overview

typing system for lambda P

decidability

Curry-Howard-De Bruijn isomorphism

typing system for lambdaP

$$\frac{}{\vdash * : \Box}\ \textit{start}$$

$$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\ \textit{var}$$

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\ \textit{weak}$$

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x{:}A \,.\, B : s}\ \textit{prod}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi x{:}A \,.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x{:}A \,.\, B}\ \textit{abs}$$

$$\frac{\Gamma \vdash F : \Pi x{:}A \,.\, B \qquad \Gamma \vdash M : A}{\Gamma \vdash (F\ M) : B[x := M]}\ \textit{app}$$

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\ \textit{conv} \qquad \text{with } B =_\beta B'$$

examples

$\mathsf{nat} : *, n : \mathsf{nat} \vdash n : \mathsf{nat}$

$\mathsf{nat} : * \vdash \Pi n : \mathsf{nat}.\, \mathsf{nat} : *$

$\mathsf{nat} : * \vdash \lambda n : \mathsf{nat}.\, n : \Pi n : \mathsf{nat}.\, \mathsf{nat}$

decidability issues

- type inhabitation problem (TIP)
  $\Gamma \vdash ? : A$
  undecidable in lambda P

- type checking problem (TCP)
  $\Gamma \vdash P : A?$
  decidable in lambda P

- type synthesis problem (TSP) or typability problem
  $\Gamma \vdash P :?$
  decidable in lambda P

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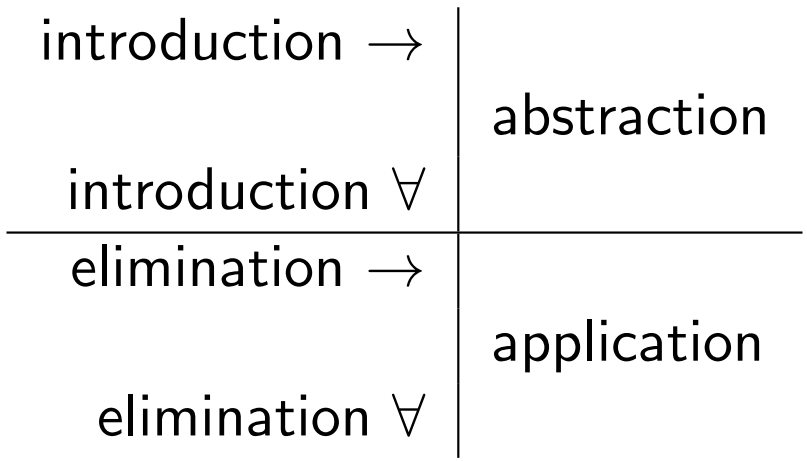
Curry-Howard-De Bruijn isomorphism

(a fragment of) λP corresponds to minimal pred1

Curry-Howard-De Bruijn isomorphism



minimal pred1: only implication and for all

formulas of minimal pred1 compared with prop1:

	prop1	pred1
a	$+$	$+$
$A \to B$	$+$	$+$
$\forall x.\, B$	$-$	$+$

minimal pred1: introduction rules

$$\frac{B}{A \to B}\ I\to \qquad \frac{B}{\forall x.\, B}\ I\forall$$

minimal pred1: elimination rules

$$\frac{A \to B \qquad A}{B}\ E\to \qquad \frac{\forall x.\, B}{B[x := M]}\ E\forall$$

minimal pred1 in λP: formulas

logic	λP	Coq
$P(M_1, \ldots, M_n)$	$PM_1 \ldots M_n : \star$	`P M1 ... Mn : Prop`
$A \to B$	$A \to B : \star$	`A -> B : Prop`
$\forall x.\, B$	$\Pi x : \mathsf{Terms}.\, B : \star$	`forall x: Terms, B : Prop`

with $P : \Pi x_1 : \mathsf{Terms}. \ldots \Pi x_n : \mathsf{Terms}. \star$
or $P : \mathsf{Terms} \to \ldots \to \mathsf{Terms} \to \star$

formulas use (algebraic) terms

minimal pred1 in λP: algebraic terms

logic	λP	Coq
	$\mathsf{Terms} : \star$	`Terms : Set`
x	$x : \mathsf{Terms}$	`x : Terms`
$f(M_1, \ldots, M_n)$	$f\ M_1\ \ldots\ M_n : \mathsf{Terms}$	`f M1 ... Mn : Terms`

with $f : \Pi x_1 : \mathsf{Terms}. \ldots \Pi x_n : \mathsf{Terms}. \mathsf{Terms}$
or $f : \mathsf{Terms} \to \ldots \to \mathsf{Terms} \to \mathsf{Terms}$

introduction rules correspond to abstraction

$$\frac{B}{A \to B}\ I{\to} \qquad \frac{B}{\forall x.\, B}\ I{\forall}$$

correspond to

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash A \to B : *}{\Gamma \vdash \lambda x : A.\, M : A \to B}$$

$$\frac{\Gamma, x : A \vdash M : A \qquad \Gamma \vdash \Pi : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}$$

in Coq: intro

elimination rules correspond to application

$$\frac{A \to B \qquad A}{B}\ E\to \qquad \frac{\forall x.\, B}{B[x := M]}\ E\forall$$

correspond to

$$\frac{F : A \to B \qquad M : A}{F\ M : B}$$

$$\frac{F : \Pi x{:}A\,.\,B \qquad M : A}{F\ M : B[x := M]}$$

in Coq: apply

logics and type theory

1st-order minimal propositional logic $\leftrightarrow$
simple type theory

1st-order minimal predicate logic $\leftrightarrow$
dependent type theory

2nd-order minimal propositional logic $\leftrightarrow$
polymorphic type theory

overview

formulas of prop2

terminology

proofs of prop2

classical prop2

impredicative definitions of logical connectives

overview

- formulas of prop2
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formulas of prop1 (already seen)

$a\ b\ c\ p\ q$

$A \to B$

$\bot$

$\top$

$A \wedge B$

$A \vee B$

formulas of pred1 (already seen)

(using terms)

$a(\ldots)$ $b(\ldots)$ $c(\ldots)$ $p(\ldots)$ $q(\ldots)$

$A \to B$

$\forall x.\, A$

$\bot$

$\top$

$A \wedge B$

$A \vee B$

$\exists x.\, A$

formulas of prop2 (new)

$a\ b\ c\ p\ q$

$A \to B$

$\forall a.\, A$

$\bot$

$\top$

$A \wedge B$

$A \vee B$

$\exists a.\, A$

examples

in prop1:
$a \to a$

in pred1:
$\forall x.\, a(x) \to a(x)$

in prop2:
$\forall a.\, a \to a$

for every proposition, that proposition implies itself

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- formulas of prop2
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higher-order

first order:
object

second order:
set of first-order objects
predicate on objects
function from objects to objects

third order:
set of second-order objects
predicate on predicates on objects
functions from second order objects

higher-order logic

first-order:
quantification over variables of order 1
$a \to a$
$\forall x.\, a(x) \to a(x)$

second-order:
quantification over variables of order 2
$\forall a.\, a \to a$
$\forall a.\, \forall x.\, a(x) \to a(x)$
$\forall f.\, \forall x.\, a(f(x)) \to a(f(x))$

third-order:
quantification over variables of order 3
$\forall b.\, \forall f.\, b(f) \to \forall x.\, a(f(x))$

quantify over predicates gives pred2
same without terms gives prop2

second-order predicate logic: example

induction principle for natural numbers

$$\forall a(a(0) \to (\forall m.\, a(m) \to a(S(m))) \to \forall n.\, a(n))$$

m	1st order variable
n	1st order variable
0	1st order constant
a	2nd order variable
S	2nd order constant

second-order predicate logic: example

there is a sorting function

$\exists f : \text{natlist} \to \text{natlist}.\, \forall l : \text{natlist}.\, \text{sorted}(f(l)) \wedge \text{permutation}(l, f(l)$

f	2nd order variable
l	1st order variable
sorted	2nd order constant
permutation	2nd order constant

examples

prop2	pred2
$\forall a.\, a \to a$	$\forall p.\, \forall x.\, p(x) \to p(x)$
prop1	**pred1**
$a \to a$	$\forall x.\, p(x) \to p(x)$

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proof rules for prop2

introduction rules	elimination rules
$I\top$	
	$E\bot$
$I[x] \to$	$E \to$
$I\wedge$	$El\wedge$, $Er\wedge$
$Il\vee$, $Ir\vee$	$E\vee$
$I\forall$	$E\forall$
$I\exists$	$E\exists$

universal quantification for prop2

$\forall$ introduction:

$$\frac{A}{\forall a.\, A}\ I\forall$$

variable condition: a not free in any open assumption
check: variable does not occur in any of the available assumptions

$\forall$ elimination:

$$\frac{\forall a.\, A}{A[a := B]}\ E\forall$$

existential quantification for prop2

$\exists$ introduction:

$$\frac{A[a := B]}{\exists a.\, A}\ I\exists$$

$\exists$ elimination:

$$\frac{\exists a.\, A \qquad \forall a.\, A \to B}{B}\ E\exists$$

variable condition: a not free in B

check: variable does not occur in the conclusion

examples of tautologies

- $(\forall b.\, b) \to a$
- $a \to \forall b.\, (b \to a)$
- $a \to \forall b.\, ((a \to b) \to b)$
- $(\exists b.\, a) \to a$
- $\exists b\, ((a \to b) \vee (b \to a))$

examples of non-tautological formulas

- $a \to (\forall a.\, a)$
- $p(x) \to (\forall x.\, p(x))$
- $(\exists a.\, a) \to a$
- $\forall a., \forall b.\, (a \to b) \vee (b \to a)$
  (classical logic needed)

minimal prop2: detour

introduction rule for a connective

immediately followed by an

elimination rule for the same connective

elimination of an implication detour (as in prop1)

$$\dfrac{\dfrac{\vdots}{B}}{\dfrac{A \to B}{B}}\, I[x] \to \qquad \dfrac{\vdots}{A}\, E \to$$

is replaced by

$$\dfrac{\vdots}{B}$$

where every occurrence of the assumption A^x is replaced by the proof

$$\dfrac{\vdots}{A}$$

elimination of an universal quantification detour

(similar to pred1)

$$\dfrac{\dfrac{B}{\forall a.\, B}\, I\forall}{B[a := A]}\, E\forall \quad \to \quad B[a := A]$$

everywhere a is replaced by A

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classical prop2

all formulas and proof rules of intuitionistic prop2
plus a classical axiom from

- $\forall a.\, a \vee \neg a$
- $\forall a.\, \neg\neg a \to a$
- $\forall a.\, \forall b.\, ((a \to b) \to a) \to a$

example of a tautology:
$\forall a.\, \forall b.\, ((a \to b) \vee (b \to a))$

expressivity of classical prop2

- $\forall a.\, B$ is equivalent to $B[a := \bot] \wedge B[a := \top]$
- $\exists a.\, B$ is equivalent to $B[a := \bot] \vee B[a := \top]$

so classical prop1 is as expressive as classical prop2

in particular classical prop2 is decidable

however intuitionistic prop2 is not decidable

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schema

polymorphic lambda calculus

Curry-Howard-de Bruijn isomorphism

schema

polymorphic lambda calculus

Curry-Howard-de Bruijn isomorphism

polymorphic identity

identity on nat:
$\lambda x : \mathsf{nat}.\, x : \mathsf{nat} \to \mathsf{nat}$

identity on bool:
$\lambda x : \mathsf{bool}.\, x : \mathsf{bool} \to \mathsf{bool}$

polymorphic identity:
$\lambda a : * \,.\, \lambda x : a.\, x : \Pi a{:} * \,.\, a \to a$

in polymorphic λ-calculus,
we can abstract over variables a in $*$
that is in Coq: variables in `Set` or `Prop`

instantiation of polymorphic types

by application and β-reduction

application:

$$\frac{\lambda a : * \,.\, \lambda x : a.\, x : \Pi a{:} * \,.\, a \to a \qquad \mathsf{nat} : *}{(\lambda a : * \,.\, \lambda x : a.\, x)\,\mathsf{nat} : \mathsf{nat} \to \mathsf{nat}}$$

beta-reduction:

$(\lambda a : * \,.\, \lambda x : a.\, x)\,\mathsf{nat} \to_\beta \lambda x : \mathsf{nat}.\, x$

polymorphic iteration

iteration on nat:

$\lambda f : \mathsf{nat} \to \mathsf{nat}.\, \lambda x : \mathsf{nat}.\, f\,(f\,x)$

iteration on bool:

$\lambda f : \mathsf{bool} \to \mathsf{bool}.\, \lambda x : \mathsf{bool}.\, f\,(f\,x)$

polymorphic iteration:

$\lambda a : \star.\, \lambda f : a \to a.\, \lambda x : a.\, f\,(f\,x)$

lambda2: examples

$$\lambda a : \mathsf{Set}.\ \lambda x : a.\ x$$
$$\lambda a : \mathsf{Prop}.\ \lambda x : a.\ x$$
$$\lambda a : \star\ .\ \lambda x : a.\ x$$

$$\lambda a : \mathsf{Set}.\ \lambda x : a.\ \lambda b : \mathsf{Set}.\ \lambda y : a \to b.\ (y\ x)$$
$$\lambda a : \mathsf{Prop}.\ \lambda x : a.\ \lambda b : \mathsf{Prop}.\ \lambda y : a \to b.\ (y\ x)$$
$$\lambda a : \star\ .\ \lambda x : a.\ \lambda b : \star\ .\ \lambda y : a \to b.\ (y\ x)$$

$$\lambda a : \mathsf{Set}.\ \lambda b : \mathsf{Set}.\ \lambda x : a \to b.\ \lambda y : a.\ (x\ y)$$
$$\lambda a : \mathsf{Prop}.\ \lambda b : \mathsf{Prop}.\ \lambda x : a \to b.\ \lambda y : a.\ (x\ y)$$
$$\lambda a : \star\ .\ \lambda b : \star\ .\ \lambda x : a \to b.\ \lambda y : a.\ (x\ y)$$

slogans

simply typed λ-calculus
terms depending on terms
$\lambda x.\, f\, x$

polymorphic λ-calculus terms depending on types $\lambda a : \star.\, \lambda x.\, x$
dependently typed λ-calculus types depending on terms
$\lambda n : \mathsf{nat}.\, \mathsf{prime}\, n$

lambda2: pseudo-terms

sorts	lambda abstraction
$*$, $\Box$	$\lambda x : A.\ N$
variables	product
$x, y, z, \ldots$	$\Pi x{:}A\,.\,N$
	function application
	$F\ N$

typing system for lambda2

selects the terms from the pseudo-terms

used to derive judgements of the form
$\Gamma \vdash M : N$

M is a term if we can derive
$\Gamma \vdash M : A$ or $\Gamma \vdash N : M$

for every kind of term there is a rule

typing system for lambda2: start and var

axiom:

$$\frac{}{\vdash * : \Box}$$

variable:

$$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}$$

typing system for lambda2: app, abs, product

application:

$$\frac{\Gamma \vdash F : \Pi x{:}A \,.\, B \qquad \Gamma \vdash N : A}{\Gamma \vdash F\ N : B[x := N]}$$

abstraction:

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi x{:}A \,.\, B : \star/\Box}{\Gamma \vdash \lambda x : A.\ M : \Pi x{:}A \,.\, B}$$

product:

$$\frac{\Gamma \vdash A : \star/\Box \qquad \Gamma, x : A \vdash B : *}{\Gamma \vdash \Pi x{:}A \,.\, B : *}$$

typing system for lambda2: weaking and conversion

weakening:

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : \star/\Box}{\Gamma, x : C \vdash A : B}$$

conversion:

$$\frac{\Gamma \vdash A : B' \qquad \Gamma \vdash B : \star/\Box}{\Gamma \vdash A : B} \qquad \text{with } B =_\beta B'$$

the three product rules

all systems:

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : *}{\Gamma \vdash \Pi x{:}A \,.\, B : *}$$

only in λP (to derive nat $\to \star : \Box$):

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : \Box}{\Gamma \vdash \Pi x{:}A \,.\, B : \Box}$$

only in $\lambda 2$ (to derive $\Pi a : \star.\, a \to a : \star$):

$$\frac{\Gamma \vdash A : \Box \qquad \Gamma, x : A \vdash B : *}{\Gamma \vdash \Pi x{:}A \,.\, B : *}$$

schema

polymorphic lambda calculus

Curry-Howard-de Bruijn isomorphism

Curry-Howard-de Bruijn isomorphism

$$\begin{array}{lcl} \text{prop1} & \sim & \lambda \rightarrow \\ \text{pred1} & \sim & \lambda P \\ \text{prop2} & \sim & \lambda 2 \end{array}$$

introduction rules correspond to abstraction

$$\frac{B}{A \to B}\ I\to \qquad \frac{B}{\forall a.\, B}\ I\forall$$

correspond to

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash A \to B : *}{\Gamma \vdash \lambda x : A.\, M : A \to B}$$

$$\frac{\Gamma, a : \star \vdash M : B \qquad \Gamma \vdash \Pi a : \star.\, B : \star}{\Gamma \vdash \lambda a : \star.\, M : \Pi a : \star.\, B}$$

in Coq: intro

elimination rules correspond to application

$$\frac{A \to B \qquad A}{B}\ E\to \qquad \frac{\forall a.\, B}{B[a := A]}\ E\forall$$

correspond to

$$\frac{F : A \to B \qquad M : A}{F\ M : B}$$

$$\frac{F : \Pi a{:}\star\ .\ B \qquad A : \star}{F\ A : B[a := A]}$$

in Coq: apply

examples

$(\forall b.\, b) \to a$

$a \to \forall b.\, (b \to a)$ (cf paradox)

$a \to \forall b.\, ((a \to b) \to b)$

more examples

$$\lambda a : \star \,.\, \lambda x : a.\, x \;:\; \Pi a{:}\star \,.\, \Pi x{:}a \,.\, a$$

$$\lambda a : \star \,.\, \lambda x : a.\, \lambda b : \star \,.\, \lambda y : a \to b.\, (y\, x) \;:\; \Pi a{:}\star \,.\, \Pi x{:}a \,.\, \Pi b{:}\star \,.\, \Pi y{:}a \to b \,.$$

$$\lambda a : \star \,.\, \lambda b : \star \,.\, \lambda x : a \to b.\, \lambda y : a.\, (x\, y) \;:\; \Pi a{:}\star \,.\, \Pi b{:}\star \,.\, \Pi x{:}a \to b \,.\, \Pi y{:}a \,.$$