Solutions 1 March 15, 2019

1 Given the formula ϕ:

$$(1 \vee \neg 3) \wedge (\neg 1 \vee \neg 4) \wedge (\neg 1 \vee \neg 3 \vee 4) \wedge (2 \vee 3 \vee 4) \wedge (\neg 2 \vee 4) \wedge (3 \vee \neg 4) \wedge (\neg 1 \vee 2 \vee 5)$$

the following DPLL inference sequence shows its unsatisfiability:

$$\begin{aligned}
\| \phi &\Longrightarrow 1^d \| \phi && \text{decide}\\
&\Longrightarrow 1^d\,\overline{4} \| \phi && \text{unit propagate}\\
&\Longrightarrow 1^d\,\overline{4}\,\overline{3} \| \phi && \text{unit propagate}\\
&\Longrightarrow 1^d\,\overline{4}\,\overline{3}\,2 \| \phi && \text{unit propagate}\\
&\Longrightarrow \overline{1} \| \phi && \text{backtrack}\\
&\Longrightarrow \overline{1}\,\overline{3} \| \phi && \text{unit propagate}\\
&\Longrightarrow \overline{1}\,\overline{3}\,\overline{4} \| \phi && \text{unit propagate}\\
&\Longrightarrow \overline{1}\,\overline{3}\,\overline{4}\,2 \| \phi && \text{unit propagate}\\
&\Longrightarrow \mathsf{FailState} && \text{fail}
\end{aligned}$$

2 We consider the following additional variables:

$a_0 \colon \neg(p \vee \neg(q \vee r)) \wedge (q \vee r)$ $a_1 \colon q \vee r$

$a_2 \colon \neg(p \vee \neg(q \vee r))$ $a_3 \colon p \vee \neg(q \vee r)$

$a_4 \colon \neg(q \vee r)$

(a) Writing $\overline{a_i}$ for $\neg a_i$, Tseitin's transformation leads to a formula with 14 clauses:

$$\begin{aligned}
\phi &\approx a_0 \wedge (a_0 \leftrightarrow a_2 \wedge a_1) \wedge (a_1 \leftrightarrow q \vee r) \wedge (a_2 \leftrightarrow \overline{a_3}) \wedge (a_3 \leftrightarrow p \vee a_4) \wedge (a_4 \leftrightarrow \overline{a_1})\\
&\equiv a_0 \wedge (\overline{a_0} \vee a_2) \wedge (\overline{a_0} \vee a_1) \wedge (\overline{a_2} \vee \overline{a_1} \vee a_0) \wedge\\
&\quad (\overline{a_1} \vee q \vee r) \wedge (\overline{q} \vee a_1) \wedge (\overline{r} \vee a_1) \wedge\\
&\quad (\overline{a_2} \vee \overline{a_3}) \wedge (a_2 \vee a_3) \wedge\\
&\quad (\overline{a_3} \vee p \vee a_4) \wedge (\overline{p} \vee a_3) \wedge (\overline{a_4} \vee a_3) \wedge\\
&\quad (\overline{a_4} \vee \overline{a_1}) \wedge (a_4 \vee a_1)
\end{aligned}$$

(b) The transformation by Plaisted and Greenbaum produces a formula ψ with 10 clauses:

$$\begin{aligned}
\phi &\approx a_0 \wedge (a_0 \rightarrow a_2 \wedge a_1) \wedge (a_1 \rightarrow q \vee r) \wedge (a_2 \rightarrow \overline{a_3}) \wedge (a_3 \leftarrow p \vee a_4) \wedge (a_4 \rightarrow \overline{a_1}) \wedge\\
&\quad (a_1 \leftarrow q \vee r)\\
&\equiv a_0 \wedge (\overline{a_0} \vee a_2) \wedge (\overline{a_0} \vee a_1) \wedge\\
&\quad (\overline{a_1} \vee q \vee r) \wedge\\
&\quad (\overline{a_2} \vee \overline{a_3}) \wedge\\
&\quad (\overline{p} \vee a_3) \wedge (\overline{a_4} \vee a_3) \wedge\\
&\quad (\overline{a_4} \vee \overline{a_1}) \wedge\\
&\quad (\overline{q} \vee a_1) \wedge (\overline{r} \vee a_1)
\end{aligned}$$

The following DPLL inference sequence shows satisfiability of ψ:

$$
\begin{aligned}
\| \psi &\Longrightarrow a_0 \| \psi && \textsf{unit propagate}\\
&\Longrightarrow a_0\, a_1 \| \psi && \textsf{unit propagate}\\
&\Longrightarrow a_0\, a_1\, a_2 \| \psi && \textsf{unit propagate}\\
&\Longrightarrow a_0\, a_1\, a_2\, \overline{a_3} \| \psi && \textsf{unit propagate}\\
&\Longrightarrow a_0\, a_1\, a_2\, \overline{a_3}\, \overline{p} \| \psi && \textsf{unit propagate}\\
&\Longrightarrow a_0\, a_1\, a_2\, \overline{a_3}\, \overline{p}\, \overline{a_4} \| \psi && \textsf{unit propagate or pure literal}\\
&\Longrightarrow a_0\, a_1\, a_2\, \overline{a_3}\, \overline{p}\, \overline{a_4}\, q^d \| \psi && \textsf{decide}\\
&\Longrightarrow a_0\, a_1\, a_2\, \overline{a_3}\, \overline{p}\, \overline{a_4}\, q^d\, r^d \| \psi && \textsf{decide}
\end{aligned}
$$

(The last step is only necessary if a total assignment is desired.)

3 We use variables a, b, and c corresponding to Adam, Brian, and Carol being the youngest kid.

(a) The statements can be modelled as follows:
Adam says "If Carol is not the youngest, then I am." $\neg c \to a$
Brian claims "If I am not the youngest, then neither is Adam." $\neg b \to \neg a$
We also need to account for the fact that only one of the variables can be true. This is for instance expressed by saying that for every pair of variables one of them must be false:

$$(\neg a \vee \neg b) \wedge (\neg a \vee \neg c) \wedge (\neg b \vee \neg c)$$

Transforming the implications to CNF leads to a problem that can be phrased in DIMACS as follows:

```
p cnf 3 5
3 1 0
2 -1 0
-1 -2 0
-1 -3 0
-2 -3 0
```

Minisat returns the solution $v(a) = v(b) = \bot$ and $v(c) = \top$, so Carol is the youngest.

(b) To exclude the above solution, we add a clause demanding that the value of at least one variable is different. Hence, we add $a \vee b \vee \neg c$ (corresponding to `1 2 -3 0` in DIMACS). The resulting problem is unsatisfiable.

4 We use the following variables to capture whether there are mines hiding in the unknown cells of the given Minesweeper board:

	2		1
	3		3
			2

x_1		x_2	
x_3	x_4	x_5	x_{11}
x_6		x_7	
x_8	x_9	x_{10}	

Every number gives rise to a constraint. From left to right, top to bottom they can, for example, be expressed as follows:

2	$\bigvee_{1\leq i<j\leq 5} x_i \wedge x_j$	two cells have a mine
	$\bigvee_{1\leq i<j<k\leq 5} \neg x_i \wedge \neg x_j \wedge \neg x_k$	three cells have no mine
1	$(x_2 \vee x_5 \vee x_{11})$	some cell has a mine
	$(\neg x_2 \wedge \neg x_5) \vee (\neg x_2 \wedge \neg x_{11}) \vee (\neg x_5 \wedge \neg x_{11})$	two cells have no mine
3	$\bigvee_{3\leq i<j<k\leq 10} x_i \wedge x_j \wedge x_k$	three cells have a mine
	$\bigvee_{3\leq i<j<k<l<m\leq 10} \neg x_i \wedge \neg x_j \wedge \neg x_k \wedge \neg x_l \wedge \neg x_m$	five cells have no mine
3	$(x_5 \wedge x_7 \wedge x_{10}) \vee (x_7 \wedge x_{10} \wedge x_{11}) \vee$	
	$(x_5 \wedge x_{10} \wedge x_{11}) \vee (x_5 \wedge x_7 \wedge x_{11})$	three cells have a mine
	$(\neg x_5 \vee \neg x_7 \vee \neg x_{10} \vee \neg x_{11})$	one cell has no mine
2	$x_7 \wedge x_{10}$	

(This is a straightforward encoding offering a variety of improvement possibilities, for instance one can use the last conjunction to simplify the earlier constraints.)

An encoding using `z3py` can be found here. The formula is satisfiable, and the according solution is that the cells with a mine are x_1, x_5, x_7, and x_{10}.