1 After collecting all subterms and merging those occurring together in an equation, we have the following sets:

(1) $\mathsf{a}, \mathsf{b}, \mathsf{f}(\mathsf{a}), \mathsf{g}(\mathsf{b})$ (2) $\mathsf{f}(\mathsf{b}), \mathsf{c}$ (3) $\mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{g}(\mathsf{b}))$ (4) $\mathsf{f}(\mathsf{c}), \mathsf{f}(\mathsf{g}(\mathsf{c}))$

We merge (1) containing $\mathsf{f}(\mathsf{a})$ and (2) containing $\mathsf{f}(\mathsf{b})$ because a and b are in the same set:

(1) $\mathsf{a}, \mathsf{b}, \mathsf{f}(\mathsf{a}), \mathsf{g}(\mathsf{b}), \mathsf{f}(\mathsf{b}), \mathsf{c}$ (3) $\mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{g}(\mathsf{b}))$ (4) $\mathsf{f}(\mathsf{c}), \mathsf{f}(\mathsf{g}(\mathsf{c}))$

Also (1) containing $\mathsf{g}(\mathsf{b})$ and (3) containing $\mathsf{g}(\mathsf{a})$ are merged as a and b are in the same set:

(1) $\mathsf{a}, \mathsf{b}, \mathsf{f}(\mathsf{a}), \mathsf{g}(\mathsf{b}), \mathsf{f}(\mathsf{b}), \mathsf{c}, \mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{g}(\mathsf{b}))$ (4) $\mathsf{f}(\mathsf{c}), \mathsf{f}(\mathsf{g}(\mathsf{c}))$

Finally, also (1) containing $\mathsf{f}(\mathsf{b})$ and (4) containing $\mathsf{f}(\mathsf{c})$ are merged because b and c are both in (1). So all the occurring terms are equal, in particular $E \vDash \mathsf{a} \approx \mathsf{c}$ and $E \vDash \mathsf{f}(\mathsf{c}) \approx \mathsf{c}$ hold.

2 Considering the formula φ:

$$(\mathsf{g}(x_3) \not\approx \mathsf{g}(x_4) \vee \mathsf{g}(x_1) \approx x_1) \wedge (x_3 \approx x_4 \vee x_1 \approx x_2) \wedge (\mathsf{f}(x_1, x_3) \approx x_1 \vee x_1 \approx \mathsf{f}(x_1, x_4)) \wedge$$
$$x_1 \not\approx x_2 \wedge (\mathsf{g}(x_3) \not\approx \mathsf{g}(x_4) \vee \mathsf{g}(\mathsf{f}(x_1, x_3)) \not\approx \mathsf{g}(x_1))$$

one can abbreviate:

$1\colon \mathsf{g}(x_3) \approx \mathsf{g}(x_4)$ $2\colon \mathsf{g}(x_1) \approx x_1$ $3\colon x_3 \approx x_4$
$4\colon x_1 \approx x_2$ $5\colon \mathsf{f}(x_1, x_3) \approx x_1$ $6\colon x_1 \approx \mathsf{f}(x_1, x_4)$
$7\colon \mathsf{g}(\mathsf{f}(x_1, x_3)) \approx \mathsf{g}(x_1)$

and apply DPLL(T) to the propositional formula $(\overline{1} \vee 2) \wedge (3 \vee 4) \wedge (5 \vee 6) \wedge \overline{4} \wedge (\overline{1} \vee \overline{7})$:

	$\parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7})$	
$\Longrightarrow$	$\overline{4} \parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7})$	unit propagate
$\Longrightarrow$	$\overline{4}\, 3 \parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7})$	unit propagate
$\Longrightarrow$	$\overline{4}\, 3\, 1 \parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7})$	T-propagate ①
$\Longrightarrow^+$	$\overline{4}\, 3\, 1\, 2\, \overline{7} \parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7})$	unit propagate
$\Longrightarrow$	$\overline{4}\, 3\, 1\, 2\, \overline{7}\, 5^d \parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7})$	decide
$\Longrightarrow$	$\overline{4}\, 3\, 1\, 2\, \overline{7}\, \overline{5} \parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7})$	T-backjump ②
$\Longrightarrow$	$\overline{4}\, 3\, 1\, 2\, \overline{7}\, \overline{5}\, 6 \parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7})$	unit propagate
$\Longrightarrow$	$\overline{4}\, 3\, 1\, 2\, \overline{7}\, \overline{5}\, 6 \parallel (\overline{1} \vee 2), (3 \vee 4), (5 \vee 6), \overline{4}, (\overline{1} \vee \overline{7}), (\overline{3} \vee 5 \vee \overline{6})$	T-learn ③
$\Longrightarrow$	FailState	

Hence φ is T-unsatisfiable. The steps ①, ②, and ③ use theory reasoning as follows:

① The T-propagation step uses the fact that $\overline{4} \wedge 3 \models_T 1$ holds, which can be seen as follows. By definition of $\models_T$, this means that $\overline{4} \wedge 3 \wedge \overline{1}$ is unsatisfiable. To this end, we use congruence closure to show $3 \models_T 1$, that is, $x_3 \approx x_4 \models \mathsf{g}(x_3) \approx \mathsf{g}(x_4)$. We skolemize all terms, replacing x_3 and x_4 by the constants c_3 and c_4, and start from term sets

(1) c_3 (2) c_4 (3) $\mathsf{g}(\mathsf{c}_3)$ (4) $\mathsf{g}(\mathsf{c}_4)$

One first merges (1) and (2) because of $\mathsf{c}_3 \approx \mathsf{c}_4$, and then (3) and (4) since the two constants are in the same set. So $\mathsf{g}(\mathsf{c}_3)$ and $\mathsf{g}(\mathsf{c}_4)$ are congruent.

② This step uses the backjump clause $C = (4 \vee \overline{3} \vee \overline{1} \vee \overline{2} \vee 7 \vee \overline{5})$ obtained from the negated assignment $M = \overline{4}\ 3\ 1\ 2\ \overline{7}\ 5$. The clause satisfies $\varphi \models_T C$ because $\neg C$ itself is already T-inconsistent. To see that, we need to find a literal in the set $N = \{5, 7\}$ occurring negatively in M which is implied by the positive literals $P = \{1, 2, 3, 5\}$. So we again use congruence closure to show that $2, 5 \models_T 7$, that is, $\mathsf{g}(x_1) \approx x_1,\ \mathsf{f}(x_1, x_3) \approx x_1 \models_T \mathsf{g}(\mathsf{f}(x_1, x_3)) \approx \mathsf{g}(x_1)$ holds (which implies $1, 2, 3, 5 \models_T 7$). Starting from the skolemized initial term sets

(1) c_1 (2) c_3 (3) $\mathsf{g}(\mathsf{c}_1)$ (4) $\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3)$ (5) $\mathsf{g}(\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3))$

one first merges (1), (3), and (4) because of the equations

(1) $\mathsf{c}_1,\ \mathsf{g}(\mathsf{c}_1),\ \mathsf{f}(\mathsf{c}_1, \mathsf{c}_3)$ (2) c_3 (5) $\mathsf{g}(\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3))$

and next (1) containing $\mathsf{g}(\mathsf{c}_1)$ and (5) containing $\mathsf{g}(\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3))$ as c_1 and $\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3)$ are in (1). Therefore $\mathsf{g}(\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3))$ and $\mathsf{g}(\mathsf{c}_1)$ are congruent, too.

③ In order to learn the clause $C = (\overline{3} \vee 5 \vee \overline{6})$ we have to establish $F \models_T C$. To that end it is again sufficient to show that $\neg C = 3 \wedge \overline{5} \wedge 6$ is T-inconsistent. So we again use congruence closure to show that $3, 6 \models_T 5$. Starting from the initial term sets

(1) c_1 (2) c_3 (3) c_4 (4) $\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3)$ (5) $\mathsf{f}(\mathsf{c}_1, \mathsf{c}_4)$

one first merges (2) and (3), as well as (1) and (5) because of the equations:

(1) $\mathsf{c}_1,\ \mathsf{f}(\mathsf{c}_1, \mathsf{c}_4)$ (2) $\mathsf{c}_3,\ \mathsf{c}_4$ (4) $\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3)$

Now (1) and (4) can be merged because c_3 and c_4 are in (2), which shows congruence of $\mathsf{f}(\mathsf{c}_1, \mathsf{c}_3)$ and c_1.

4 See the file `rabbit.py`.