



SAT and SMT Solving

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lecture 3
SS 2019

Outline

- Summary of Last Week
- Maximum Satisfiability
- Application: Automotive Configuration
- Algorithms for Minimum Unsatisfiability

Definition (Implication Graph)

For derivation $\| F' \implies_B^* M \| F$ **implication graph** is constructed as follows:

- ▶ add node labelled l for every decision literal l in M
- ▶ repeat until there is no change:
 - if \exists clause $l_1 \vee \dots \vee l_m \vee l'$ in F such that there are already nodes l_1^c, \dots, l_m^c
 - ▶ add node l' if not yet present
 - ▶ add edges $l_i^c \rightarrow l'$ for all $1 \leq i \leq m$ if not yet present
 - if \exists clause $l'_1 \vee \dots \vee l'_k$ in F such that there are nodes l_1^c, \dots, l_k^c
 - ▶ add conflict node labeled C
 - ▶ add edges $l_i^c \rightarrow C$

Definitions

- ▶ **cut** of implication graph has at least all decision literals on the left, and at least the conflict node on the right
- ▶ literal l in implication graph is **unique implication point (UIP)** if all paths from last decision literal to conflict node go through l

Lemma

if edges intersected by cut are $l_1 \rightarrow l'_1, \dots, l_k \rightarrow l'_k$ then $F' \models l_1^c \vee \dots \vee l_k^c$

Backjump Clauses by Resolution

- ▶ set C_0 to conflict clause
- ▶ let l be last assigned literal such that l^c is in C_0
- ▶ while l is no decision literal:
 - ▶ C_{i+1} is resolvent of C_i and clause D that led to assignment of l
 - ▶ let l be last assigned literal such that l^c is in C_{i+1}

Observation

every clause C_i corresponds to cut in implication graph

Definition (DPLL with Learning and Restarts)

DPLL with learning and restarts \mathcal{R} extends system \mathcal{B} by following three rules:

- ▶ **learn** $M \parallel F \implies M \parallel F, C$
if $F \models C$ and all atoms of C occur in M or F
- ▶ **forget** $M \parallel F, C \implies M \parallel F$
if $F \models C$
- ▶ **restart** $M \parallel F \implies \parallel F$

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Theorem (Termination)

any derivation $\parallel F \implies_{\mathcal{R}} S_1 \implies_{\mathcal{R}} S_2 \implies_{\mathcal{R}} \dots$ is finite if

- ▶ it contains no infinite subderivation of learn and forget steps, and
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Theorem (Correctness)

for $\parallel F \implies_{\mathcal{R}} S_1 \implies_{\mathcal{R}} S_2 \implies_{\mathcal{R}} \dots \implies_{\mathcal{R}} S_n$ with final state S_n :

- ▶ if $S_n = \text{FailState}$ then F is unsatisfiable
- ▶ if $S_n = M \parallel F'$ then F is satisfiable and $M \models F'$

Outline

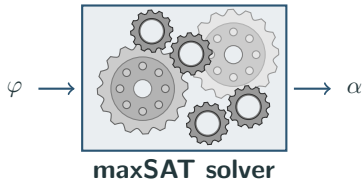
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maxSAT

maxSAT Problem

input: propositional formula φ in CNF

output: valuation α such that α satisfies maximal number of clauses in φ

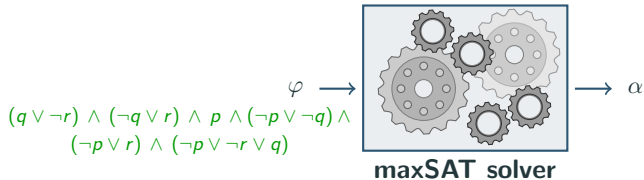


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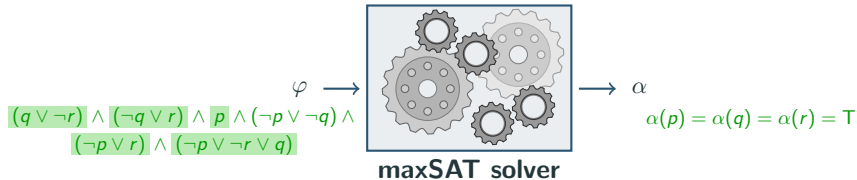


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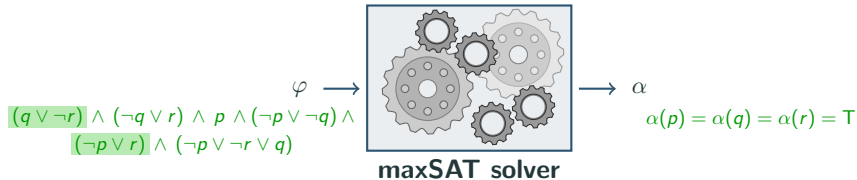


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Terminology

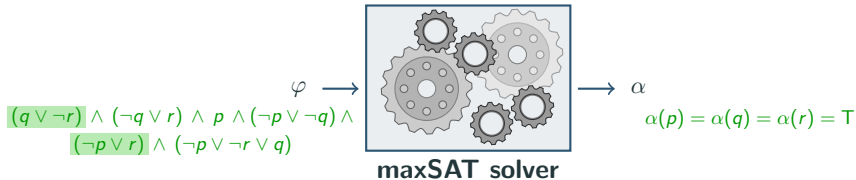
- **optimization problem** P asks to find “best” solution among all solutions

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Terminology

- ▶ optimization problem P asks to find “best” solution among all solutions
- ▶ **maxSAT encoding** transforms optimization problem P into formula φ such that optimal solution to P corresponds to maxSAT solution to φ

Remark

many real world problems have optimization component

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Examples

- ▶ find **shortest path** to goal state
 - ▶ planning
 - ▶ model checking

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 - ▶ planning
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- ▶ find smallest explanation
 - ▶ debugging
 - ▶ configuration

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- ▶ find least resource-consuming schedule
 - ▶ scheduling
 - ▶ logistics

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- ▶ find most probable explanation
 - ▶ probabilistic inference
- ▶ ...

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- ▶ ...

Notation

for valuation v let $\bar{v}(\varphi) = \begin{cases} 1 & \text{if } v(\varphi) = \text{T} \\ 0 & \text{if } v(\varphi) = \text{F} \end{cases}$

Maximal Satisfiability

Consider CNF formula φ as set of clauses $C \in \varphi$

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Example

$$\varphi = \left\{ \begin{array}{llllll} 6 \vee 2, & \bar{6} \vee 2, & \bar{2} \vee 1, & \bar{1}, & \bar{6} \vee 8, & 6 \vee \bar{8}, \\ 2 \vee 4, & \bar{4} \vee 5, & 7 \vee 5, & \bar{7} \vee 5, & \bar{3}, & \bar{5} \vee 3 \end{array} \right\}$$

► $\text{maxSAT}(\varphi) = 10$, e.g. for valuation $\bar{1} 2 \bar{3} 4 5 6 \bar{7} 8$

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instance: CNF formulas χ and φ

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Terminology

- ▶ φ are **soft** constraints
- ▶ χ are **hard** constraints

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write **maxSAT**(φ) and **pmaxSAT**(χ, φ) for solutions to these problems

Weighted Maximal Satisfiability (maxSAT_w)

instance: CNF formula φ with weight $w_C \in \mathbb{Z}$ for all $C \in \varphi$

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using $v(x) = v(y) = \text{T}$ and $v(z) = \text{F}$ have

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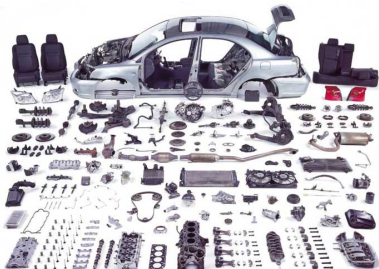
maxSAT and minUNSAT are **dual notions**

Application: Automotive Configuration (1)

Manufacturer's Constraints on Components

component family	components
engine	E_1, E_2, E_3
gearbox	G_1, G_2, G_3
control unit	C_1, \dots, C_5
dashboard	D_1, \dots, D_4
navigation system	N_1, N_2, N_3
air conditioner	AC_1, AC_2, AC_3
alarm system	AS_1, AS_2
radio	R_1, \dots, R_5

Component families with limitations

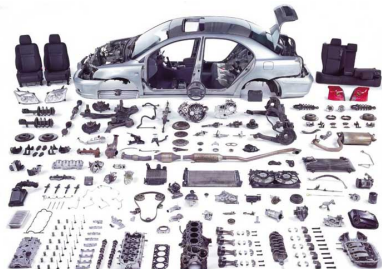


Application: Automotive Configuration (1)

Manufacturer's Constraints on Components

component family	components limit
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gearbox	$G_1, G_2, G_3 = 1$
control unit	$C_1, \dots, C_5 = 1$
dashboard	$D_1, \dots, D_4 = 1$
navigation system	$N_1, N_2, N_3 \leq 1$
air conditioner	$AC_1, AC_2, AC_3 \leq 1$
alarm system	$AS_1, AS_2 \leq 1$
radio	$R_1, \dots, R_5 \leq 1$

Component families with limitations



Application: Automotive Configuration (1)

Manufacturer's Constraints on Components

component family	components limit
engine	$E_1, E_2, E_3 = 1$
gearbox	$G_1, G_2, G_3 = 1$
control unit	$C_1, \dots, C_5 = 1$
dashboard	$D_1, \dots, D_4 = 1$
navigation system	$N_1, N_2, N_3 \leq 1$
air conditioner	$AC_1, AC_2, AC_3 \leq 1$
alarm system	$AS_1, AS_2 \leq 1$
radio	$R_1, \dots, R_5 \leq 1$

premise	conclusion
G_1	$E_1 \vee E_2$
$N_1 \vee N_2$	D_1
N_3	$D_2 \vee D_3$
$AC_1 \vee AC_3$	$D_1 \vee D_2$
AS_1	$D_2 \vee D_3$
$R_1 \vee R_2 \vee R_5$	$D_1 \vee D_4$

Component dependencies

Component families with limitations

Application: Automotive Configuration (1)

Manufacturer's Constraints on Components

component family	components limit	premise	conclusion
engine	$E_1, E_2, E_3 = 1$	G_1	$E_1 \vee E_2$
gearbox	$G_1, G_2, G_3 = 1$	$N_1 \vee N_2$	D_1
control unit	$C_1, \dots, C_5 = 1$	N_3	$D_2 \vee D_3$
dashboard	$D_1, \dots, D_4 = 1$	$AC_1 \vee AC_3$	$D_1 \vee D_2$
navigation system	$N_1, N_2, N_3 \leq 1$	AS_1	$D_2 \vee D_3$
air conditioner	$AC_1, AC_2, AC_3 \leq 1$	$R_1 \vee R_2 \vee R_5$	$D_1 \vee D_4$
alarm system	$AS_1, AS_2 \leq 1$		
radio	$R_1, \dots, R_5 \leq 1$		

Component dependencies

Component families with limitations

Encoding

- ▶ for every component c use variable x_c which is assigned T iff c is used
- ▶ require manufacturer's constraints φ_{car} by adding respective clauses

Application: Automotive Configuration (1)

Manufacturer's Constraints on Components

component family	components limit	premise	conclusion
engine	$E_1, E_2, E_3 = 1$	G_1	$E_1 \vee E_2$
gearbox	$G_1, G_2, G_3 = 1$	$N_1 \vee N_2$	D_1
control unit	$C_1, \dots, C_5 = 1$	N_3	$D_2 \vee D_3$
dashboard	$D_1, \dots, D_4 = 1$	$AC_1 \vee AC_3$	$D_1 \vee D_2$
navigation system	$N_1, N_2, N_3 \leq 1$	AS_1	$D_2 \vee D_3$
air conditioner	$AC_1, AC_2, AC_3 \leq 1$	$R_1 \vee R_2 \vee R_5$	$D_1 \vee D_4$
alarm system	$AS_1, AS_2 \leq 1$		
radio	$R_1, \dots, R_5 \leq 1$		

Component dependencies

Component families with limitations

Encoding

- ▶ for every component c use variable x_c which is assigned T iff c is used
- ▶ require manufacturer's constraints φ_{car} by adding respective clauses

Problem 1: Validity of Configuration

- ▶ is desired configuration valid?

SAT encoding

Application: Automotive Configuration (1)

Manufacturer's Constraints on Components

component family	components limit	premise	conclusion
engine	$E_1, E_2, E_3 = 1$	G_1	$E_1 \vee E_2$
gearbox	$G_1, G_2, G_3 = 1$	$N_1 \vee N_2$	D_1
control unit	$C_1, \dots, C_5 = 1$	N_3	$D_2 \vee D_3$
dashboard	$D_1, \dots, D_4 = 1$	$AC_1 \vee AC_3$	$D_1 \vee D_2$
navigation system	$N_1, N_2, N_3 \leq 1$	AS_1	$D_2 \vee D_3$
air conditioner	$AC_1, AC_2, AC_3 \leq 1$	$R_1 \vee R_2 \vee R_5$	$D_1 \vee D_4$
alarm system	$AS_1, AS_2 \leq 1$		
radio	$R_1, \dots, R_5 \leq 1$		

Component dependencies

Component families with limitations

Encoding

- ▶ for every component c use variable x_c which is assigned T iff c is used
- ▶ require manufacturer's constraints φ_{car} by adding respective clauses

Problem 1: Validity of Configuration

- ▶ is desired configuration valid?

e.g. $E_1 \wedge G_1 \wedge C_5 \wedge (D_2 \vee D_3) \checkmark$

SAT encoding

$E_3 \wedge G_1 \wedge C_5 \wedge D_2 \vee AC_1 \times$

Application: Automotive Configuration (2)

Problem 2: Maximization of Chosen Components

- ▶ find maximal valid subset of configuration c_1, \dots, c_n

partial maxSAT

$$\underbrace{\varphi_{\text{car}}}_{\text{hard clauses}} \wedge \underbrace{x_{c_1} \wedge \dots \wedge x_{c_n}}_{\text{soft clauses}}$$

Application: Automotive Configuration (2)

Problem 2: Maximization of Chosen Components

- ▶ find maximal valid subset of configuration c_1, \dots, c_n

partial maxSAT

$$\underbrace{\varphi_{\text{car}}}_{\text{hard clauses}} \wedge \underbrace{x_{c_1} \wedge \dots \wedge x_{c_n}}_{\text{soft clauses}}$$

Example

component family	choice	result
engine	E_1	E_1
gearbox	G_2	G_2
control unit	C_2	C_2
dashboard	$D_1 \vee D_3$	D_1
navigation system	N_2	N_2
air conditioner	AC_1	AC_1
alarm system	AS_1	–
radio	R_2	R_2

Application: Automotive Configuration (2)

Problem 2: Maximization of Chosen Components

- ▶ find maximal valid subset of configuration c_1, \dots, c_n partial maxSAT
- ▶ possibly with priorities p_i for component c_i weighted partial maxSAT

$$\underbrace{\varphi_{\text{car}}}_{\text{hard clauses}} \wedge \underbrace{(x_{c_1}, p_1) \wedge \dots \wedge (x_{c_n}, p_n)}_{\text{soft clauses}}$$

Example

component family	choice	result
engine	E_1	E_1
gearbox	G_2	G_2
control unit	C_2	C_2
dashboard	$D_1 \vee D_3$	D_1
navigation system	N_2	N_2
air conditioner	AC_1	AC_1
alarm system	AS_1	–
radio	R_2	R_2

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- ▶ find maximal valid subset of configuration c_1, \dots, c_n partial maxSAT
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Application: Automotive Configuration (2)

Problem 2: Maximization of Chosen Components

- ▶ find maximal valid subset of configuration c_1, \dots, c_n partial maxSAT
- ▶ possibly with priorities p_i for component c_i weighted partial maxSAT

$$\underbrace{\varphi_{\text{car}}}_{\text{hard clauses}} \wedge \underbrace{(x_{c_1}, p_1) \wedge \dots \wedge (x_{c_n}, p_n)}_{\text{soft clauses}}$$

Problem 3: Minimization of Costs

- ▶ given cost q_i for each component c_i , find cheapest valid configuration weighted partial maxSAT

$$\underbrace{\varphi_{\text{car}}}_{\text{hard clauses}} \wedge \underbrace{(c_1, -q_1) \wedge \dots \wedge (c_n, -q_n)}_{\text{soft clauses}}$$

Application: Automotive Configuration (2)

Problem 2: Maximization of Chosen Components

- ▶ find maximal valid subset of configuration c_1, \dots, c_n partial maxSAT
- ▶ possibly with priorities p_i for component c_i weighted partial maxSAT

$$\underbrace{\varphi_{\text{car}}}_{\text{hard clauses}} \wedge \underbrace{(x_{c_1}, p_1) \wedge \dots \wedge (x_{c_n}, p_n)}_{\text{soft clauses}}$$

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$$\underbrace{\varphi_{\text{car}}}_{\text{hard clauses}} \wedge \underbrace{(c_1, -q_1) \wedge \dots \wedge (c_n, -q_n)}_{\text{soft clauses}}$$

Result

collaboration with BMW: evaluated on configuration formulas of 2013 product line

Outline

- Summary of Last Week
- Maximum Satisfiability
- Application: Automotive Configuration
- Algorithms for Minimum Unsatisfiability
 - Branch and Bound
 - Binary Search

Branch & Bound

Idea

- ▶ gets **list** of clauses φ as input and returns **minUNSAT**(φ)

Branch & Bound

Idea

- ▶ gets list of clauses φ as input and returns $\text{minUNSAT}(\varphi)$
- ▶ explores assignments in depth-first search

Branch & Bound

Idea

- ▶ gets list of clauses φ as input and returns $\text{minUNSAT}(\varphi)$
- ▶ explores assignments in depth-first search

Ingredients

- ▶ **UB** is minimal number of unsatisfied clauses found so far (**upper bound**)

Branch & Bound

Idea

- ▶ gets list of clauses φ as input and returns $\text{minUNSAT}(\varphi)$
- ▶ explores assignments in depth-first search

Ingredients

- ▶ UB is minimal number of unsatisfied clauses found so far (upper bound)
- ▶ φ_x is formula φ with all occurrences of x replaced by **T**
- ▶ $\varphi_{\bar{x}}$ is formula φ with all occurrences of x replaced by **F**

Branch & Bound

Idea

- ▶ gets list of clauses φ as input and returns $\text{minUNSAT}(\varphi)$
- ▶ explores assignments in depth-first search

Ingredients

- ▶ UB is minimal number of unsatisfied clauses found so far (upper bound)
- ▶ φ_x is formula φ with all occurrences of x replaced by T
- ▶ $\varphi_{\bar{x}}$ is formula φ with all occurrences of x replaced by F
- ▶ for list of clauses φ , function $\text{simp}(\varphi)$
 - ▶ replaces $\neg T$ by F and $\neg F$ by T
 - ▶ drops all clauses which contain T
 - ▶ removes F from all remaining clauses

Branch & Bound

Idea

- ▶ gets list of clauses φ as input and returns $\text{minUNSAT}(\varphi)$
- ▶ explores assignments in depth-first search

Ingredients

- ▶ UB is minimal number of unsatisfied clauses found so far (upper bound)
- ▶ φ_x is formula φ with all occurrences of x replaced by T
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- ▶ for list of clauses φ , function $\text{simp}(\varphi)$
 - ▶ replaces $\neg T$ by F and $\neg F$ by T
 - ▶ drops all clauses which contain T
 - ▶ removes F from all remaining clauses

Example

$$\begin{array}{l} \varphi = y \vee \neg F, \quad x \vee y \vee F, \quad F, \quad x \vee \neg y \vee T, \quad x \vee \neg z \\ \text{simp}(\varphi) = \quad \quad \quad x \vee y, \quad \square, \quad \quad \quad x \vee \neg z \end{array}$$

Branch & Bound

Idea

- ▶ gets list of clauses φ as input and returns $\text{minUNSAT}(\varphi)$
- ▶ explores assignments in depth-first search

Ingredients

- ▶ UB is minimal number of unsatisfied clauses found so far (upper bound)
- ▶ φ_x is formula φ with all occurrences of x replaced by T
- ▶ $\varphi_{\bar{x}}$ is formula φ with all occurrences of x replaced by F
- ▶ for list of clauses φ , function $\text{simp}(\varphi)$
 - ▶ replaces $\neg T$ by F and $\neg F$ by T
 - ▶ drops all clauses which contain T
 - ▶ removes F from all remaining clauses
- ▶ \square denotes empty clause and $\#\text{empty}(\varphi)$ number of empty clauses in φ

Example

$$\begin{array}{l} \varphi = y \vee \neg F, \quad x \vee y \vee F, \quad F, \quad x \vee \neg y \vee T, \quad x \vee \neg z \\ \text{simp}(\varphi) = \quad \quad \quad x \vee y, \quad \square, \quad \quad \quad x \vee \neg z \end{array}$$

Algorithm (Branch & Bound)

```
function BnB( $\varphi$ , UB)
   $\varphi$  = simp( $\varphi$ )
  if  $\varphi$  contains only empty clauses then
    return #empty( $\varphi$ )
  if #empty( $\varphi$ )  $\geq$  UB then
    return UB
  x = selectVariable( $\varphi$ )
  UB' = min(UB, BnB( $\varphi_x$ , UB))
  return min(UB', BnB( $\varphi_{\bar{x}}$ , UB'))
```

Algorithm (Branch & Bound)

```
function BnB( $\varphi$ , UB)
   $\varphi$  = simp( $\varphi$ )
  if  $\varphi$  contains only empty clauses then
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```

- note that number of clauses falsified by any valuation is $\leq |\varphi|$

Algorithm (Branch & Bound)

```
function BnB( $\varphi$ , UB)
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  x = selectVariable( $\varphi$ )
  UB' = min(UB, BnB( $\varphi_x$ , UB))
  return min(UB', BnB( $\varphi_{\bar{x}}$ , UB'))
```

- ▶ note that number of clauses falsified by any valuation is $\leq |\varphi|$
- ▶ start by calling **BnB(φ , $|\varphi|$)**

Algorithm (Branch & Bound)

```
function BnB( $\varphi$ , UB)
   $\varphi$  = simp( $\varphi$ )
  if  $\varphi$  contains only empty clauses then
    return #empty( $\varphi$ )
  if #empty( $\varphi$ )  $\geq$  UB then
    return UB
  x = selectVariable( $\varphi$ )
  UB' = min(UB, BnB( $\varphi_x$ , UB))
  return min(UB', BnB( $\varphi_{\bar{x}}$ , UB'))
```

- ▶ note that number of clauses falsified by any valuation is $\leq |\varphi|$
- ▶ start by calling $\text{BnB}(\varphi, |\varphi|)$
- ▶ **idea:** #empty(φ) is number of clauses falsified by current valuation

Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$

Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$

Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

$\text{BnB}(\varphi, 6)$

Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

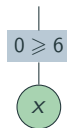
$\text{BnB}(\varphi, 6)$

|
0 \geq 6

Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

$\text{BnB}(\varphi, 6)$

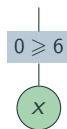


Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

- ▶ $\varphi_x = \text{T}, \neg \text{T} \vee y, z \vee \neg y, \text{T} \vee z, \text{T} \vee y, \neg y$

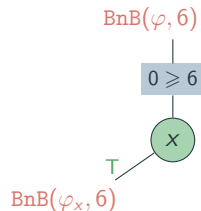
$\text{BnB}(\varphi, 6)$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

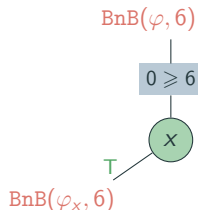
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

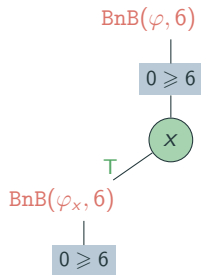
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
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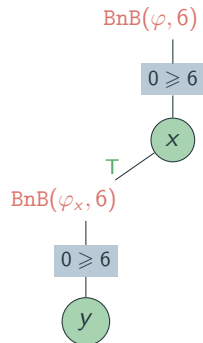
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Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
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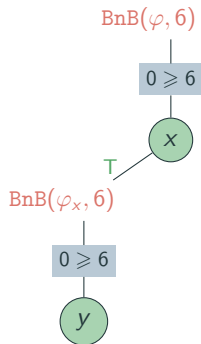
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

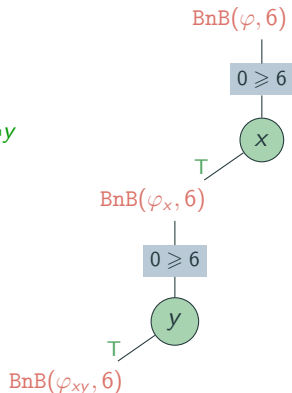
- ▶ $\varphi_x = T, \neg T \vee y, z \vee \neg y, T \vee z, T \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = T, z \vee \neg T, \neg T$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

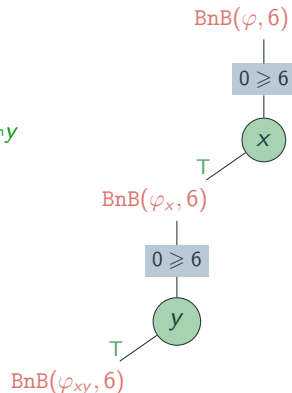
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
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 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

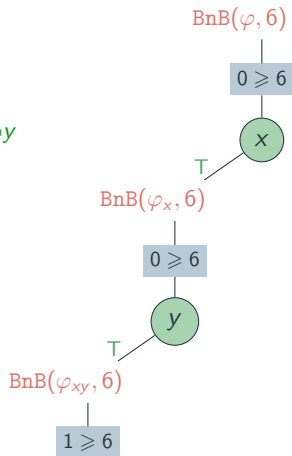
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

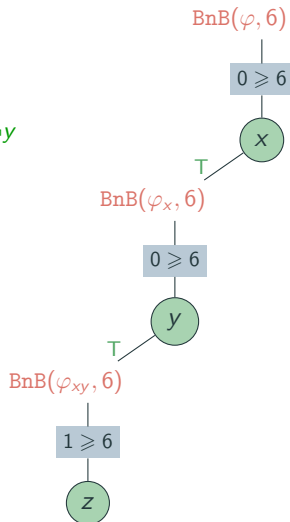
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 $\text{simp}(\varphi_{xy}) = z, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
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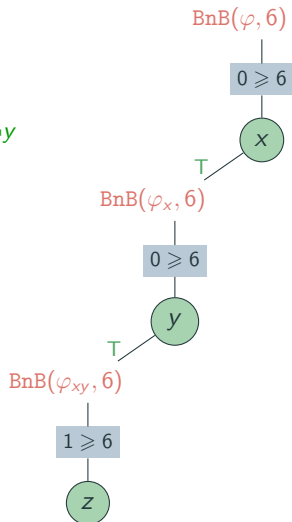
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

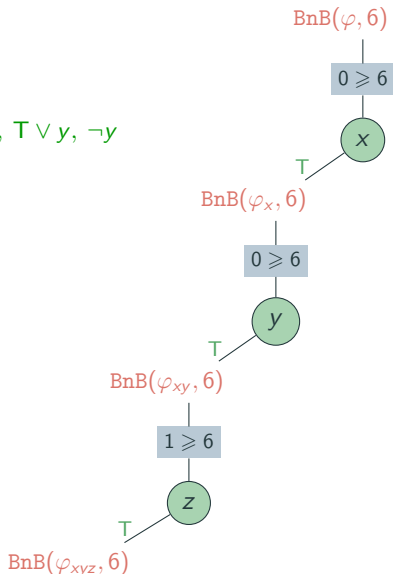
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

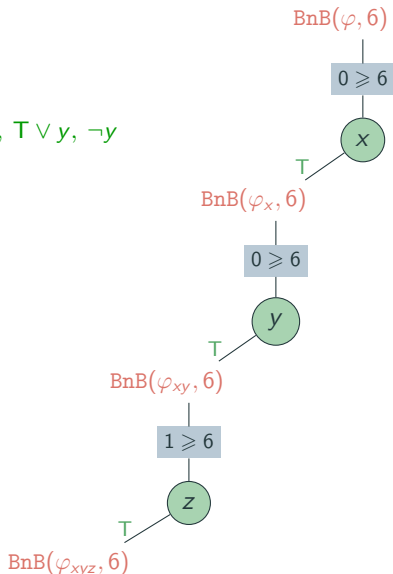
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
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- ▶ $\text{simp}(\varphi) = \varphi$

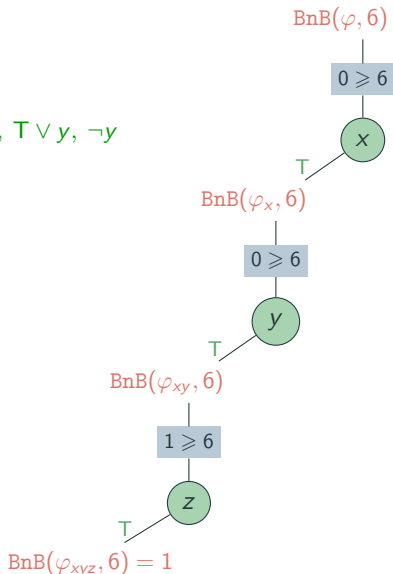
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

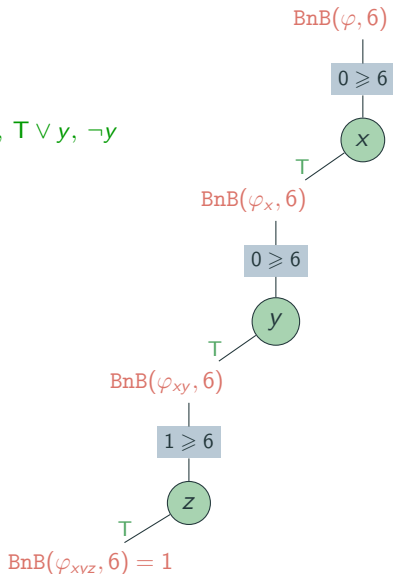
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

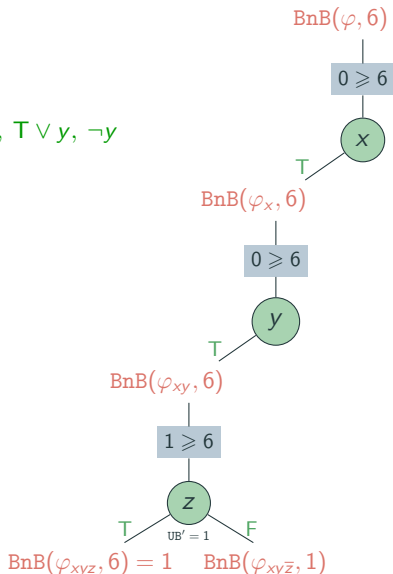
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

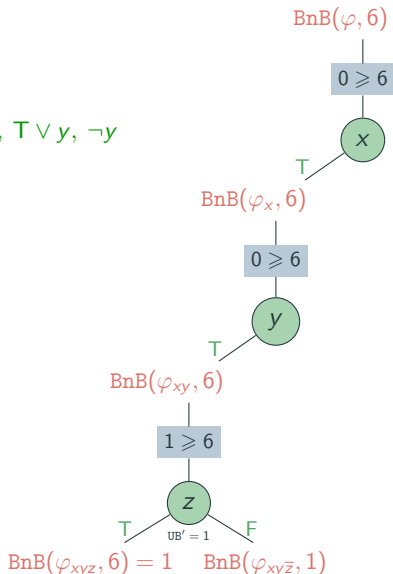
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

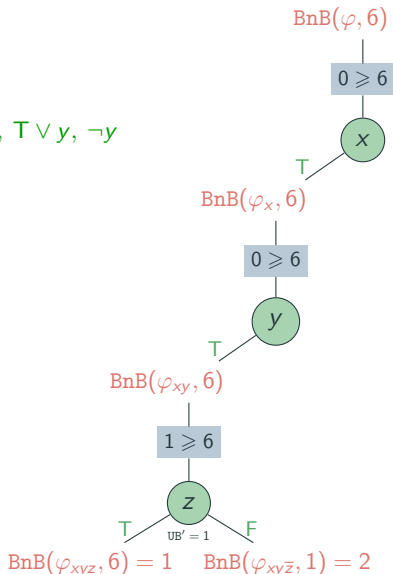
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

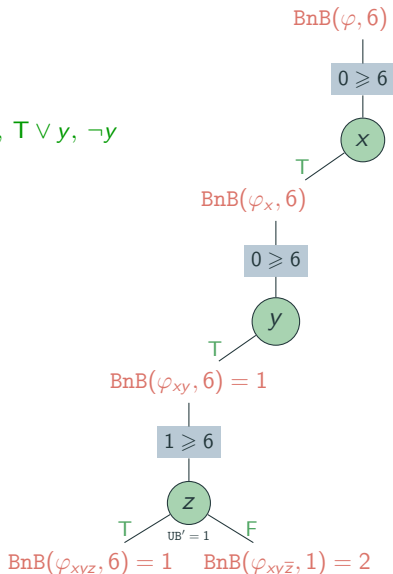
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

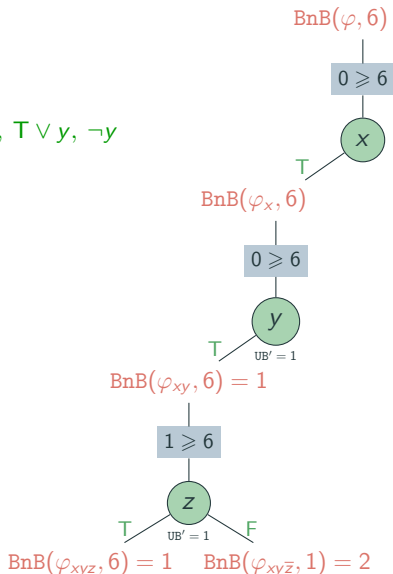
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

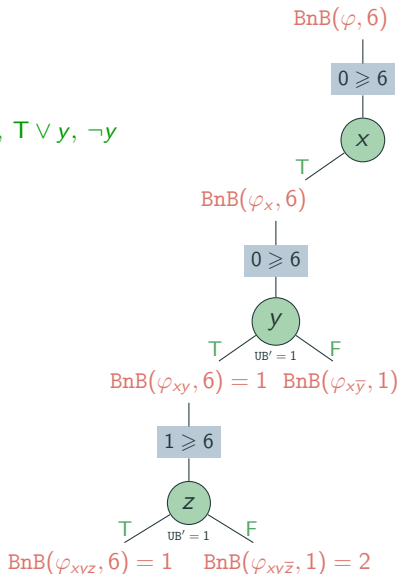
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \text{F}, z \vee \neg \text{F}, \neg \text{F}$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

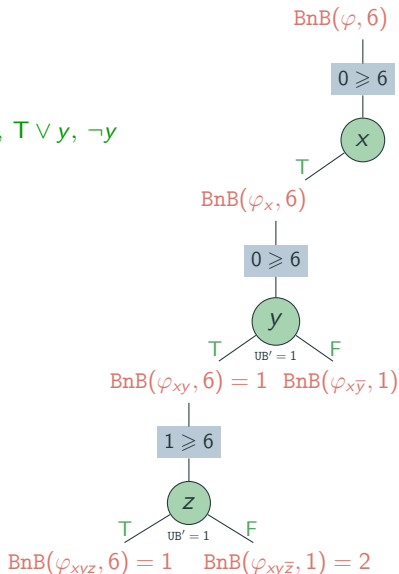
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \text{F}, z \vee \neg \text{F}, \neg \text{F}$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

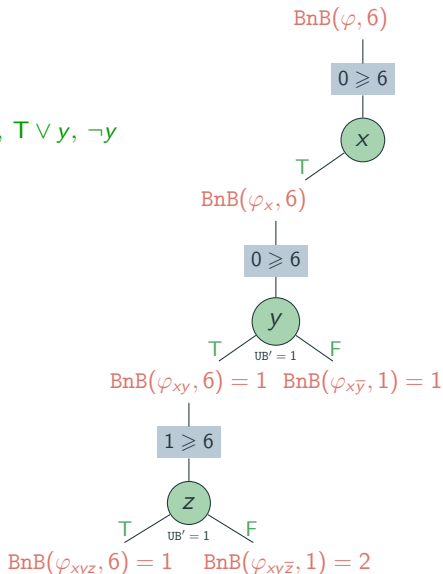
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \text{F}, z \vee \neg \text{F}, \neg \text{F}$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

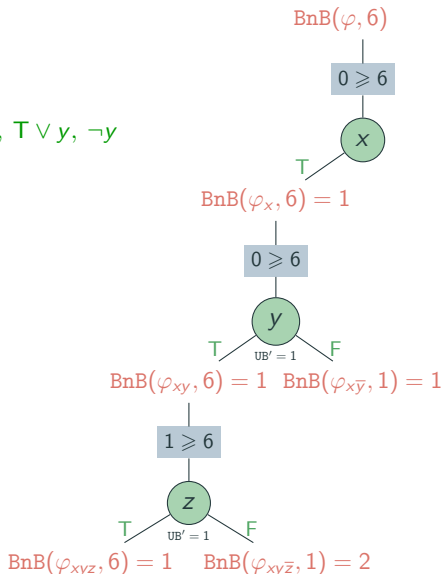
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \text{F}, z \vee \neg \text{F}, \neg \text{F}$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

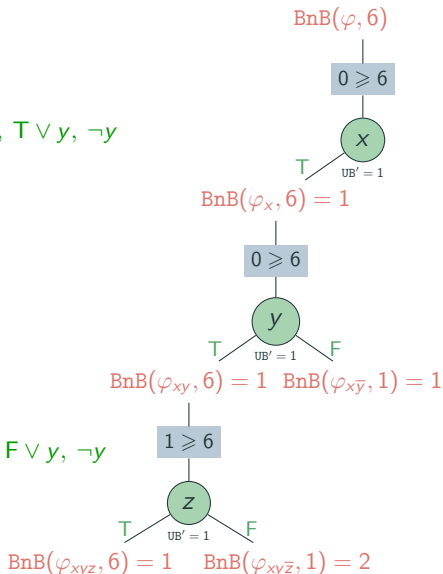
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \text{F}, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \text{F}, z \vee \neg \text{F}, \neg \text{F}$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

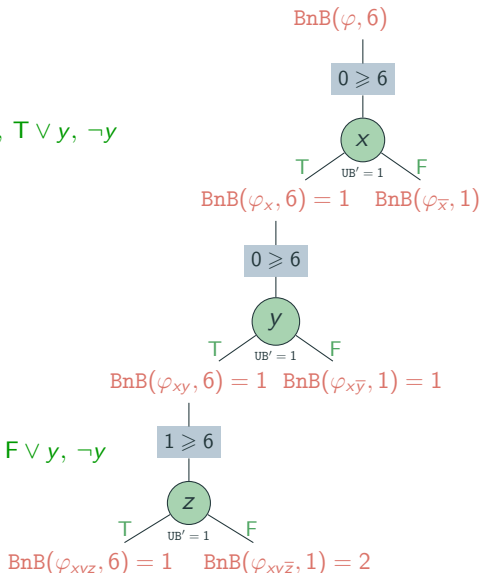
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \top, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$
- ▶ $\varphi_{\bar{x}} = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

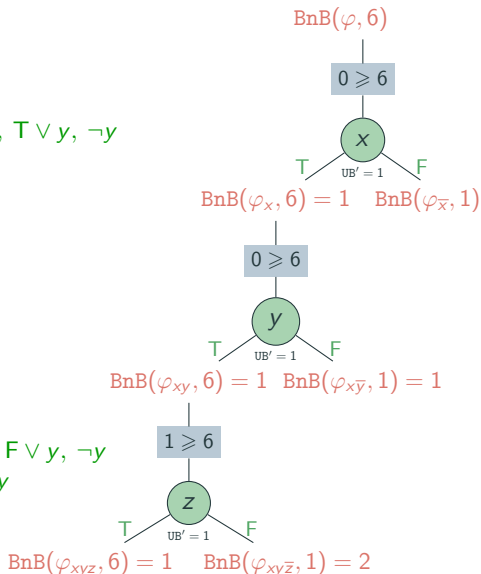
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \top, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$
- ▶ $\varphi_{\bar{x}} = \text{F}, \neg \text{F} \vee y, z \vee \neg y, \text{F} \vee z, \text{F} \vee y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

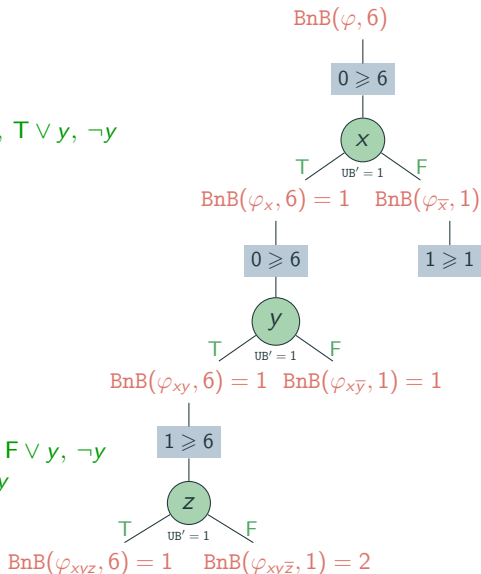
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \top, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$
- ▶ $\varphi_{\bar{x}} = \text{F}, \neg \text{F} \vee y, z \vee \neg y, \text{F} \vee z, \text{F} \vee y, \neg y$
 $\text{simp}(\varphi_{\bar{x}}) = \square, z \vee \neg y, z, y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

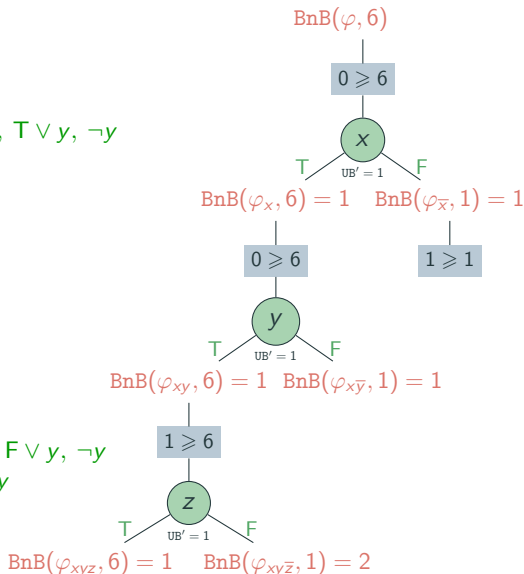
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \top, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$
- ▶ $\varphi_{\bar{x}} = \text{F}, \neg \text{F} \vee y, z \vee \neg y, \text{F} \vee z, \text{F} \vee y, \neg y$
 $\text{simp}(\varphi_{\bar{x}}) = \square, z \vee \neg y, z, y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

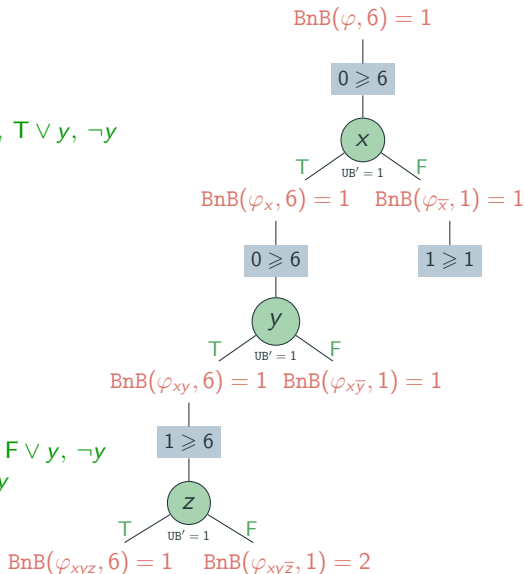
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \top, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{x\bar{y}}) = z, \square$
- ▶ $\varphi_{\bar{x}} = \text{F}, \neg \text{F} \vee y, z \vee \neg y, \text{F} \vee z, \text{F} \vee y, \neg y$
 $\text{simp}(\varphi_{\bar{x}}) = \square, z \vee \neg y, z, y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

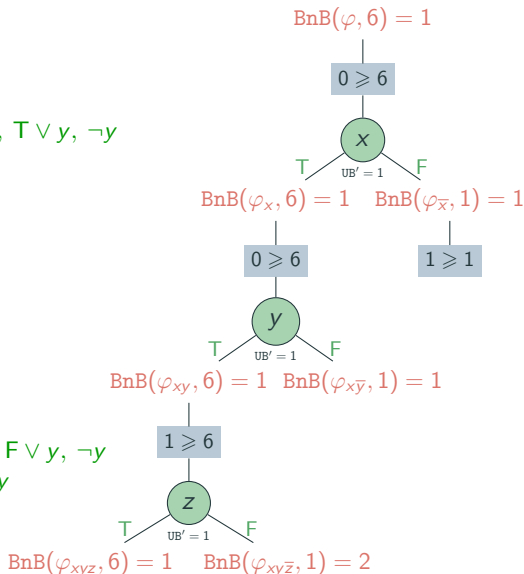
- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \top, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$
- ▶ $\varphi_{\bar{x}} = \bar{x}, \neg \bar{x} \vee y, z \vee \neg y, \bar{x} \vee z, \bar{x} \vee y, \neg y$
 $\text{simp}(\varphi_{\bar{x}}) = \square, z \vee \neg y, z, y, \neg y$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$
 - ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \top, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$
 - ▶ $\varphi_{x\bar{y}} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$
- ▶ $\varphi_{\bar{x}} = \text{F}, \neg \text{F} \vee y, z \vee \neg y, \text{F} \vee z, \text{F} \vee y, \neg y$
 $\text{simp}(\varphi_{\bar{x}}) = \square, z \vee \neg y, z, y, \neg y$
- ▶ $\text{minUNSAT}(\varphi) = 1$



Example

- ▶ $\varphi = x, \neg x \vee y, z \vee \neg y, x \vee z, x \vee y, \neg y$
- ▶ call $\text{BnB}(\varphi, 6)$
- ▶ $\text{simp}(\varphi) = \varphi$

- ▶ $\varphi_x = \top, \neg \top \vee y, z \vee \neg y, \top \vee z, \top \vee y, \neg y$
 $\text{simp}(\varphi_x) = y, z \vee \neg y, \neg y$

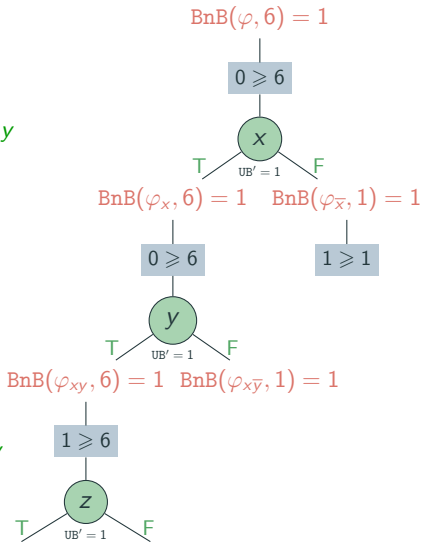
- ▶ $\varphi_{xy} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{xy}) = z, \square$
 - ▶ $\varphi_{xyz} = \top, \square$
 $\text{simp}(\varphi_{xyz}) = \square$
 - ▶ $\varphi_{xy\bar{z}} = \top, \square$
 $\text{simp}(\varphi_{xy\bar{z}}) = \square, \square$

- ▶ $\varphi_{x\bar{y}} = \top, z \vee \neg \top, \neg \top$
 $\text{simp}(\varphi_{x\bar{y}}) = \square$

- ▶ $\varphi_{\bar{x}} = \bar{x}, \neg \bar{x} \vee y, z \vee \neg y, \bar{x} \vee z, \bar{x} \vee y, \neg y$
 $\text{simp}(\varphi_{\bar{x}}) = \square, z \vee \neg y, z, y, \neg y$

- ▶ $\text{minUNSAT}(\varphi) = 1$

- ▶ e.g. $v(x) = v(y) = v(z) = \top$ $\text{BnB}(\varphi_{xyz}, 6) = 1$ $\text{BnB}(\varphi_{xy\bar{z}}, 1) = 2$



Algorithm (Branch & Bound, improved)

```
function BnB'( $\varphi$ , UB)
   $\varphi$  = simp( $\varphi$ )
  if  $\varphi$  contains only empty clauses then
    return #empty( $\varphi$ )
  LB = #empty( $\varphi$ ) + underapproximate( $\varphi$ )
  if LB  $\geq$  UB then
    return UB
  x = selectVariable( $\varphi$ )
  UB' = min(UB, BnB'( $\varphi_x$ , UB))
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► $ic(x)$ is number of unit clauses x in φ

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Theorem

$$\text{BnB}(\varphi, |\varphi|) = \text{BnB}'(\varphi, |\varphi|) = \text{minUNSAT}(\varphi)$$

Binary Search

Idea

- ▶ gets list of clauses φ as input and returns $\text{minUNSAT}(\varphi)$

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- ▶ repeatedly call SAT solver in binary search fashion

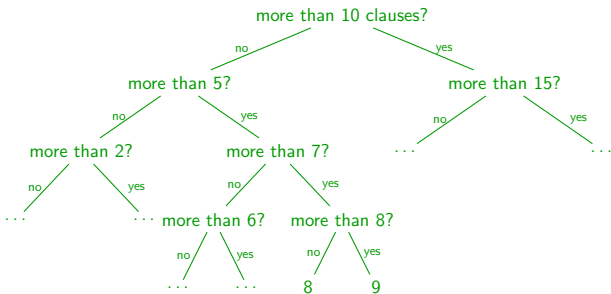
Binary Search

Idea

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Example

Suppose given formula with 20 clauses. Can we satisfy ...



Cardinality Constraints

Definitions

- ▶ **cardinality constraint** has form $(\sum_{x \in X} x) \bowtie N$ where \bowtie is $=, <, >, \leq,$ or $\geq,$
 X is set of propositional variables and $N \in \mathbb{N}$

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Example

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- ▶ $x_1 + x_2 + \dots + x_8 \leq 3$

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Example

- ▶ $x + y + z = 1$ satisfied by $v(x) = v(y) = F$, $v(z) = T$
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$O(2^{|X|})$

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$$\mathcal{O}(N \cdot |X|)$$

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- ▶ write $\text{CNF}(\sum_{x \in X} x \bowtie N)$ for CNF encoding
- ▶ cardinality constraints occur very frequently! (n -queens, Minesweeper, ...)

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Algorithm (Binary Search)

```
function BinarySearch( $\{C_1, \dots, C_m\}$ )  
   $\varphi := \{C_1 \vee b_1, \dots, C_m \vee b_m\}$   
  return search( $\varphi, 0, m$ )
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```

Theorem

$\text{BinarySearch}(\psi) = \min\text{UNSAT}(\psi)$

Example

$$\varphi = \{ 6 \vee 2, \bar{6} \vee 2, \bar{2} \vee 1, \bar{1}, \bar{6} \vee 8, \\ 6 \vee \bar{8}, 2 \vee 4, \bar{4} \vee 5, 7 \vee 5, \bar{7} \vee 5, \\ \bar{3}, \bar{5} \vee 3 \}$$

Example

$$\varphi = \{ 6 \vee 2 \vee b_1, \quad \bar{6} \vee 2 \vee b_2, \quad \bar{2} \vee 1 \vee b_3, \quad \bar{1} \vee b_4, \quad \bar{6} \vee 8 \vee b_5, \\ 6 \vee \bar{8} \vee b_6, \quad 2 \vee 4 \vee b_7, \quad \bar{4} \vee 5 \vee b_8, \quad 7 \vee 5 \vee b_9, \quad \bar{7} \vee 5 \vee b_{10}, \\ \bar{3} \vee b_{11}, \quad \bar{5} \vee 3 \vee b_{12} \}$$

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► $L = 0, U = 12, \text{mid} = 6$ $\text{SAT}(\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 6))?$ ✓

Example

$$\varphi = \{ 6 \vee 2 \vee b_1, \quad \bar{6} \vee 2 \vee b_2, \quad \bar{2} \vee 1 \vee b_3, \quad \bar{1} \vee b_4, \quad \bar{6} \vee 8 \vee b_5, \\ 6 \vee \bar{8} \vee b_6, \quad 2 \vee 4 \vee b_7, \quad \bar{4} \vee 5 \vee b_8, \quad 7 \vee 5 \vee b_9, \quad \bar{7} \vee 5 \vee b_{10}, \\ \bar{3} \vee b_{11}, \quad \bar{5} \vee 3 \vee b_{12} \}$$

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- ▶ $L = 0, U = 6, \text{mid} = 3$ $\text{SAT}(\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 3))?$ ✓
- ▶ $L = 0, U = 3, \text{mid} = 1$ $\text{SAT}(\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 1))?$ ✗

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- | | | |
|--------------------------|--|---|
| ▶ L = 0, U = 12, mid = 6 | SAT($\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 6)$)? | ✓ |
| ▶ L = 0, U = 6, mid = 3 | SAT($\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 3)$)? | ✓ |
| ▶ L = 0, U = 3, mid = 1 | SAT($\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 1)$)? | ✗ |
| ▶ L = 2, U = 3, mid = 2 | SAT($\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 2)$)? | ✓ |

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- ▶ $L = 0, U = 3, \text{mid} = 1$ $\text{SAT}(\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 1))?$ ✗
- ▶ $L = 2, U = 3, \text{mid} = 2$ $\text{SAT}(\varphi \wedge \text{CNF}(\sum_{i=1}^m b_i \leq 2))?$ ✓
- ▶ $L = 2, U = 2$ **return 2**

Cardinality Constraints in Z3

```
from z3 import *

xs = [ Bool("x"+str(i)) for i in range (0,10)]
ys = [ Bool("y"+str(i)) for i in range (0,10)]

def card(ps):
    return sum([If(x, 1, 0) for x in ps])

solver = Solver()
solver.add(card(xs) == 5, card(ys) > 2, card(ys) <= 4)

if solver.check() == sat:
    model = solver.model()
    for i in range(0,10):
        print xs[i], "=", model[xs[i]], ys[i], "=", model[ys[i]]
```

Remark

maxSAT is not a decision problem

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FP^{NP} is class of functions computable in polynomial time with access to NP oracle

Complexity

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Remarks

- ▶ FP^{NP} allows polynomial number of oracle calls (which is e.g. SAT solver)
- ▶ other members of FP^{NP} are function versions of travelling salesperson and Knapsack



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