1 Consider the formula

$$(1 \vee \overline{2}) \wedge (2 \vee \overline{3} \vee 4) \wedge (2 \vee \overline{4} \vee \overline{5}) \wedge (\overline{4} \vee 6 \vee 7) \wedge (6 \vee 8 \vee 9) \wedge (\overline{9} \vee \overline{10}) \wedge (\overline{7} \vee 10 \vee 11) \wedge (5 \vee \overline{7} \vee \overline{9} \vee \overline{11})$$

and suppose a DPLL inference sequence reached the state $\overline{1}^d\, \overline{2}\, 3^d\, 4\, \overline{5}\, \overline{6}^d\, 7\, \overline{8}^d\, 9\, \overline{10}\, 11$.

[3] (a) Construct an implication graph and give three different cuts together with the induced implied clauses. Which nodes are UIPs?

[3] (b) Give an implied clause derived from a cut that has as few literals as possible. Derive this clause by resolution from the conflict.

2 Consider the following formula φ:

$$\mathsf{a} \approx \mathsf{b} \wedge \mathsf{c} \approx \mathsf{g}(\mathsf{a}) \wedge (\mathsf{f}(\mathsf{a}) \not\approx \mathsf{f}(\mathsf{b}) \vee \mathsf{f}(\mathsf{g}(\mathsf{a})) \not\approx \mathsf{g}(\mathsf{a})) \wedge \mathsf{f}(\mathsf{a}) \approx \mathsf{c} \wedge \mathsf{f}(\mathsf{b}) \approx \mathsf{f}(\mathsf{c})$$

[1] (a) Construct a propositional skeleton of φ.

[5] (b) Determine satisfiability of φ using DPLL(T). Explain which DPLL(T) inference rules are used, which EUF problems appear and how they are solved using congruence closure, and why the formula is eventually determined to be (un)satisfiable.

3 Consider the following set of linear constraints:

$$-x + y \leqslant 2 \qquad 2y \leqslant 2 \qquad -x - y \leqslant -4 \qquad 2x - y \leqslant 8$$

Suppose a run of the Simplex algorithm with variable order $s_1 > s_2 > s_3 > s_4 > x > y$ reached the following intermediate state:

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & s_3 \end{array} \\ \begin{pmatrix} -2 & -1 \\ -2 & -2 \\ -1 & -1 \\ 3 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ s_1 \leqslant 2 \\ s_2 \leqslant 2 \\ s_3 \leqslant -4 \\ s_4 \leqslant 8 \end{array}$$

with assignment $x = 0$, $y = 4$, $s_1 = 4$, $s_2 = 8$, $s_3 = -4$, $s_4 = -4$.

[2] (a) Which basic variables violate their bounds? Which variable pairs are suitable for a pivot step, and why?

[4] (b) Perform the pivot step according to Bland's rule, and compute the new assignment. Does it satisfy the constraints?

[2] (c) Is the (initial) problem bounded? (Explain why!)

[3] 4 (a) Consider bitvector variables $\mathbf{a}_3$, $\mathbf{b}_3$ of three bits each. Suppose that $\mathbf{a}_3 = a_2 a_1 a_0$, with a_2 being the most significant bit (so $\mathbf{a} = 2^0 \cdot a_0 + 2^1 \cdot a_1 + 2^2 \cdot a_2$), and similarly for $\mathbf{b}_3$. Give a propositional formula which encodes that the addition $\mathbf{a}_3 + \mathbf{b}_3$ overflows, i.e., the result is no longer representable in 3 bits.

[2] (b) Consider a bitvector variable $\mathbf{c}_8$ of eight bits. Use bitvector operations to express the exponentiation $\mathbf{exp2}(\mathbf{c}_8) = 2^{\mathbf{c}_8}$. For instance, $\mathbf{exp2}(\mathbf{3}_8) = \mathbf{exp2}(00000011) = 00001000 = \mathbf{8}_8$ as $2^3 = 8$.

In case of an overflow, your encoding can return anything, but it should work correctly as long as no overflow occurs. You can use all bitvector operations introduced in the lecture as auxiliaries to build your encoding.

5 The following formula ψ combines uninterpreted functions and linear integer arithmetic:

$$(z = 0) \land (x + z \geqslant y) \land (y \geqslant x) \land (\mathsf{f}(z) > z) \land (\mathsf{f}(x) = \mathsf{f}(z) + \mathsf{f}(y))$$

[2] (a) Purify ψ.

[3] (b) Use the Nelson-Oppen procedure to determine satisfiability of ψ. You can use either the deterministic or the nondeterministic version.

$\star$ 6 Consider the formula φ:

$$(\neg x \lor \neg y) \land \neg x \land (x \lor y) \land (x \lor \neg y \lor z) \land x \land (x \lor \neg y) \land \neg z$$

[3] (a) Use the Branch & Bound algorithm to determine minUNSAT(φ) and maxSAT(φ).

[1] (b) Determine a smallest unsatisfiable core (SUC) of φ.

Exercises marked with a $\star$ are optional and give extra points.