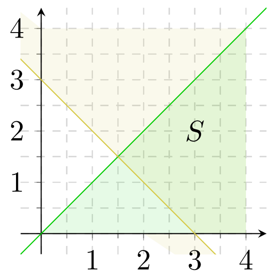
1 We consider the inequalities $y \leq x$ and $3 - x \leq y$. The solutions space illustrated below is related to the solutions of the inequality next to the diagram:



$$\underbrace{\begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 3 \\ 0 & 0 & -1 \end{pmatrix}}_{A} \cdot \begin{pmatrix} x \\ y \\ \tau \end{pmatrix} \leq 0$$

Using the proof of the FMW theorem, we compute $cone\,(W)$ for W consisting of the row vectors of the matrix A:

$$w_1 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T \qquad w_2 = \begin{pmatrix} -1 & -1 & 3 \end{pmatrix}^T \qquad w_3 = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}^T .$$

To that end, normal vectors to the space spanned by any two different vectors of W are needed:

- $c_{12} = w_1 \times w_2 = (3 \quad 3 \quad 2)$ is normal to w_1 and w_2
  $c_{12} \cdot w_1 = 0 \qquad c_{12} \cdot w_2 = 0 \qquad c_{12} \cdot w_3 = -2$
- $c_{13} = w_1 \times w_3 = (-1 \quad -1 \quad 0)$ is normal to w_1 and w_3
  $c_{13} \cdot w_1 = 0 \qquad c_{13} \cdot w_2 = 2 \qquad c_{13} \cdot w_3 = 0$
- $c_{23} = w_2 \times w_3 = (1 \quad -1 \quad 0)$ is normal to w_2 and w_3
  $c_{23} \cdot w_1 = -2 \qquad c_{23} \cdot w_2 = 0 \qquad c_{23} \cdot w_3 = 0$

We thus use c_{12}, $-c_{13}$, and c_{23} as rows in the matrix

$$B = \begin{pmatrix} 3 & 3 & 2 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

and have $cone\,(W) = \{\vec{x} \mid B\vec{x} \leq 0\}$ as well as

$$\{\vec{x} \mid A\vec{x} \leq 0\} = cone\,(\{\begin{pmatrix} \frac{3}{2} & \frac{3}{2} & 1 \end{pmatrix}^T, \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T\})$$

The solution space S to the original inequalities are thus given by

$$hull\,\begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} + cone\,\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.$$

In particular $S \cap \mathbb{Z}$ has bound $B := b \cdot (1 + n) = 2 \cdot 3 = 6$.

2 (a) From the initial tableau (left) a solution to the problem over $\mathbb{R}^2$ can be obtained with the Simplex algorithm, together with a final tableau (right):

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -4 & -5 \\ 2 & -5 \\ -1 & 5 \\ -7 & 1 \\ 5 & 2 \end{pmatrix} \end{array} \begin{array}{l} \\ s_1 \leq -10 \\ s_2 \leq 0 \\ s_3 \leq 21 \\ s_4 \leq -4 \\ s_5 \leq 25 \end{array} \qquad \begin{array}{c} \\ s_4 \\ x \\ s_3 \\ y \\ s_5 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -1\frac{3}{10} & 1\frac{1}{10} \\ \frac{1}{6} & -\frac{1}{6} \\ -\frac{5}{6} & -\frac{1}{6} \\ -0.13 & -0.07 \\ 0.57 & -0.97 \end{pmatrix} \end{array} \begin{array}{r} x = 1\frac{2}{3} \\ y = \frac{2}{3} \\ s_1 = -10 \\ s_2 = 0 \\ s_3 = 1\frac{2}{3} \\ s_4 = -11 \\ s_5 = 9\frac{2}{3} \end{array}$$

We pick the variable x which is assigned $1\frac{2}{3} \notin Z$. Since there are only upper bounds, the set of nonbasic variables L is empty and the Gomory cut inequality simplifies to the following form:

$$-\sum_{j \in U^-} \frac{A_{ij}}{1-c}(u_j - x_j) + \sum_{j \in U^+} \frac{A_{ij}}{c}(u_j - x_j) \geq 1 \qquad (\star)$$

Due to the coefficients in the tableau we have $U^- = \{s_1\}$ and $U^+ = \{s_2\}$, and $c = \frac{2}{3}$. So $(\star)$ amounts to

$$-\left(-\frac{1}{6} \cdot 3 \cdot (-10 - s_1)\right) + \frac{1}{6} \cdot \frac{3}{2} \cdot (0 - s_2) \geq 1$$

or equivalently

$$-\frac{1}{2}s_1 - \frac{1}{4}s_2 \geq 6$$

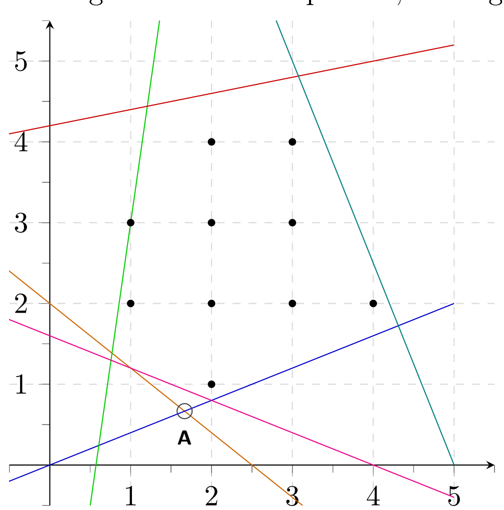
Using the equations $s_1 = -4x - 5y$ and $s_2 = 2x - 5y$, this is equivalent to

$$-\frac{1}{2}(-4x - 5y) - \frac{1}{4}(2x - 5y) = \frac{3}{2}x + \frac{15}{4}y \geq 6$$

Multiplying the inequality by 4 renders it more readable:

$$6x + 15y \geq 24$$

This cut corresponds to the magenta line in the picture, cutting off the last solution **A**.



3 See the file `loanly_officers.py`.