Understanding IPL

Theorem (Glivenko)

ϕ tautology in classical logic iff $\vdash \neg\neg\phi$ in IPL.

Theorem (Disjunction Property of IPL)

$\vdash \phi \vee \psi$ iff $\vdash \phi$ or $\vdash \psi$ in IPL.

Example

$p \vee \neg p$ tautology in classical logic even if neither p nor $\neg p$ is.

Lattice

Implicational formulae ϕ having only propositional variable p, ordered by implication.

- In classical logic: 4-point diamond.
- In intuitionitistic logic: Rieger-Nishimura lattice.

Kripke semantics

Definition: Kripke model

Forcing relation $\Vdash$ relating worlds $c \in C$ to propositional variables such that if $c \Vdash p$ and $c \leq c'$, then $c' \Vdash p$, with $\leq$ partial order.

Definition: Forcing for Formulas

- $c \Vdash \phi \to \psi$ if $c' \Vdash \psi$ for all $c' \geq c$ with $c' \Vdash \phi$;
- $c \Vdash \bot$ does not hold;
- $c \Vdash \phi \vee \psi$ if $c \Vdash \phi$ or $c \Vdash \psi$; ...

Theorem

$\Vdash$ *iff* $\vdash$

Kripke semantics

of Disjunction Property.

- Suppose $\nVdash \phi$ and $\nVdash \psi$;
- Then there are models C and D in which ϕ, ψ are not forced;
- Glue these models into $E = \{e\} \sqcup C \sqcup D$.
- E does not force $\phi \vee \psi$.

□