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SAT and SMT Solving

Sarah Winkler

Computational Logic Group
Department of Computer Science
University of Innsbruck

lecture 1
SS 2019

Outline

- Introduction
  - Organisation
  - Why SAT and SMT?
  - Contents
- Propositional Logic
- DPLL
- Transformations to CNF
- Using SAT Solvers

Important Information

- LVA 703048 (PS 2)

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Time and Place

PS	Friday	13:15 – 15:00	HSB9

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- 70% weekly exercises, 30% proseminar test on June 28
- attendence required

Exercises

- 10 points per week
- indicate solved exercises before Friday 11:00 in OLAT, submit solutions

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Consultation Hours

Sarah Winkler	3M03	Thursday 14:00 – 16:00

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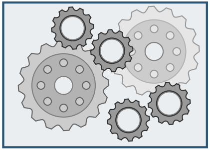
SAT Solving

input: propositional formula φ

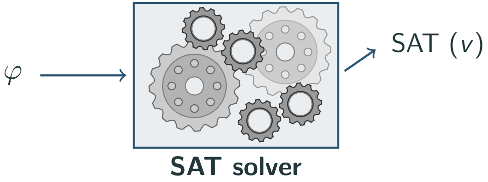
φ ⟶ 

SAT solver

SAT Solving

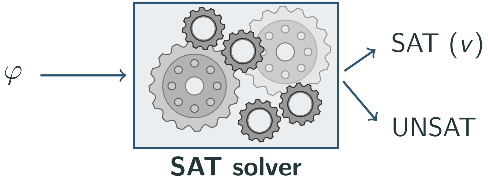
input: propositional formula φ

output: SAT + valuation v such that $v(\varphi) = T$ if φ satisfiable



SAT Solving

input:	propositional formula φ	
output:	SAT + valuation v such that $v(\varphi) = T$	if φ satisfiable
	UNSAT	otherwise



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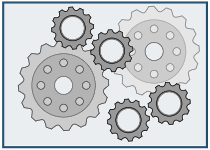
φ $\longrightarrow$

$(q \vee \neg r) \wedge (\neg q \vee r) \wedge p$



SAT solver

SAT (v) $\quad v(p) = \mathrm{T}$, $v(q) = \mathrm{F}$, $v(r) = \mathrm{F}$

UNSAT

SAT Solving

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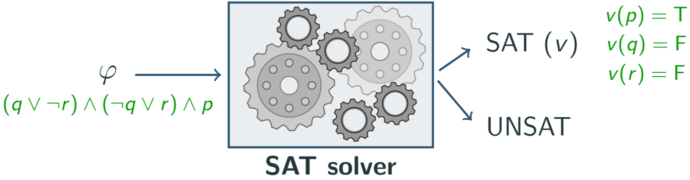

Terminology

- decision problem P is problem with answer yes or no

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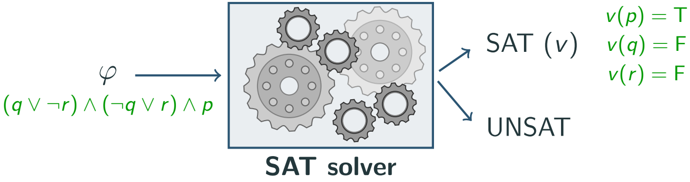


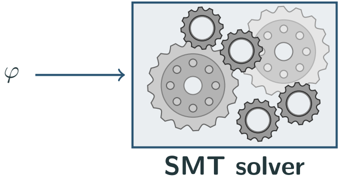
Terminology

- decision problem P is problem with answer yes or no
- SAT encoding of decision problem P is propositional formula φ_P such that

$$\text{answer to } P \text{ is yes} \iff \varphi_P \text{ is satisfiable}$$

SMT Solving

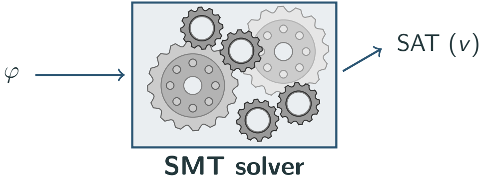
input: formula φ involving theory T



SMT Solving

input: formula φ involving theory T

output: SAT + valuation v such that $v(\varphi) = T$ if φ is T-satisfiable



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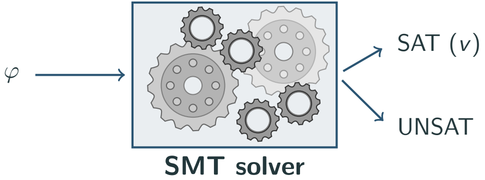

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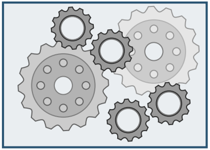
UNSAT otherwise

φ

$a + b \geqslant c \vee (a = 0 \wedge p)$



SMT solver

SAT (v) $v(a) = 3$ $v(b) = 0$ $v(c) = 0$ $v(p) = \mathrm{T}$

UNSAT

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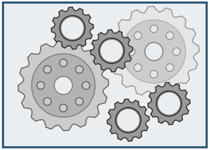

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Example (Theories)

- arithmetic $2a + b \geqslant c \vee (a = 0 \wedge p)$

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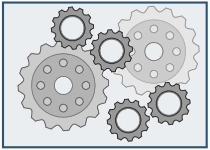

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Example (Theories)

- arithmetic $2a + b \geqslant c \vee (a = 0 \wedge p)$
- uninterpreted functions $\mathrm{f}(x, y) \neq \mathrm{f}(y, x) \wedge \mathrm{g}(\mathrm{f}(x, x)) = \mathrm{g}(y)$

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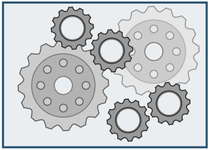

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- bit vectors $((\mathrm{zext}_{32}\ a_8) + b_{32}) \times c_{32} >_u 0_{32}$

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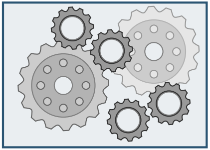

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Terminology

- SMT encoding over theory T of decision problem P is formula φ_P such that

  answer to P is yes $\iff$ φ_P is satisfiable

Application: Driving License Test

Problem

Austrian driving license test consists of 80 questions out of 1500 such that the following conditions are satisfied:

- 30 questions "main questions" with 3 sub-questions each
- at least 12 main questions must be about crossroads
- at least 12 questions must have pictures
- at least 5 "hard", "medium", and "easy" main questions

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▶ how can software find valid question set?

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- variables q_i for $1 \leqslant i \leqslant 1500$

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Result

easy generation of valid question sets (with some random preselection)

Problem

Can one color all natural numbers with two colors such that whenever $x^2 + y^2 = z^2$ not all of x, y, and z have same color?

Application: Pythagorean Triples

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Example

$$3^2 + 4^2 = 5^2 \qquad 5^2 + 12^2 = 13^2$$

(a)	1	2	3	4	5	6	7	8	9	10	11	12	13	...	✓
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Result: No. Coloring exists only up to 7,825.

Problem

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$x^2 + y^2 =$

Example

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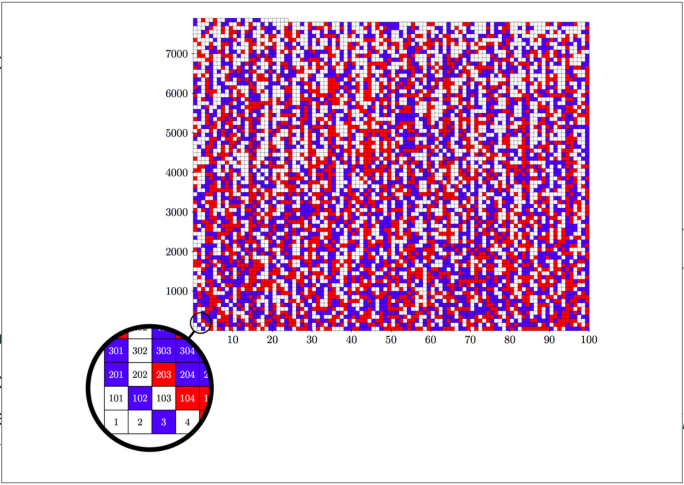
SAT Enc

- variab … red
- SAT … $_b \vee \bar{x}_c)$
  (+ sy



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1000s of variables, solving time 2 days with 800 processors, 200 TB of proof

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Zahlenrätsel

Der längste Mathe-Beweis der Welt

Drei Mathematiker haben ein Zahlenrätsel geknackt - mithilfe eines Supercomputers. Der Beweis umfasst 200 Terabyte. Sie wollen wissen, worum es geht? Okay, versuchen wir es.

Von Holger Dambeck

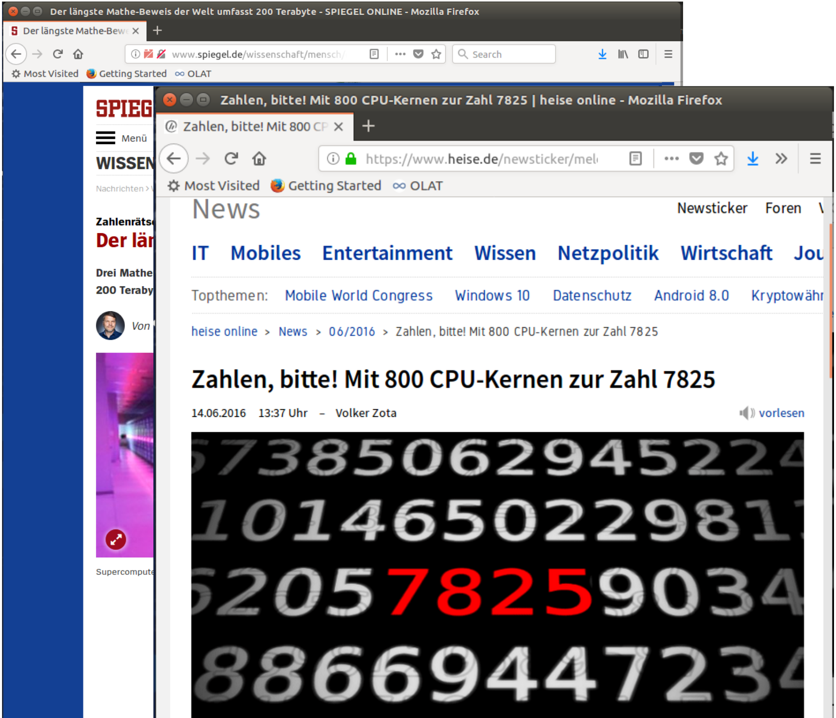
DPA

Supercomputer als Mathematiker

Teilen Twittern E-Mail

Montag, **30.05.2016** 18:41 Uhr Drucken Nutzungsrechte Feedback Kommentieren

Das Preisgeld war nur symbolisch: 100 Dollar hatte der US-Mathematiker Ronald Graham in den Achtzigerjahren demjenigen versprochen, der ein bis dahin ungelöstes Zahlenrätsel entschlüsselt. Es ging dabei um sogenannte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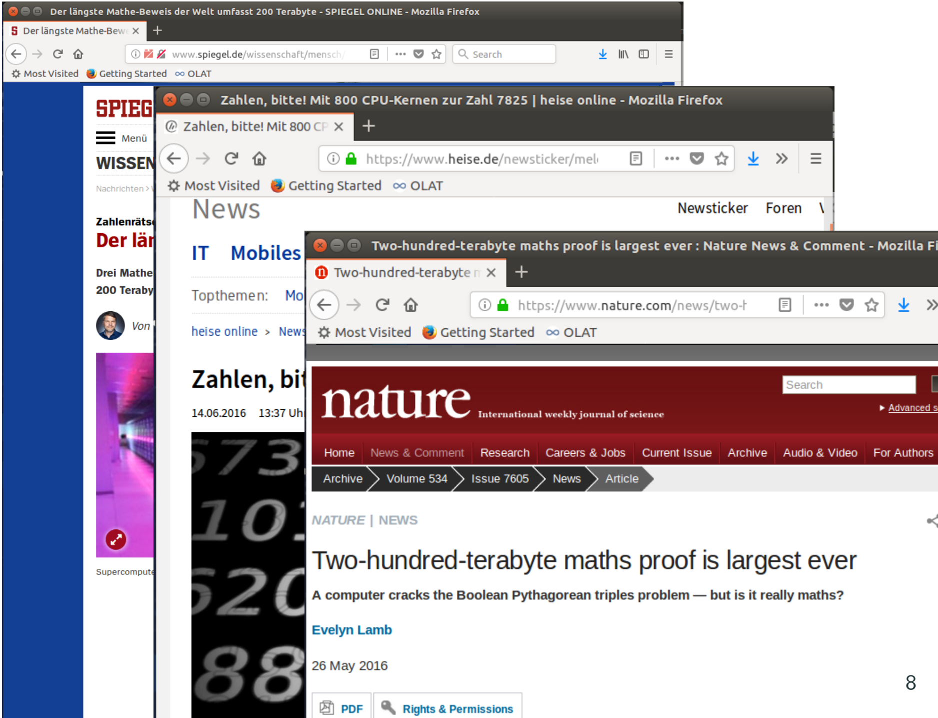
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Application: Tournament Scheduling

Problem: Round Robin Scheduling

Schedule sports league tournament for n teams, p periods of $n-1$ rounds each
($+$ venue restrictions, break restrictions, ...)

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Schedule sports league tournament for n teams, p periods of $n - 1$ rounds each (+ venue restrictions, break restrictions, . . .)

Example (Österreichische Fußball-Bundesliga)

10 teams play in 4 periods (9 rounds each), periods 1 & 2 and 3 & 4 mirrored

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(Part of) SAT Encoding

- variable x_{ijpr} is true if team i plays team j at home in period p, round r

Application: Tournament Scheduling

Problem: Round Robin Scheduling

Schedule sports league tournament for n teams, p periods of $n-1$ rounds each (+ venue restrictions, break restrictions, ...)

Example (Österreichische Fußball-Bundesliga)

10 teams play in 4 periods (9 rounds each), periods 1 & 2 and 3 & 4 mirrored

(Part of) SAT Encoding

- variable x_{ijpr} is true if team i plays team j at home in period p, round r
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Result

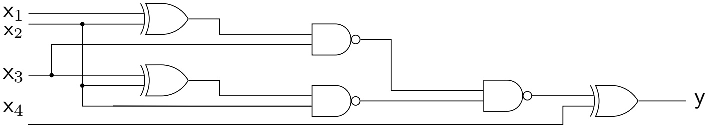
SAT scheduling is 100x faster than previous industrial scheduling tools

Application: Hardware Verification

Problem

- errors in hardware chips are costly (Intel paid $475 million for FDIV bug)

Example (Formal Circuit Model)



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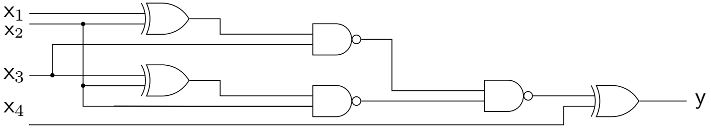

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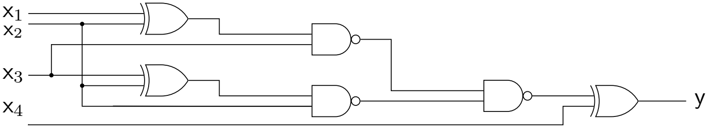

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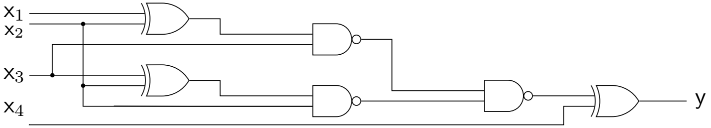

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Impact

- ensured correctness, more reliable hardware components (formal verification)
- manufacturers rely on SAT-based verification since beginning of 2000s
  e.g., Intel Core i7 implements over 2700 distinct verified microinstructions

Outline

- Introduction
  - Organisation
  - Why SAT and SMT?
  - Contents
- Propositional Logic
- DPLL
- Transformations to CNF
- Using SAT Solvers

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Part 1: SAT

DPLL, conflict analysis, CDCL, heuristics, unsatisfiable cores, maxSAT, symmetry breaking

Part 2: SMT

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Propositional Logic Revisited

Concepts

- literal
- formula
- assignment
- satisfiability and validity
- negation normal form (NNF)
- conjunctive normal form (CNF)
- disjunctive normal form (DNF)

Definition (Propositional Logic: Syntax)

propositional formulas are built form

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satisfiability and validity are decidable

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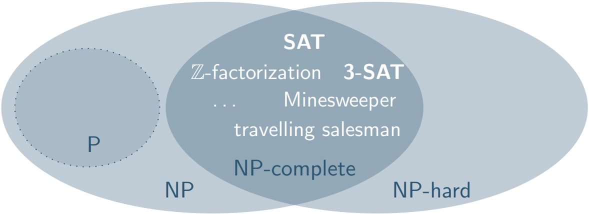

- 1 million \$ prize money awarded for solution to **P** $\stackrel{?}{=}$ **NP**

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- can be described abstractly by transition system
  (Nieuwenhuis, Oliveras, Tinelli 2006)

Approach

- most state-of-the-art SAT solvers use variation of Davis - Putnam - Logemann - Loveland (DPLL) procedure (1962)
- DPLL is sound and complete backtracking-based search algorithm
- can be described abstractly by transition system (Nieuwenhuis, Oliveras, Tinelli 2006)

Definition (Abstract DPLL)

- decision literal is annotated literal l^d

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- decision literal is annotated literal l^d
- state is pair $M \parallel F$ for
  - list M of (decision) literals — partial assignment
  - formula F in CNF

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Definition (Abstract DPLL)

- decision literal is annotated literal l^d
- state is pair $M \parallel F$ for
  - list M of (decision) literals — partial assignment
  - formula F in CNF
- transition rules

$$M \parallel F \quad \Longrightarrow \quad M' \parallel F' \quad \text{or} \quad \text{FailState}$$

Example

$\varphi = (\overline{1} \vee \overline{2}) \wedge (2 \vee 3) \wedge (\overline{1} \vee \overline{3} \vee 4) \wedge (2 \vee \overline{3} \vee \overline{4}) \wedge (1 \vee 4)$

$$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$$

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$$\begin{array}{rll} & \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \\ \Longrightarrow & 1^d \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{decide} \end{array}$$

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$$
\begin{array}{rrl}
 & \| \ \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \\
\Longrightarrow & 1^d \ \| \ \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{decide} \\
\Longrightarrow & 1^d\ \overline{2} \ \| \ \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate}
\end{array}
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\Longrightarrow & 1^d\ \overline{2} & \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & 1^d\ \overline{2}\ 3 & \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate}
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\Longrightarrow & 1^d\ \overline{2} \parallel & \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & 1^d\ \overline{2}\ 3 \parallel & \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & 1^d\ \overline{2}\ 3\ 4 \parallel & \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate}
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\Longrightarrow & 1^d\ \overline{2} \| & \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & 1^d\ \overline{2}\ 3 \| & \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & 1^d\ \overline{2}\ 3\ 4 \| & \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & \overline{1} \| & \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{backtrack}
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 & \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \\
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\Longrightarrow & 1^d\ \overline{2}\ 3 \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4 & \text{unit propagate} \\
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		$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	
$\Longrightarrow$	1^d	$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	decide
$\Longrightarrow$	$1^d\ \overline{2}$	$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	unit propagate
$\Longrightarrow$	$1^d\ \overline{2}\ 3$	$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	unit propagate
$\Longrightarrow$	$1^d\ \overline{2}\ 3\ 4$	$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	unit propagate
$\Longrightarrow$	$\overline{1}$	$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	backtrack
$\Longrightarrow$	$\overline{1}\ 4$	$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	unit propagate
$\Longrightarrow$	$\overline{1}\ 4\ 3^d$	$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	decide

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$$\varphi = (\overline{1} \vee \overline{2}) \wedge (2 \vee 3) \wedge (\overline{1} \vee \overline{3} \vee 4) \wedge (2 \vee \overline{3} \vee \overline{4}) \wedge (1 \vee 4)$$

	$\parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$		
$\Longrightarrow$	$1^d \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	decide
$\Longrightarrow$	$1^d\ \overline{2} \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	unit propagate
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$\Longrightarrow$	$\overline{1}\ 4\ 3^d\ 2 \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ 2 \vee \overline{3} \vee \overline{4},\ 1 \vee 4$	unit propagate

Definition (DPLL Transition Rules)

- unit propagation $\quad M \parallel F,\ C \vee l \quad \Longrightarrow \quad M\, l \parallel F,\ C \vee l$
  if $M \models \neg C$ and l is undefined in M

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- pure literal $M \parallel F \quad \Longrightarrow \quad M\, l \parallel F$
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Definition (DPLL Transition Rules)

- unit propagation $\quad M \parallel F,\ C \vee l \implies M\, l \parallel F,\ C \vee l$
  if $M \models \neg C$ and l is undefined in M
- pure literal $\quad M \parallel F \implies M\, l \parallel F$
  if l occurs in F but l^c does not occur in F, and l is undefined in M
- decide $\quad M \parallel F \implies M\, l^d \parallel F$
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  if l or l^c occurs in F, and l is undefined in M
- backtrack $M\ l^d\ N \parallel F, C \quad \Longrightarrow \quad M\ l^c \parallel F, C$
  if $M\ l^d\ N \models \neg C$ and N contains no decision literals

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- backtrack $M\, l^d\, N \parallel F, C \quad \Longrightarrow \quad M\, l^c \parallel F, C$
  if $M\, l^d\, N \models \neg C$ and N contains no decision literals

- fail $M \parallel F, C \quad \Longrightarrow \quad$ FailState
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- backjump $M\, l^d\, N \parallel F, C \implies M\, l' \parallel F, C$

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  if $M\ l^d\ N \models \neg C$ and N contains no decision literals
- fail $M \parallel F, C \quad \Longrightarrow \quad$ FailState
  if $M \models \neg C$ and M contains no decision literals
- backjump $M\ l^d\ N \parallel F, C \quad \Longrightarrow \quad M\ l' \parallel F, C$
  if $M\ l^d\ N \models \neg C$ and $\exists$ clause $C' \vee l'$ such that

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  - $F, C \models C' \vee l'$ backjump clause

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  - $F, C \models C' \vee l'$ — backjump clause
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  - $F, C \models C' \vee l'$ — backjump clause
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Example (Backjump)

$\varphi = (\overline{1} \vee 2) \wedge (\overline{1} \vee \overline{3} \vee 4 \vee 5) \wedge (\overline{2} \vee \overline{4} \vee \overline{5}) \wedge (4 \vee \overline{5}) \wedge (\overline{4} \vee 5)$

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$$\begin{array}{lrl}
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\Longrightarrow & 1^d \; \| \; \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{decide} \\
\Longrightarrow & 1^d\, 2 \; \| \; \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{unit propagate}
\end{array}$$

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\Longrightarrow & 1^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{decide} \\
\Longrightarrow & 1^d\ 2 \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{unit propagate} \\
\Longrightarrow & 1^d\ 2\ 3^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{decide}
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Example (Backjump)

$\varphi = (\overline{1} \vee 2) \wedge (\overline{1} \vee \overline{3} \vee 4 \vee 5) \wedge (\overline{2} \vee \overline{4} \vee \overline{5}) \wedge (4 \vee \overline{5}) \wedge (\overline{4} \vee 5)$

$$\begin{array}{lrl}
 & \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \\
\Longrightarrow & 1^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{decide} \\
\Longrightarrow & 1^d\ 2 \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{unit propagate} \\
\Longrightarrow & 1^d\ 2\ 3^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{decide} \\
\Longrightarrow & 1^d\ 2\ 3^d\ 4^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & \text{decide}
\end{array}$$

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	$\parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	
$\Longrightarrow$	$1^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2 \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d\ \overline{5} \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate

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$$
\begin{array}{llr r}
 & \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & & \\
\Longrightarrow & 1^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & & \text{decide} \\
\Longrightarrow & 1^d\ 2 \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & & \text{unit propagate} \\
\Longrightarrow & 1^d\ 2\ 3^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & & \text{decide} \\
\Longrightarrow & 1^d\ 2\ 3^d\ 4^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & & \text{decide} \\
\Longrightarrow & 1^d\ 2\ 3^d\ 4^d\ \overline{5} \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & & \text{unit propagate} \\
\Longrightarrow & 1^d\ 2\ 3^d\ \overline{4} \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5 & & \text{backtrack}
\end{array}
$$

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	$\parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	
$\Longrightarrow$	$1^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2 \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d\ \overline{5} \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4} \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	backtrack
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4}\ 5 \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate

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	$\parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$		
$\Longrightarrow$	$1^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2 \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d\ \overline{5} \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4} \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	backtrack
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4}\ 5 \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$\ldots$	backtrack

Example (Backjump)

$\varphi = (\overline{1} \vee 2) \wedge (\overline{1} \vee \overline{3} \vee 4 \vee 5) \wedge (\overline{2} \vee \overline{4} \vee \overline{5}) \wedge (4 \vee \overline{5}) \wedge (\overline{4} \vee 5)$

	$\parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	
$\Longrightarrow$	$1^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2 \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d\ \overline{5} \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4} \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	backtrack
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4}\ 5 \parallel$	$\overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$\ldots$		backtrack

Decisions 1 and 3 are incompatible: $\varphi \models \overline{1} \vee \overline{3}$

Example (Backjump)

$\varphi = (\overline{1} \lor 2) \land (\overline{1} \lor \overline{3} \lor 4 \lor 5) \land (\overline{2} \lor \overline{4} \lor \overline{5}) \land (4 \lor \overline{5}) \land (\overline{4} \lor 5)$

	$\|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	
$\Longrightarrow$	$1^d \ \|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	decide
$\Longrightarrow$	$1^d\ 2 \ \|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d \ \|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d \ \|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d\ \overline{5} \ \|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4} \ \|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	backtrack
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4}\ 5 \ \|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	unit propagate
$\Longrightarrow$	...		backtrack
$\Longrightarrow$	$1^d\ 2\ \overline{3} \ \|\|$	$\overline{1} \lor 2,\ \overline{1} \lor \overline{3} \lor 4 \lor 5,\ \overline{2} \lor \overline{4} \lor \overline{5},\ 4 \lor \overline{5},\ \overline{4} \lor 5$	backjump

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Example (Backjump)

$\varphi = (\overline{1} \vee 2) \wedge (\overline{1} \vee \overline{3} \vee 4 \vee 5) \wedge (\overline{2} \vee \overline{4} \vee \overline{5}) \wedge (4 \vee \overline{5}) \wedge (\overline{4} \vee 5)$

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$\Longrightarrow$	$1^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2 \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	decide
$\Longrightarrow$	$1^d\ 2\ 3^d\ 4^d\ \overline{5} \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4} \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	backtrack
$\Longrightarrow$	$1^d\ 2\ 3^d\ \overline{4}\ 5 \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	unit propagate
$\Longrightarrow$	$\ldots$	backtrack
$\Longrightarrow$	$1^d\ 2\ \overline{3} \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	backjump
$\Longrightarrow^+$	$1^d\ 2\ \overline{3}\ \overline{4}\ \overline{5} \parallel \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4 \vee 5,\ \overline{2} \vee \overline{4} \vee \overline{5},\ 4 \vee \overline{5},\ \overline{4} \vee 5$	

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Definition

basic DPLL $\mathcal{B}$ consists of unit propagation, decide, fail, and backjump

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Lemma (Model Entailment)

Suppose $\parallel F \Longrightarrow^*_{\mathcal{B}} M \parallel F$ *such that*

- $M = M_0\ l_1^d\ M_1\ l_2^d\ M_2\ \dots\ l_k^d\ M_k$ *and*
- *there are no decision literals in* M_i.

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Then $F, \; l_1, \dots, l_i \vDash M_i$ *for all* $0 \leqslant i \leqslant k$.

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decisions imply
all other literals in M

Then $F,\ l_1, \ldots, l_i \vDash M_i$ *for all* $0 \leqslant i \leqslant k$.

Theorem (Termination)

for any formula F there are no infinite derivations

$$\| F \quad \Longrightarrow_{\mathcal{B}} \quad S_1 \quad \Longrightarrow_{\mathcal{B}} \quad S_2 \quad \Longrightarrow_{\mathcal{B}} \quad \dots$$

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Proof.

- for list of distinct literals M, define $a(M) = n - |M|$ where
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missing literals in M

Proof.

- for list of distinct literals M, define $a(M) = n - |M|$ where
  - n is total number of atoms
  - $|M|$ is length of M
- measure state $M_0\ l_1^d\ M_1\ l_2^d\ M_2 \dots l_k^d\ M_k \parallel F$ by tuple

$$(a(M_0),\ a(M_1),\ \dots,\ a(M_k),\ \underbrace{\infty, \dots, \infty}_{n-k})$$

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- compare tuples lexicographically by extension of $>_{\mathbb{N}}$ with ∞ maximal

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$$(a(M_0),\ a(M_1),\ \dots,\ a(M_k),\ \underbrace{\infty, \dots, \infty}_{n-k})$$

- compare tuples lexicographically by extension of $>_{\mathbb{N}}$ with ∞ maximal
- every transition step decreases measure

∎

Example (Revisited for Termination)

$\varphi = (\overline{1} \vee \overline{2}) \wedge (2 \vee 3) \wedge (\overline{1} \vee \overline{3} \vee 4) \wedge (2 \vee \overline{3} \vee \overline{4}) \wedge (1 \vee 4)$

$$\| \; \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots \qquad\qquad (n, \infty, \ldots)$$

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$$\begin{array}{llll} & \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots & & (n, \infty, \ldots) \\ \Longrightarrow & 1^d \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots & \text{decide} & (n, n, \infty, \ldots) \end{array}$$

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$\Longrightarrow$	$\overline{1} \ \| \ \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	backtrack	$(n-1, \infty, \ldots)$

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$\Longrightarrow$	$\overline{1}\ 4 \parallel \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	unit propagate	$(n-2, \infty, \ldots)$

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Observation

- decide replaces ∞ by n
- unit propagate, backtrack, and backjump replace m by $m-1$

Consider maximal derivation with final state S_n:

$$\| F \quad \Longrightarrow_{\mathcal{B}} \quad S_1 \quad \Longrightarrow_{\mathcal{B}} \quad S_2 \quad \Longrightarrow_{\mathcal{B}} \quad \ldots \quad \Longrightarrow_{\mathcal{B}} \quad S_n$$

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Theorem

if $S_n = \mathrm{FailState}$ *then* F *is unsatisfiable*

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Theorem

if $S_n = \text{FailState}$ then F is unsatisfiable

Proof.

- must have $\| F \Longrightarrow_{\mathcal{B}}^{*} M \| F \Longrightarrow_{\text{fail}} \text{FailState}$
  such that M contains no decision literals and $M \models \neg C$ for some C in F

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Theorem

if $S_n = M \| F'$ *then* F *is satisfiable and* $M \models F$

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Proof.

- must have $\| F \Longrightarrow_{\mathcal{B}}^{*} M \| F \Longrightarrow_{\text{fail}} \text{FailState}$
  such that M contains no decision literals and $M \vDash \neg C$ for some C in F
- by Model Entailment Lemma $F \vDash M$, so $F \vDash \neg C$
- also have $F \vDash C$ because C is in F, so F is unsatisfiable ∎

Theorem

if $S_n = M \| F'$ *then* F *is satisfiable and* $M \vDash F$

Proof.

- have $F = F'$

Consider maximal derivation with final state S_n:

$$\| F \quad \Longrightarrow_{\mathcal{B}} \quad S_1 \quad \Longrightarrow_{\mathcal{B}} \quad S_2 \quad \Longrightarrow_{\mathcal{B}} \quad \dots \quad \Longrightarrow_{\mathcal{B}} \quad S_n$$

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Theorem

if $S_n = M \| F'$ then F is satisfiable and $M \vDash F$

Proof.

- have $F = F'$
- S_n is final, so all literals of F are defined in M (otherwise decide applicable)

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Theorem

if $S_n = \text{FailState}$ *then* F *is unsatisfiable*

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  such that M contains no decision literals and $M \models \neg C$ for some C in F
- by Model Entailment Lemma $F \models M$, so $F \models \neg C$
- also have $F \models C$ because C is in F, so F is unsatisfiable ■

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if $S_n = M \| F'$ *then* F *is satisfiable and* $M \models F$

Proof.

- have $F = F'$
- S_n is final, so all literals of F are defined in M (otherwise decide applicable)
- $\nexists$ clause C in F such that $M \models \neg C$ (otherwise backjump or fail applicable)

Consider maximal derivation with final state S_n:

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Theorem

if $S_n =$ FailState then F is unsatisfiable

Proof.

- must have $\| \; F \Longrightarrow_{\mathcal{B}}^{*} M \; \| \; F \Longrightarrow_{\text{fail}}$ FailState
  such that M contains no decision literals and $M \vDash \neg C$ for some C in F
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- also have $F \vDash C$ because C is in F, so F is unsatisfiable ■

Theorem

if $S_n = M \; \| \; F'$ then F is satisfiable and $M \vDash F$

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- $\nexists$ clause C in F such that $M \vDash \neg C$ (otherwise backjump or fail applicable)
- so M satisfies F ($M \vDash F$) ■

DPLL

Martin Davis and Hilary Putnam.
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Marijn Heule, Oliver Kullmann, and Victor Marek.
Solving and Verifying the Boolean Pythagorean Triples Problem via Cube-and-Conquer.
Proc. 16th International Conference on Theory and Applications of Satisfiability Testing, pp. 228–245, 2016.

Outline

- Introduction
- Propositional Logic
- DPLL
- Transformations to CNF
- Using SAT Solvers

Fact

most SAT solvers require input to be in CNF

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- transforming formula to equivalent CNF can cause exponential blowup
- transforming formula into equisatisfiable CNF is possible in linear time

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Definition

formulas φ and ψ are equisatisfiable ($\varphi \approx \psi$) if

$$\varphi \text{ is satisfiable} \quad\iff\quad \psi \text{ is satisfiable}$$

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Example

$$p \vee q \ \approx\ \top \qquad\qquad p \wedge \neg p \ \approx\ q \wedge \neg q \qquad\qquad p \wedge \neg p \ \not\approx\ p \wedge \neg q$$

Example (Tseitin's Transformation)

- $\varphi = \neg(p \lor q) \lor (p \land (p \lor q))$

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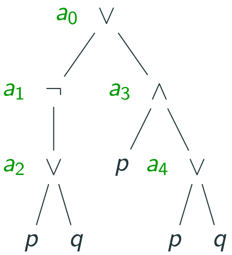


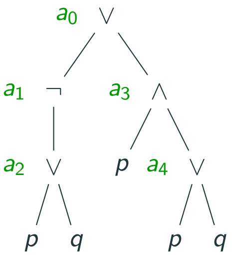
Example (Tseitin's Transformation)

- $\varphi = \neg(p \vee q) \vee (p \wedge (p \vee q))$
- use fresh propositional variable for every connective

  $a_0 : \neg(p \vee q) \vee (p \wedge (p \vee q))$ $\quad a_1 : \neg(p \vee q)$

  $a_2 : p \vee q$ $\quad a_3 : p \wedge (p \vee q)$

  $a_4 : p \vee q$



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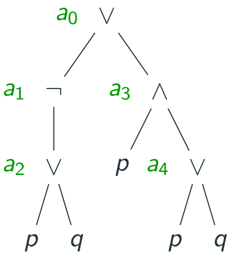

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  $a_2 \colon p \vee q$ $\quad a_3 \colon p \wedge (p \vee q)$

  $a_4 \colon p \vee q$

- $\varphi \approx \quad a_0 \wedge (a_0 \leftrightarrow a_1 \vee a_3) \wedge (a_1 \leftrightarrow \neg a_2) \wedge (a_2 \leftrightarrow p \vee q) \wedge$
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- $\varphi = \neg(p \vee q) \vee (p \wedge (p \vee q))$
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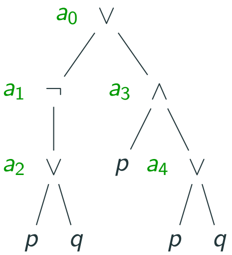

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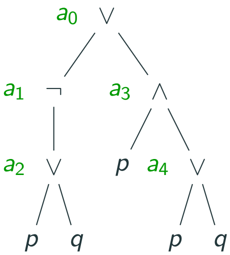

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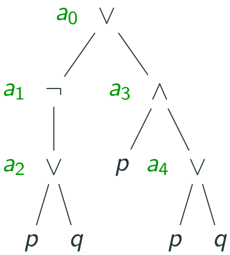

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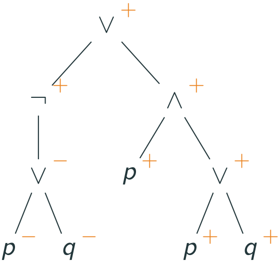

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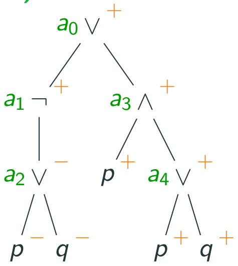

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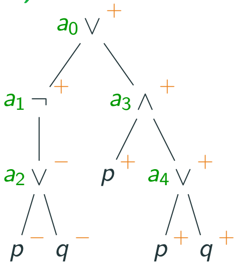

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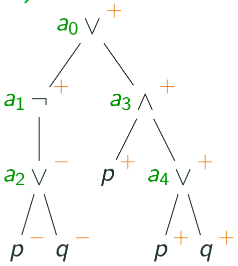

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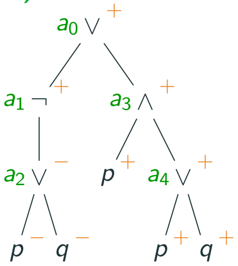

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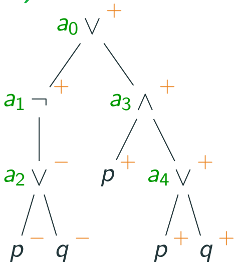

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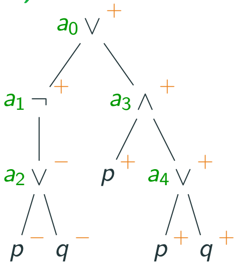

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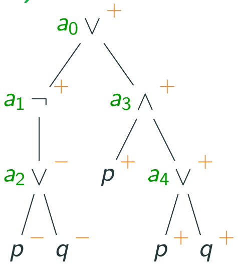

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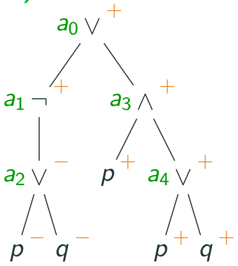

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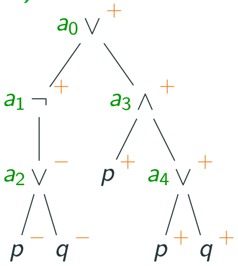

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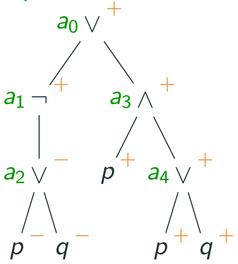

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Minisat

- minimalistic open source solver (`http://minisat.se/` or apt, yum,...)

```
$ minisat test.sat result.txt
```

- web interface

Example (DIMACS)

formula $(x_1 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee \neg x_1) \wedge (\neg x_1 \vee x_2 \vee x_4)$ can be expressed by

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- lines starting with `c` are considered comments

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common open source SAT/SMT solver

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- `And(`$a_1, \ldots, a_n$`)` conjunction with arbitrarily many arguments

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Building Formulas

- `True`, `False` boolean constants
- `Bool(`*name*`)` propositional variable named *name*
  (calling `Bool(`*name*`)` twice yields same variable)
- `FreshBool(`*pre*`)` new propositional variable with name prefix *name*
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- `And`$(a_1, \ldots, a_n)$ conjunction with arbitrarily many arguments
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Z3

common open source SAT/SMT solver

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`Xor(`a, b`)`	exclusive or

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Moreover ...

- `simplify(`φ`)` simplifies formula φ
- `Solver.statistics()` is map of solving statistics

Example

```
from z3 import *

p = Bool('p') # create variable named 'p'
foo1 = FreshBool('foo') # create fresh variables prefixed 'foo'
foo2 = FreshBool('foo')

phi = Or(p, p, And(foo2, Xor(foo1, Not(foo1)), True), False)
print(phi) # Or(p, p, And(foo!1, Xor(foo!0, Not(foo!0)), True), False)
psi = simplify(phi)
print(psi) # Or(p, foo!1)

solver = Solver()
solver.add(psi) # assert that psi should be true
solver.add(Implies(foo1,p), Or(foo1, foo2)) # assert something else

print solver # [Or(p, foo!1), Implies(foo!0, p), Or(foo!0, foo!1)]
result = solver.check() # check for satisfiability

if result:
  model = solver.model() # get valuation
  print model[p], model[foo1], model[foo2] # False False True
```

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	8		3
			2

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x_7		x_8	
x_9	x_{10}	x_{11}	

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- variable x_i for each unknown cell i, $v(x_i) = \mathrm{T}$ iff cell i has mine
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  2 $x_8 \wedge x_{11}$

  3 $\bigvee_{1 \leqslant i,j \leqslant 5, i \neq j} \neg x_i \wedge \neg x_j \qquad \bigvee_{1 \leqslant i,j,k \leqslant 5, i \neq j, i \neq k, j \neq k} x_i \wedge x_j \wedge x_k$