thesis, Institut für Informatik, Technische Universität München, 2009.

## Important Concepts

- `best` *(method)*
- `bestsimp` *(method)*
- `blast` *(method)*
- `del` *(modifiers)*
- `dest` *(attribute/modifier)*
- `elim` *(attribute/method/modifier)*
- `fast` *(method)*
- `fastforce` *(method)*
- `force` *(method)*
- `fundef_cong` *(attribute)*
- `intro` *(attribute/method/modifier)*
- `lexicographic_order` *(method)*
- `of` *(attribute)*
- `OF` *(attribute)*
- `rule_format` *(attribute)*
- `simp` *(attribute)*
- `symmetric` *(attribute)*
- `THEN` *(attribute)*
- `termination_simp` *(attribute)*
- `unfold` *(method)*