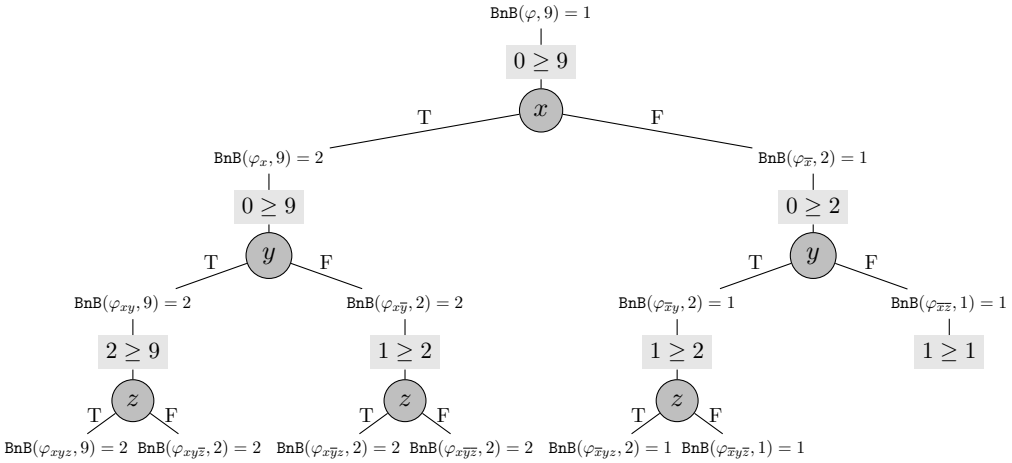
Selected Solutions 3 | March 29, 2018

1 The following computation yields $\texttt{BnB}(\varphi, 9) = 1$, hence $\text{maxSAT}(\varphi) = 8$.



$$
\begin{aligned}
\texttt{simp}(\varphi_x) =& \neg y,\ (\mathrm{T} \vee y),\ (\neg\mathrm{T} \vee z),\ (\neg\mathrm{T} \vee y),\ (\neg\mathrm{T} \vee \neg y),\ (\mathrm{T} \vee z),\ (\neg\mathrm{T} \vee \neg y \vee z),\\
& (y \vee \neg z),\ z\\
=& \neg y,\ z,\ y,\ \neg y,\ (\neg y \vee z),\ (y \vee \neg z),\ z\\
\texttt{simp}(\varphi_{xy}) =& \neg\mathrm{T},\ z,\ \mathrm{T},\ \neg\mathrm{T},\ (\neg\mathrm{T} \vee z),\ (\mathrm{T} \vee \neg z),\ z = \square,\ z,\ \square,\ z,\ z\\
\texttt{simp}(\varphi_{xyz}) =& \square,\ \square\\
\texttt{simp}(\varphi_{xy\bar{z}}) =& \square,\ \square,\ \square,\ \square,\ \square\\
\texttt{simp}(\varphi_{x\bar{y}}) =& \neg\mathrm{F},\ z,\ \mathrm{F},\ \neg\mathrm{F},\ (\neg\mathrm{F} \vee z),\ (\mathrm{F} \vee \neg z),\ z = z,\ \square,\ \neg z,\ z\\
\texttt{simp}(\varphi_{x\bar{y}z}) =& \square,\ \square,\ \square\\
\texttt{simp}(\varphi_{x\bar{y}\bar{z}}) =& \square,\ \square\\
\texttt{simp}(\varphi_{\bar{x}}) =& \neg y,\ (\mathrm{F} \vee y),\ (\neg\mathrm{F} \vee z),\ (\neg\mathrm{F} \vee y),\ (\neg\mathrm{F} \vee \neg y),\ (\mathrm{F} \vee z),\ (\neg\mathrm{F} \vee \neg y \vee z),\\
& (y \vee \neg z),\ z\\
=& \neg y,\ y,\ z,\ (y \vee \neg z),\ z\\
\texttt{simp}(\varphi_{\bar{x}y}) =& \square,\ z,\ z\\
\texttt{simp}(\varphi_{\bar{x}yz}) =& \square\\
\texttt{simp}(\varphi_{\bar{x}y\bar{z}}) =& \square,\ \square,\ \square\\
\texttt{simp}(\varphi_{\bar{x}\bar{y}}) =& \square,\ z,\ \neg z,\ z
\end{aligned}
$$

2 See the file binSearch.py.

4 Let φ be a list of clauses. We prove by induction on the number of variables in φ that if the number of empty clauses in φ is smaller or equal to k then $\texttt{BnB}(\varphi, k)$ returns $\min(\text{minUNSAT}(\varphi, k), k)$,

for any number k, which implies the desired statement. Below we will use the fact that $\texttt{simp}(\psi) \equiv \psi$ for any formula ψ, which is easy to show.

For the base case, suppose there are no variables. Then $\texttt{simp}(\varphi) \equiv \varphi$ can contain only empty clauses, which is exactly the number of clauses that evaluate to false in any assignment. Since the number of empty clauses is by assumption smaller or equal to k, the statement follows.

Now suppose φ contains at least one variable x. Since $\texttt{simp}(C) \equiv C$, simplification does not affect the result. By the assumption on the the number of variables, φ cannot contain only empty clauses, so the first `if` statement cannot apply. Let m be the number of empty clauses in $\texttt{simp}(\varphi)$. No valuation can satisfy empty clauses, so in the case where there $m \geqslant k$ we have $\min(\text{minUNSAT}(\varphi, k), k) = k$.

By the induction hypothesis $\texttt{BnB}(\varphi_x, k') = \min(\text{minUNSAT}(\varphi_x, k'), k')$ and $\texttt{BnB}(\varphi_{\overline{x}}, k') = \min(\text{minUNSAT}(\varphi_{\overline{x}}, k'), k')$ hold for all $k' \leqslant k$ because the formulas φ_x and $\varphi_{\overline{x}}$ have one variable less, and the number of empty clauses in these formulas cannot exceed k. Let N be $\text{minUNSAT}(\varphi, k)$, which is the minimum number of clauses falsified by any assignment by definition. This assignment must assign T of F to x. In the former case we have $N = \text{minUNSAT}(\varphi_x, k)$ and in the latter $N = \text{minUNSAT}(\varphi_{\overline{x}}, k)$, but in either case N is the minimum of the two. We have that $\texttt{BnB}(\varphi, k)$ returns $\text{minUNSAT}(\varphi_x, k)$, $\text{minUNSAT}(\varphi_{\overline{x}}, k)$, and k, which thus indeed is $\min(\text{minUNSAT}(\varphi, k), k)$.