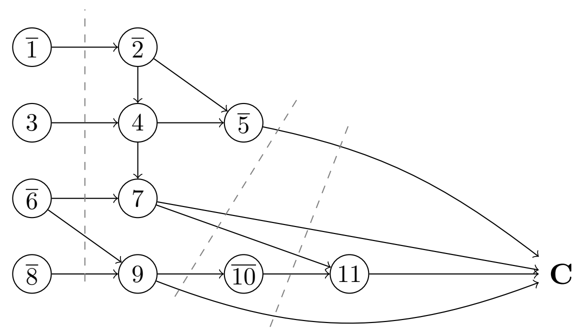
SAT and SMT Solving SS 2019 LVA 703048

Solutions to Test Exercises July 1, 2019

1 (a) The implication graph looks as follows:



The UIPs are $\overline{8}$ and 9. The indicated cuts lead to the implied clauses $1 \vee \overline{3} \vee 6 \vee 8$, $5 \vee \overline{7} \vee \overline{9}$, and $5 \vee \overline{7} \vee \overline{9} \vee 10$, from left to right.

(b) The clause $5 \vee \overline{7} \vee \overline{9}$ is minimal. Using resolution it can be derived as follows: The conflict clause is $5 \vee \overline{7} \vee \overline{9} \vee \overline{11}$, its literal whose complement was assigned last is $\overline{11}$. The clause responsible for this assignment is $\overline{7} \vee 10 \vee 11$. We thus resolve

$$\frac{5 \vee \overline{7} \vee \overline{9} \vee \overline{11} \qquad \overline{7} \vee 10 \vee 11}{5 \vee \overline{7} \vee \overline{9} \vee 10}$$

The literal in the resulting clause whose complement was assigned last is 10. The clause responsible for this assignment is $\overline{9} \vee \overline{10}$. We hence get

$$\frac{5 \vee \overline{7} \vee \overline{9} \vee 10 \qquad \overline{9} \vee \overline{10}}{5 \vee \overline{7} \vee \overline{9}}$$

2 (a) For example, substituting literals as follows:

$$1\colon \mathsf{a} \approx \mathsf{b} \quad 2\colon \mathsf{c} \approx \mathsf{g}(\mathsf{a}) \quad 3\colon \mathsf{f}(\mathsf{a}) \approx \mathsf{f}(\mathsf{b}) \quad 4\colon \mathsf{f}(\mathsf{g}(\mathsf{a})) \approx \mathsf{g}(\mathsf{a}) \quad 5\colon \mathsf{f}(\mathsf{a}) \approx \mathsf{c} \quad 6\colon \mathsf{f}(\mathsf{b}) \approx \mathsf{f}(\mathsf{c})$$

the propositional skeleton is $1 \wedge 2 \wedge (\overline{3} \vee \overline{4}) \wedge 5 \wedge 6$.

(b) We apply DPLL(T) as follows:

	$\parallel 1,\ 2,\ \overline{3} \vee \overline{4},\ 5,\ 6$		
$\Longrightarrow^+$	$1\,2\,5\,6 \parallel 1,\ 2,\ \overline{3} \vee \overline{4},\ 5,\ 6$	unit propagate
$\Longrightarrow$	$1\,2\,5\,6\,3 \parallel 1,\ 2,\ \overline{3} \vee \overline{4},\ 5,\ 6$	T-propagate
$\Longrightarrow$	$1\,2\,5\,6\,3\,\overline{4} \parallel 1,\ 2,\ \overline{3} \vee \overline{4},\ 5,\ 6$	unit propagate

(Here the literal 3 is T-learned from literal 1 in a single step of the congruence closure algorithm.) At this point the SAT solver claims satisfiability with model $1\ 2\ 5\ 6\ 3\ \overline{4}$, corresponding to

$$\mathsf{a} \approx \mathsf{b} \wedge \mathsf{c} \approx \mathsf{g}(\mathsf{a}) \wedge \mathsf{f}(\mathsf{a}) \approx \mathsf{c} \wedge \mathsf{f}(\mathsf{b}) \approx \mathsf{f}(\mathsf{c}) \wedge \mathsf{f}(\mathsf{a}) \approx \mathsf{f}(\mathsf{b}) \wedge \mathsf{f}(\mathsf{g}(\mathsf{a})) \not\approx \mathsf{g}(\mathsf{a})$$

We apply congruence closure to check that the model is T-consistent. This is the case if the positive literals do not imply $\mathsf{f}(\mathsf{g}(\mathsf{a})) \approx \mathsf{g}(\mathsf{a})$, which is the negation of the negative literal. We start by putting all subterms into different sets:

$$1\colon \{\mathsf{a}\} \quad 2\colon \{\mathsf{b}\} \quad 3\colon \{\mathsf{c}\} \quad 4\colon \{\mathsf{f}(\mathsf{a})\} \quad 5\colon \{\mathsf{f}(\mathsf{b})\} \quad 6\colon \{\mathsf{f}(\mathsf{c})\} \quad 7\colon \{\mathsf{g}(\mathsf{a})\} \quad 8\colon \{\mathsf{f}(\mathsf{g}(\mathsf{a}))\}$$

Merging sets according to equations results in

$$1\colon \{\mathsf{a},\ \mathsf{b}\} \qquad 3\colon \{\mathsf{c},\ \mathsf{g}(\mathsf{a}),\ \mathsf{f}(\mathsf{a}),\ \mathsf{f}(\mathsf{b}),\ \mathsf{f}(\mathsf{c})\} \qquad 8\colon \{\mathsf{f}(\mathsf{g}(\mathsf{a}))\}$$

Since $\mathsf{c} \approx \mathsf{g}(\mathsf{a})$ implies $\mathsf{f}(\mathsf{c}) \approx \mathsf{f}(\mathsf{g}(\mathsf{a}))$ the sets 3 and 8 must be merged:

$$1\colon \{\mathsf{a},\ \mathsf{b}\} \qquad 3\colon \{\mathsf{c},\ \mathsf{g}(\mathsf{a}),\ \mathsf{f}(\mathsf{a}),\ \mathsf{f}(\mathsf{b}),\ \mathsf{f}(\mathsf{c}),\ \mathsf{f}(\mathsf{g}(\mathsf{a}))\}$$

No more merge steps are possible. As $\mathsf{f}(\mathsf{g}(\mathsf{a}))$ and $\mathsf{g}(\mathsf{a})$ are in the same set the model is inconsistent, and the clause $\overline{1} \vee \overline{2} \vee \overline{5} \vee \overline{6} \vee \overline{3} \vee 4$ negating the current model is implied. We can continue the DPLL(T) run by a T-learn step to add the new clause:

$$\Longrightarrow \quad 1\,2\,5\,6\,3\,\overline{4} \parallel 1,\ 2,\ \overline{3} \vee \overline{4},\ 5,\ 6,\ \overline{1} \vee \overline{2} \vee \overline{5} \vee \overline{6} \vee \overline{3} \vee 4 \qquad T\text{-learn}$$

$$\Longrightarrow \quad \mathsf{FailState} \qquad \text{fail}$$

and conclude unsatisfiability after a final step.

3 (a) Both s_1 and s_2 violate their (upper) bound, so we want to decrease the value of one of them. Because the coefficients for both s_1 and s_2 with x and s_3 are negative, the value of one of x or s_3 has to be increased. But s_3 is at its upper bound, so not suitable.

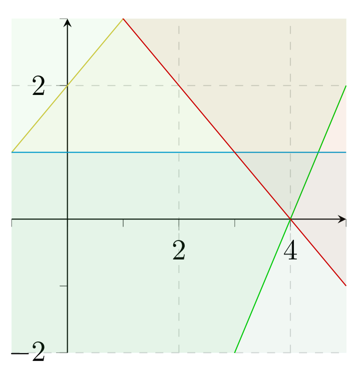
So only (s_1, x) and (s_2, x) are suitable.

(b) We pivot s_2 with x, because $s_2 < s_1$. This yields the following updated tableau:

$$\begin{array}{c} \\ s_1 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 \\ -\frac{3}{2} & -2 \end{pmatrix} \end{array}$$

The variable s_2 is set to its upper bound 2, and the nonbasic variable s_3 is still assigned -4. The remaining variables are updated to $x = 3$, $y = 1$, $s_1 = -2$, $s_4 = 5$, which satisfies the constraints.

(c) The solution space is given by the small triangle, hence it is bounded.



4 (a) We use auxiliary bitvectors $\mathbf{c}_3$ and (optionally) $\mathbf{s}_3$ to encode addition:

$$c_0 = a_0 \wedge b_0 \qquad s_0 = a_0 \oplus b_0$$
$$c_1 = (a_1 \wedge b_1) \vee (a_1 \wedge c_0) \vee (b_1 \wedge c_0) \qquad s_1 = a_1 \oplus b_1 \oplus c_0$$
$$c_2 = (a_2 \wedge b_2) \vee (a_2 \wedge c_1) \vee (b_2 \wedge c_1) \qquad s_2 = a_2 \oplus b_2 \oplus c_1$$

Then c_2 encodes occurrence of an overflow. (The sum $\mathbf{s}_3$ is actually not needed for that.)

(b) We can simply set $\mathbf{exp2}(\mathbf{c}_8) = \mathbf{1}_8 \ll \mathbf{c}_8$.

5 (a) The formula $(z = 0) \wedge (x + z \geqslant y) \wedge (y \geqslant x) \wedge (\mathsf{f}(z) > z) \wedge (\mathsf{f}(x) = \mathsf{f}(z) + \mathsf{f}(y))$ can be purified to

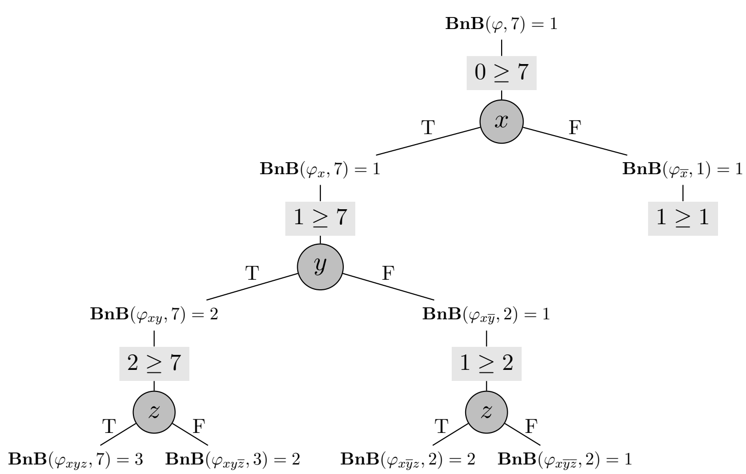
$$\psi_1 = (z = 0) \wedge (x + z \geqslant y) \wedge (y \geqslant x) \wedge (c > z) \wedge (a = c + b)$$
$$\psi_2 = (a = \mathsf{f}(x)) \wedge (b = \mathsf{f}(y)) \wedge (c = \mathsf{f}(z))$$

using fresh variables a, b, and c.

(b) We use the deterministic version of the Nelson-Oppen procedure. Initially, the set E of inferred equations is empty.

- First, LIA infers $x = y$ from ψ_1 and therefore sets $E = \{x = y\}$.
- Now EUF can infer $a = b$ from ψ_2 together with E, and hence updates $E = \{x = y,\ a = b\}$.
- At this point LRA concludes unsatisfiability from ψ_1 and E because $a = b$ and $a = c + b$ but $c > 0$.

6 (a) The following computation yields $\mathtt{BnB}(\varphi, 7) = 1$, hence $\text{maxSAT}(\varphi) = 6$.



(b) An SUC is given by the clause set $\{\neg x,\ x\}$.