universität innsbruck

SAT and SMT Solving

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lecture 4
SS 2019

Outline

- Summary of Last Week
- Unsatisfiable Cores
- Algorithm by Fu and Malik
- Unsatisfiable Cores in Practice

Maximum Satisfiability

Consider CNF formulas χ and φ as sets of clauses.

Definitions

- maxSAT(φ) is maximal $\sum_{C\in\varphi} \overline{v}(C)$ for valuation v
- pmaxSAT(φ, χ) is maximal $\sum_{C\in\varphi} \overline{v}(C)$ for valuation v with $v(\chi) = \mathsf{T}$

Definitions

given weights $w_C \in \mathbb{Z}$ for all $C \in \varphi$,

- $\text{maxSAT}_w(\varphi)$ is maximal $\sum_{C\in\varphi} w_C \cdot \overline{v}(C)$ for valuation v?
- $\text{pmaxSAT}_w(\varphi, \chi)$ is maximal $\sum_{C\in\varphi} w_C \cdot \overline{v}(C)$ for valuation v with $v(\chi) = \mathsf{T}$

Definition

minUNSAT(φ) is minimal $\sum_{C\in\varphi} \overline{v}(\neg C)$ for valuation v

Lemma

$$|\varphi| = |\text{minUNSAT}(\varphi)| + |\text{maxSAT}(\varphi)|$$

Branch & Bound

Idea

- gets list of clauses φ as input return minUNSAT(φ)
- explores assignments in depth-first search

```
function BnB(φ, UB)
  φ = simp(φ)
  if φ contains only empty clauses then
      return #empty(φ)
  if #empty(φ) ⩾ UB then
      return UB
  x = selectVariable(φ)
  UB′ = min(UB, BnB(φ_x, UB))
  return min(UB′, BnB(φ_x̄, UB′))
```

Theorem

$\mathtt{BnB}(\varphi, |\varphi|) = \mathsf{minUNSAT}(\varphi)$

Binary Search

Idea

- gets list of clauses φ as input and returns minUNSAT(φ)
- repeatedly call SAT solver in binary search fashion

Definitions

- cardinality constraint is
$$\sum_{x \in X} x \bowtie N$$
where $\bowtie$ is $=$, $<$, $>$, $\leqslant$, or $\geqslant$, X is set of propositional variables, and $N \in \mathbb{N}$
- valuation v satisfies $\sum_{x \in X} x \bowtie N$ iff $k \bowtie N$
where k is number of variables $x \in X$ such that $v(x) = \mathrm{T}$

Remark

cardinality constraints are expressible in CNF

Algorithm (Binary Search)

```
function BinarySearch({C_1, ..., C_m})
  φ := {C_1 ∨ b_1, ..., C_m ∨ b_m}
  return search(φ,0,m)
```

$b_1, \ldots, b_m$ are fresh variables

```
function search(φ, L, U)
  if L ⩾ U then
    return U
  mid := ⌊(U+L)/2⌋
  if SAT(φ ∧ CNF(∑_{i=1}^{m} b_i ⩽ mid)) then
    return search(φ, L, mid)
  else
    return search(φ, mid + 1, U)
```

Theorem

$\texttt{BinarySearch}(\varphi) = \text{minUNSAT}(\varphi)$

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$$\varphi = \{\neg x, \qquad x \vee z, \qquad \neg y \vee \neg z, \qquad x, \qquad y \vee \neg z\}$$

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Remark

SUC is always minimal unsatisfiable core

Example

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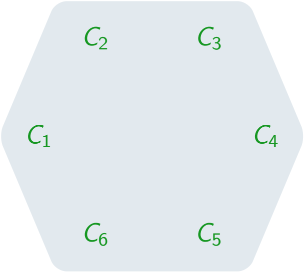

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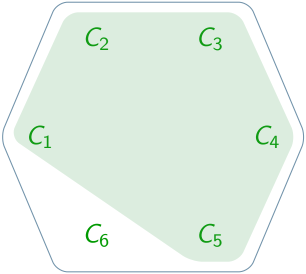

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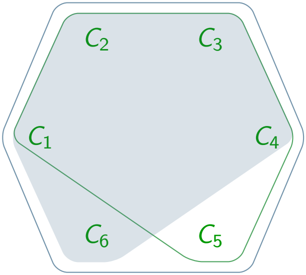

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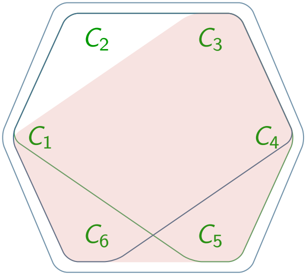

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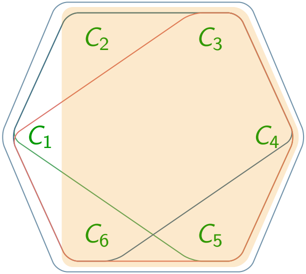

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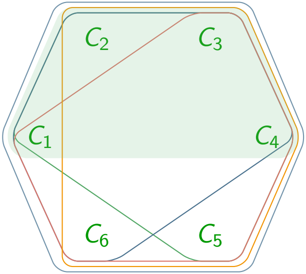

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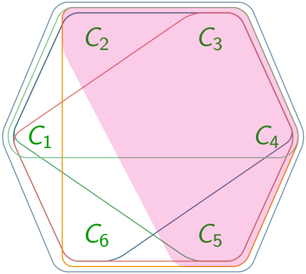

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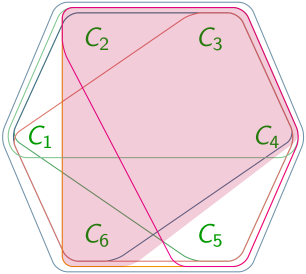

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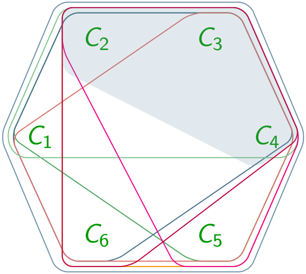

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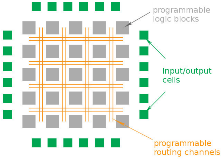

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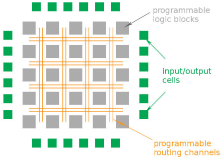

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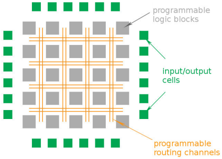
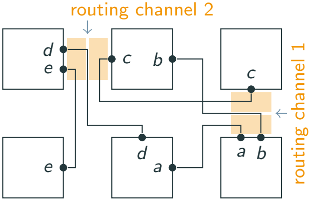


Example (Encoding Routing Requirements)

- consider connections a, b, c, d, e of 2 bits each



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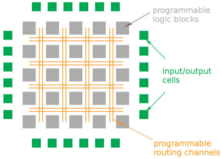

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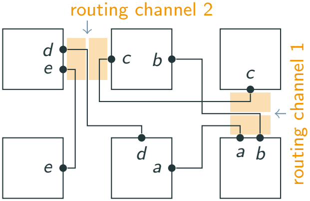

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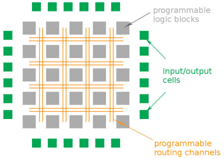

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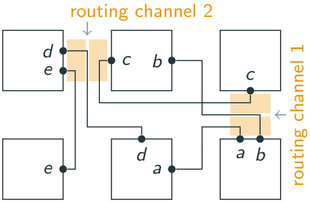
- consider connections a, b, c, d, e of 2 bits each
- liveness: want to route $\geqslant 1$ bit of a, b, c, d, e
- 2 routing channels of 2 tracks each
- exclusivity: each channel has only 2 tracks



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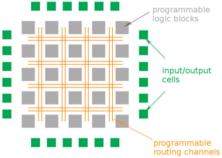

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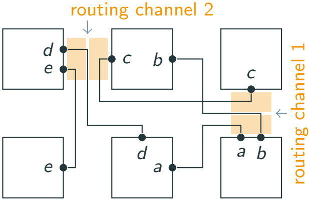
- unsatisfiable: UCs indicate problems



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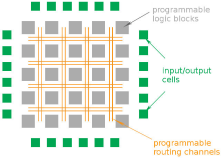
$b_0 \vee b_1$	$\neg a_0 \vee \neg c_0$	$\neg c_0 \vee \neg e_0$
$c_0 \vee c_1$	$\neg b_0 \vee \neg c_0$	$\neg d_0 \vee \neg e_0$
$d_0 \vee d_1$	$\neg a_1 \vee \neg b_1$	$\neg c_1 \vee \neg d_1$
$e_0 \vee e_1$	$\neg a_1 \vee \neg c_1$	$\neg c_1 \vee \neg e_1$
	$\neg b_1 \vee \neg c_1$	$\neg d_1 \vee \neg e_1$

Application: FPGA Routing

Field Programmable Gate Arrays (FPGAs)

- can simulate microprocessors but faster for special tasks (from complex combinatorics to mere logic)
- logic blocks connected by “routing channels”
- “routing”: determine which channels are used for what



Example (Encoding Routing Requirements)

- consider connections a, b, c, d, e of 2 bits each
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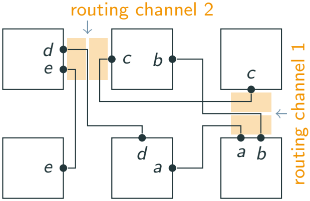

UC_1: channel 1 capacity exceeded

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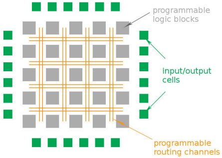

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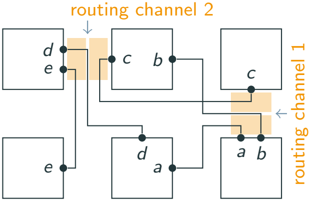

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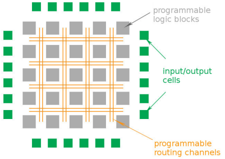

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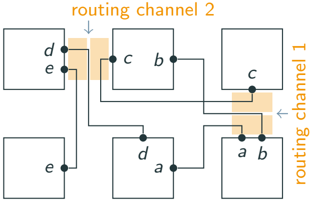

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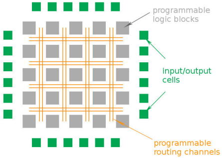

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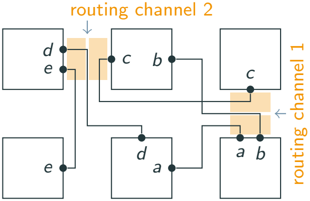

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$e_0 \vee e_1$ ●●●

$\neg a_0 \vee \neg b_0$ ●●
$\neg a_0 \vee \neg c_0$ ●●
$\neg b_0 \vee \neg c_0$ ●●
$\neg a_1 \vee \neg b_1$ ●●
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$\neg b_1 \vee \neg c_1$ ●●

$\neg c_0 \vee \neg d_0$ ●●
$\neg c_0 \vee \neg e_0$ ●●
$\neg d_0 \vee \neg e_0$ ●●
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UC_1: channel 1 capacity exceeded
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UC_3: c is overconstrained
UC_4: c is overconstrained

Finding Minimal Unsatisfiable Cores by Resolution

Idea

- repeatedly pick clause C from φ and check satisfiability:
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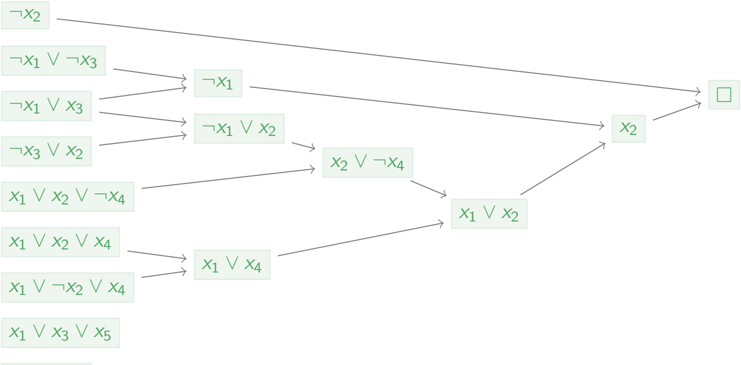

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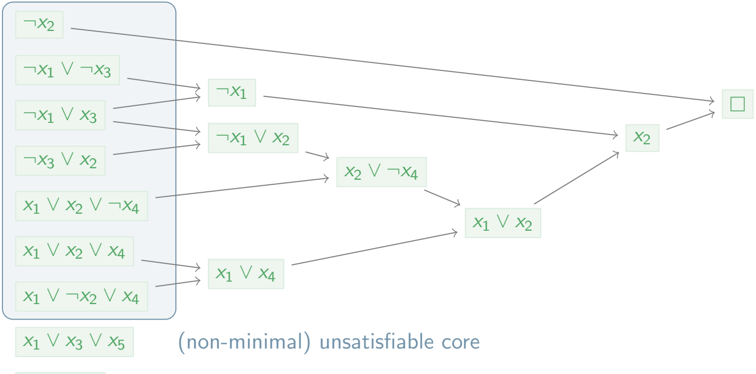

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Notation

- $Reach_G(C)$ is set of nodes reachable from C in G

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- $Reach_G(C)$ is set of nodes reachable from C in G
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Notation

- $\mathit{Reach}_G(C)$ is set of nodes reachable from C in G
- $\mathit{Reach}^E_G(C)$ is set of edges reachable from C in G
- $\overline{N}$ is $V \setminus N$ for any set of nodes N

Algorithm minUnsatCore(φ)

Input: unsatisfiable formula φ
Output: minimal unsatisfiable core of φ

```
build resolution graph G = (V_i ⊎ V_c, E) for φ
while ∃ unmarked clause in V_i do
    C ← unmarked clause in V_i
    if SAT(¬Reach_G(C)) then          ▷ subgraph without C satisfiable?
        mark C                          ▷ C is UC member
    else
        build resolution graph G' = (V'_i ⊎ V'_c, E') for ¬Reach_G(C)
        V_i ← V_i \ {C} and V_c ← V'_c ∪ (V_c \ Reach_G(C))
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        G ← (V_i ∪ V_c, E)
        G ← G|□                         ▷ restrict to nodes with path to □
return V_i
```

(Here $\overline{Reach_G(C)}$ denotes the complement of $Reach_G(C)$, written ¬Reach_G(C) above.)

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In the listing, $\overline{Reach_G(C)}$ is printed with an overline (shown here as ¬Reach_G(C)).

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return V_i

Algorithm minUnsatCore(φ)

Input: unsatisfiable formula φ
Output: minimal unsatisfiable core of φ

build resolution graph $G = (V_i \uplus V_c, E)$ for φ
while $\exists$ unmarked clause in V_i **do**
  $C \leftarrow$ unmarked clause in V_i
  if SAT($\overline{Reach_G(C)}$) **then** ▷ subgraph without C satisfiable?
    mark C ▷ C is UC member
  else
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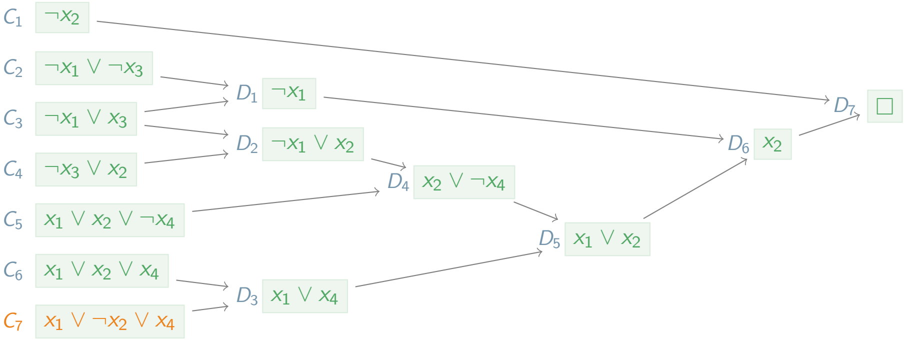
Theorem

if φ unsatisfiable then minUnsatCore(φ) is minimal unsatisfiable core of φ

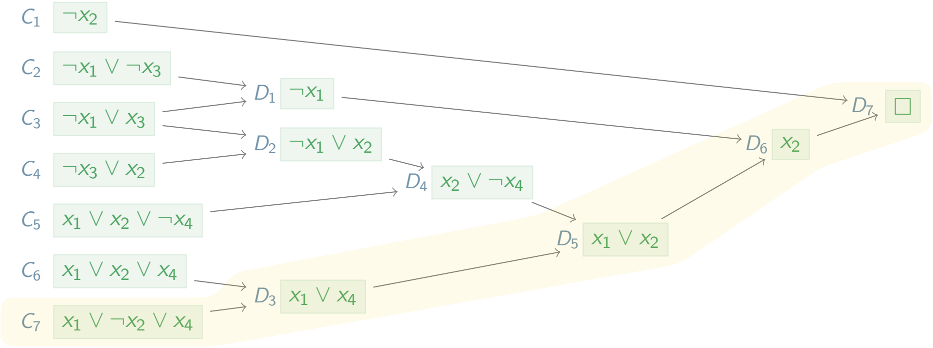
Example



minUnsatCore(φ)

- pick C_7

Example



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- pick C_7
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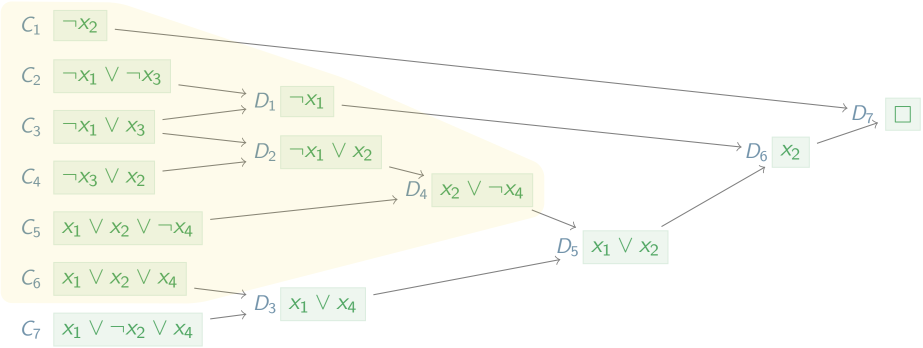

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Example

C_1 $\neg x_2$

C_2 $\neg x_1 \lor \neg x_3$

C_3 $\neg x_1 \lor x_3$

C_4 $\neg x_3 \lor x_2$

C_5 $x_1 \lor x_2 \lor \neg x_4$

C_6 $x_1 \lor x_2 \lor x_4$

D_1 $\neg x_1$

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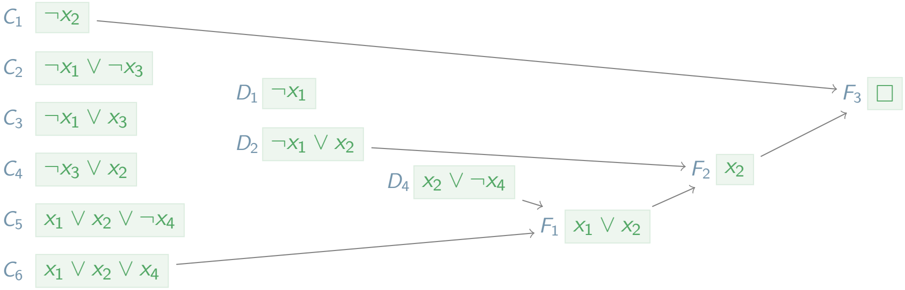

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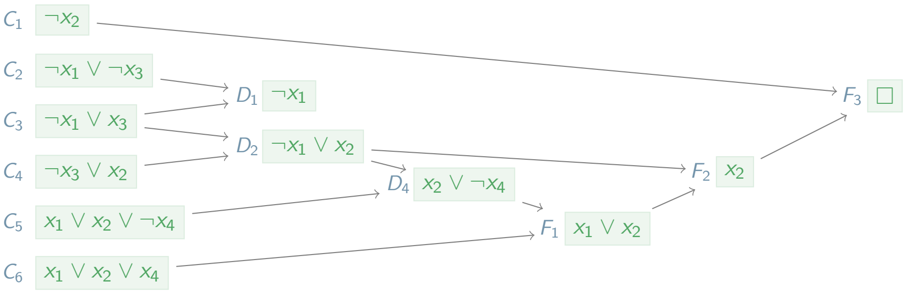

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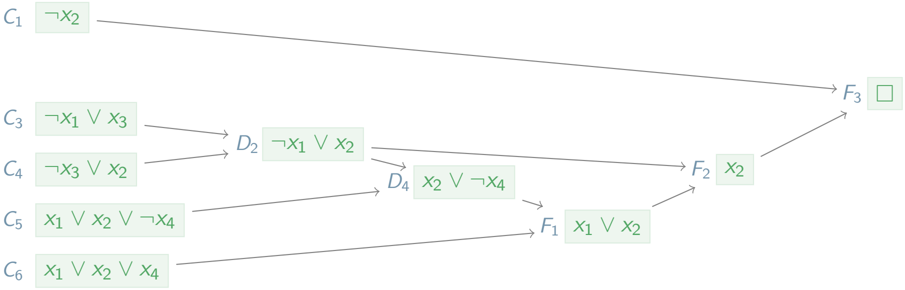

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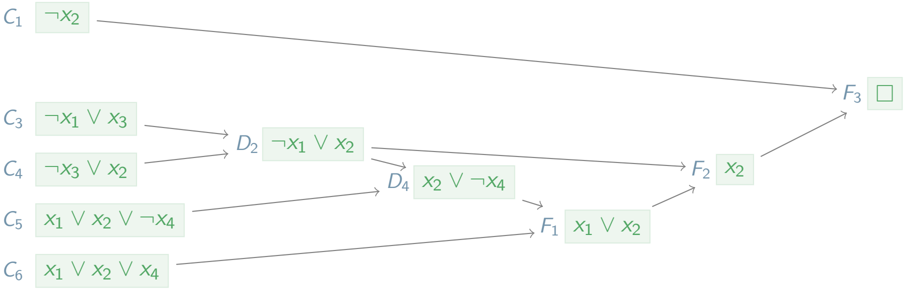

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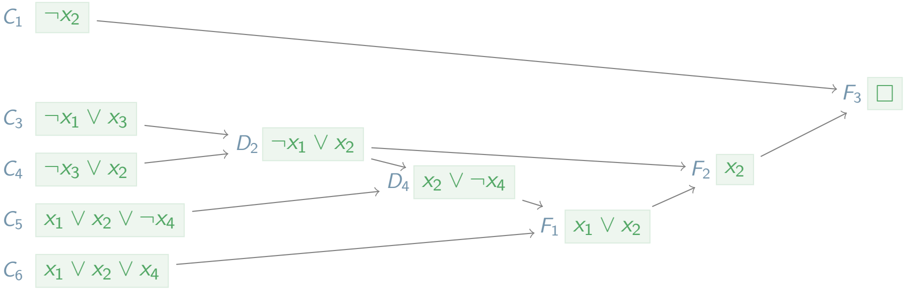
- after 5 more loop iterations: return $\{C_1, C_3, \ldots, C_6\}$

Example



minUnsatCore(φ)

re-use relevant resolvents: fewer steps to $\square$

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Outline

- Summary of Last Week
- Unsatisfiable Cores
- Algorithm by Fu and Malik
- Unsatisfiable Cores in Practice

Bounds for Maximum Satisfiability

consider CNF formula $\varphi = C_1 \wedge \cdots \wedge C_m$

Definition

blocked formula is $\varphi_B = (C_1 \vee b_1) \wedge \cdots \wedge (C_m \vee b_m)$ for fresh variables $b_1, \ldots, b_m$

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Example (Upper Bound)

$\neg x_3 \vee \neg x_4$ $\quad$ $\neg x_3 \vee x_4$ $\quad$ x_7

x_3 $\quad$ $\neg x_7 \vee x_8$ $\quad$ $\neg x_1 \vee x_8$ $\quad$ $x_4 \vee x_5$ $\quad$ $x_1 \vee \neg x_5 \vee x_6$

$x_5 \vee \neg x_6$

x_1 $\quad$ $\neg x_1 \vee \neg x_3$ $\quad$ $\neg x_7 \vee \neg x_8 \vee x_6$ $\quad$ $\neg x_9 \vee x_2$ $\quad$ $\neg x_4 \vee x_5$ $\quad$ $\neg x_1 \vee \neg x_5$

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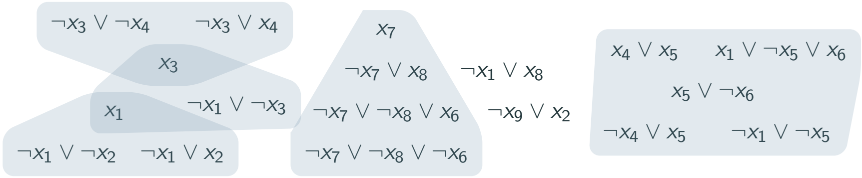

unsatisfiable cores

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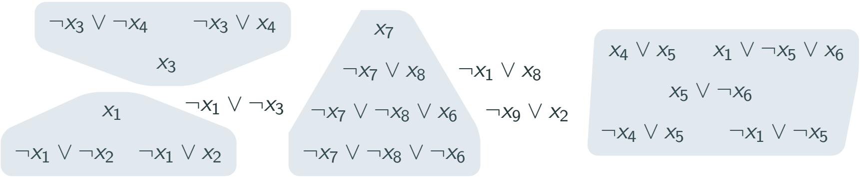

4 disjoint unsatisfiable cores

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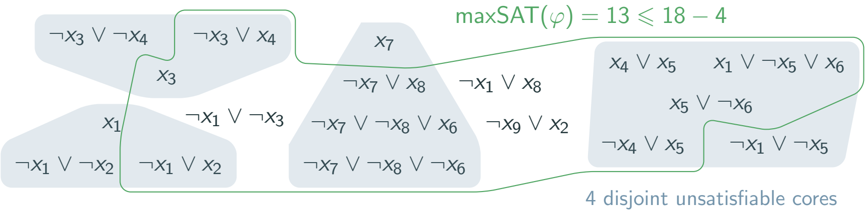

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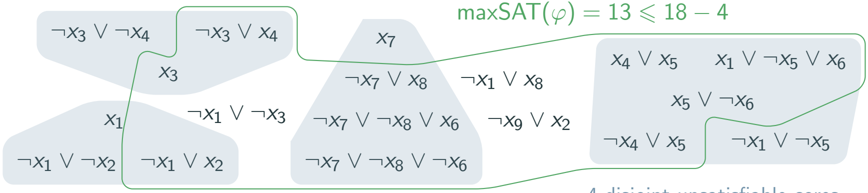
Example (Upper Bound)

must miss at least one clause from every core!

$\mathsf{maxSAT}(\varphi) = 13 \leqslant 18 - 4$



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Lemma

$$|\varphi| = |\text{pminUNSAT}(\chi, \varphi)| + |\text{pmaxSAT}(\chi, \varphi)|$$

Algorithm FuMalik(χ, φ)

Input: soft clauses φ and satisfiable hard clauses χ
Output: pminUNSAT(χ, φ)

```
cost ← 0
while ¬SAT(χ ∪ φ) do
    UC ← unsatCore(χ ∪ φ)
    B ← ∅
    for C ∈ UC ∩ φ do                          ▷ loop over soft clauses in core
        b ← fresh "blocking" variable
        φ ← φ \ {C} ∪ {C ∨ b}
        B ← B ∪ {b}
    χ ← χ ∪ CNF(∑_{b∈B} b = 1)                 ▷ cardinality constraint is hard
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```

Theorem

FuMalik(χ, φ) = pminUNSAT(χ, φ)

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2$	$\neg x_1 \vee x_2$	$\neg x_1 \vee x_7$	x_1
	$\neg x_3 \vee x_4$	x_3	$\neg x_3 \vee \neg x_4$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	x_7
	$\neg x_7 \vee x_8$	$\neg x_7 \vee \neg x_8 \vee x_6$	$\neg x_7 \vee \neg x_8 \vee \neg x_6$	$\neg x_1 \vee \neg x_3$

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2$	$\neg x_1 \vee x_2$	$\neg x_1 \vee x_7$	x_1
	$\neg x_3 \vee x_4$	x_3	$\neg x_3 \vee \neg x_4$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	x_7
	$\neg x_7 \vee x_8$	$\neg x_7 \vee \neg x_8 \vee x_6$	$\neg x_7 \vee \neg x_8 \vee \neg x_6$	$\neg x_1 \vee \neg x_3$

- unsatisfiable core: $\neg x_1 \vee \neg x_2$, $\neg x_1 \vee x_2$, x_1

Example

χ:	$\neg x_1 \lor x_3$	$\neg x_7 \lor x_2$	$x_7 \lor x_2$	$x_1 \lor \neg x_2$
φ:	$\neg x_1 \lor \neg x_2 \lor b_1$	$\neg x_1 \lor x_2 \lor b_2$	$\neg x_1 \lor x_7$	$x_1 \lor b_3$
	$\neg x_3 \lor x_4$	x_3	$\neg x_3 \lor \neg x_4$	$x_4 \lor x_5$
	$\neg x_4 \lor x_5$	$x_1 \lor \neg x_5 \lor x_6$	$x_5 \lor \neg x_6$	x_7
	$\neg x_7 \lor x_8$	$\neg x_7 \lor \neg x_8 \lor x_6$	$\neg x_7 \lor \neg x_8 \lor \neg x_6$	$\neg x_1 \lor \neg x_3$

- unsatisfiable core: $\neg x_1 \lor \neg x_2,\ \neg x_1 \lor x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2 \vee b_1$	$\neg x_1 \vee x_2 \vee b_2$	$\neg x_1 \vee x_7$	$x_1 \vee b_3$
	$\neg x_3 \vee x_4$	x_3	$\neg x_3 \vee \neg x_4$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	x_7
	$\neg x_7 \vee x_8$	$\neg x_7 \vee \neg x_8 \vee x_6$	$\neg x_7 \vee \neg x_8 \vee \neg x_6$	$\neg x_1 \vee \neg x_3$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2 \vee b_1$	$\neg x_1 \vee x_2 \vee b_2$	$\neg x_1 \vee x_7$	$x_1 \vee b_3$
	$\neg x_3 \vee x_4 \vee c_1$	$x_3 \vee c_2$	$\neg x_3 \vee \neg x_4 \vee c_3$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	x_7
	$\neg x_7 \vee x_8$	$\neg x_7 \vee \neg x_8 \vee x_6$	$\neg x_7 \vee \neg x_8 \vee \neg x_6$	$\neg x_1 \vee \neg x_3$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \mathrm{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2 \vee b_1$	$\neg x_1 \vee x_2 \vee b_2$	$\neg x_1 \vee x_7$	$x_1 \vee b_3$
	$\neg x_3 \vee x_4 \vee c_1$	$x_3 \vee c_2$	$\neg x_3 \vee \neg x_4 \vee c_3$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	x_7
	$\neg x_7 \vee x_8$	$\neg x_7 \vee \neg x_8 \vee x_6$	$\neg x_7 \vee \neg x_8 \vee \neg x_6$	$\neg x_1 \vee \neg x_3$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2 \vee b_1$	$\neg x_1 \vee x_2 \vee b_2$	$\neg x_1 \vee x_7$	$x_1 \vee b_3$
	$\neg x_3 \vee x_4 \vee c_1$	$x_3 \vee c_2$	$\neg x_3 \vee \neg x_4 \vee c_3$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	$x_7 \vee d_1$
	$\neg x_7 \vee x_8 \vee d_2$	$\neg x_7 \vee \neg x_8 \vee x_6 \vee d_3$	$\neg x_7 \vee \neg x_8 \vee \neg x_6 \vee d_4$	$\neg x_1 \vee \neg x_3$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$
  $\chi = \chi \cup \text{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$

Example

χ:	$\neg x_1 \lor x_3$	$\neg x_7 \lor x_2$	$x_7 \lor x_2$	$x_1 \lor \neg x_2$
φ:	$\neg x_1 \lor \neg x_2 \lor b_1$	$\neg x_1 \lor x_2 \lor b_2$	$\neg x_1 \lor x_7$	$x_1 \lor b_3$
	$\neg x_3 \lor x_4 \lor c_1$	$x_3 \lor c_2$	$\neg x_3 \lor \neg x_4 \lor c_3$	$x_4 \lor x_5$
	$\neg x_4 \lor x_5$	$x_1 \lor \neg x_5 \lor x_6$	$x_5 \lor \neg x_6$	$x_7 \lor d_1$
	$\neg x_7 \lor x_8 \lor d_2$	$\neg x_7 \lor \neg x_8 \lor x_6 \lor d_3$	$\neg x_7 \lor \neg x_8 \lor \neg x_6 \lor d_4$	$\neg x_1 \lor \neg x_3$

- unsatisfiable core: $\neg x_1 \lor \neg x_2,\ \neg x_1 \lor x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \lor x_4,\ x_3, \neg x_3 \lor \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \lor x_8,\ \neg x_7 \lor \neg x_8 \lor x_6,\ \neg x_7 \lor \neg x_8 \lor \neg x_6$
  $\chi = \chi \cup \text{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \lor x_3,\ \neg x_7 \lor x_2,\ x_7 \lor x_2,\ x_1 \lor \neg x_2,\ \neg x_1 \lor \neg x_3$

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2 \vee b_1$	$\neg x_1 \vee x_2 \vee b_2$	$\neg x_1 \vee x_7$	$x_1 \vee b_3$
	$\neg x_3 \vee x_4 \vee c_1$	$x_3 \vee c_2$	$\neg x_3 \vee \neg x_4 \vee c_3$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	$x_7 \vee d_1$
	$\neg x_7 \vee x_8 \vee d_2$	$\neg x_7 \vee \neg x_8 \vee x_6 \vee d_3$	$\neg x_7 \vee \neg x_8 \vee \neg x_6 \vee d_4$	$\neg x_1 \vee \neg x_3 \vee e_1$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \mathrm{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$
  $\chi = \chi \cup \mathrm{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \vee x_3,\ \neg x_7 \vee x_2,\ x_7 \vee x_2,\ x_1 \vee \neg x_2,\ \neg x_1 \vee \neg x_3$
  $\chi = \chi \cup \mathrm{CNF}(e_1 = 1)$
  $cost = 4$

Example

χ:	$\neg x_1 \lor x_3$	$\neg x_7 \lor x_2$	$x_7 \lor x_2$	$x_1 \lor \neg x_2$
φ:	$\neg x_1 \lor \neg x_2 \lor b_1$	$\neg x_1 \lor x_2 \lor b_2$	$\neg x_1 \lor x_7$	$x_1 \lor b_3$
	$\neg x_3 \lor x_4 \lor c_1$	$x_3 \lor c_2$	$\neg x_3 \lor \neg x_4 \lor c_3$	$x_4 \lor x_5$
	$\neg x_4 \lor x_5$	$x_1 \lor \neg x_5 \lor x_6$	$x_5 \lor \neg x_6$	$x_7 \lor d_1$
	$\neg x_7 \lor x_8 \lor d_2$	$\neg x_7 \lor \neg x_8 \lor x_6 \lor d_3$	$\neg x_7 \lor \neg x_8 \lor \neg x_6 \lor d_4$	$\neg x_1 \lor \neg x_3 \lor e_1$

- unsatisfiable core: $\neg x_1 \lor \neg x_2,\ \neg x_1 \lor x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \lor x_4,\ x_3, \neg x_3 \lor \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \lor x_8,\ \neg x_7 \lor \neg x_8 \lor x_6,\ \neg x_7 \lor \neg x_8 \lor \neg x_6$
  $\chi = \chi \cup \text{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \lor x_3,\ \neg x_7 \lor x_2,\ x_7 \lor x_2,\ x_1 \lor \neg x_2,\ \neg x_1 \lor \neg x_3$
  $\chi = \chi \cup \text{CNF}(e_1 = 1)$
  $cost = 4$
- satisfiable

Example

χ:	$\neg x_1 \lor x_3$	$\neg x_7 \lor x_2$	$x_7 \lor x_2$	$x_1 \lor \neg x_2$
φ:	$\neg x_1 \lor \neg x_2 \lor b_1$	$\neg x_1 \lor x_2 \lor b_2$	$\neg x_1 \lor x_7$	$x_1 \lor b_3$
	$\neg x_3 \lor x_4 \lor c_1$	$x_3 \lor c_2$	$\neg x_3 \lor \neg x_4 \lor c_3$	$x_4 \lor x_5$
	$\neg x_4 \lor x_5$	$x_1 \lor \neg x_5 \lor x_6$	$x_5 \lor \neg x_6$	$x_7 \lor d_1$
	$\neg x_7 \lor x_8 \lor d_2$	$\neg x_7 \lor \neg x_8 \lor x_6 \lor d_3$	$\neg x_7 \lor \neg x_8 \lor \neg x_6 \lor d_4$	$\neg x_1 \lor \neg x_3 \lor e_1$

- unsatisfiable core: $\neg x_1 \lor \neg x_2,\ \neg x_1 \lor x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \lor x_4,\ x_3, \neg x_3 \lor \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \lor x_8,\ \neg x_7 \lor \neg x_8 \lor x_6,\ \neg x_7 \lor \neg x_8 \lor \neg x_6$
  $\chi = \chi \cup \text{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \lor x_3,\ \neg x_7 \lor x_2,\ x_7 \lor x_2,\ x_1 \lor \neg x_2,\ \neg x_1 \lor \neg x_3$
  $\chi = \chi \cup \text{CNF}(e_1 = 1)$
  $cost = 4$
- satisfiable
- $\text{pminUNSAT}(\chi, \varphi) = 4$

Example

χ:	$\neg x_1 \vee x_3$	$\neg x_7 \vee x_2$	$x_7 \vee x_2$	$x_1 \vee \neg x_2$
φ:	$\neg x_1 \vee \neg x_2 \vee b_1$	$\neg x_1 \vee x_2 \vee b_2$	$\neg x_1 \vee x_7$	$x_1 \vee b_3$
	$\neg x_3 \vee x_4 \vee c_1$	$x_3 \vee c_2$	$\neg x_3 \vee \neg x_4 \vee c_3$	$x_4 \vee x_5$
	$\neg x_4 \vee x_5$	$x_1 \vee \neg x_5 \vee x_6$	$x_5 \vee \neg x_6$	$x_7 \vee d_1$
	$\neg x_7 \vee x_8 \vee d_2$	$\neg x_7 \vee \neg x_8 \vee x_6 \vee d_3$	$\neg x_7 \vee \neg x_8 \vee \neg x_6 \vee d_4$	$\neg x_1 \vee \neg x_3 \vee e_1$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$
  $\chi = \chi \cup \text{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \vee x_3,\ \neg x_7 \vee x_2,\ x_7 \vee x_2,\ x_1 \vee \neg x_2,\ \neg x_1 \vee \neg x_3$
  $\chi = \chi \cup \text{CNF}(e_1 = 1)$
  $cost = 4$
- satisfiable: $v(x_1) = v(x_2) = v(x_3) = v(x_5) = v(x_7) = \text{T}$ and $v(x_i) = \text{F}$ otherwise
- $\text{pminUNSAT}(\chi, \varphi) = 4$ and $\text{pmaxSAT}(\chi, \varphi) = 12$

Unsatisfiable Cores in z3

```
from z3 import *

x1,x2,x3 = Bool("x1"), Bool("x2"), Bool("x3")
phi = [ Or(Not(x1), Not(x2)), Or(Not(x1), x2),\
  Or(Not(x1), x3), x1, Or(Not(x3), x2)]

solver = Solver()
solver.set(unsat_core=True)

# assert clauses in phi with names phi0 ... phi4
for i,c in enumerate(phi):
  solver.assert_and_track(c, "phi" + str(i))

if solver.check() == z3.unsat:
  uc = solver.unsat_core()
  print(uc) # [phi0, phi1, phi3]
```

Literature

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