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SAT and SMT Solving

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lecture 5
SS 2019

Outline

- Summary of Last Week
- Satisfiability Modulo Theories
- DPLL(T)
- Equality and Uninterpreted Functions in Practice

Definitions

for unsatisfiable CNF formula φ given as set of clauses

- $\psi \subseteq \varphi$ such that $\bigwedge_{C \in \psi} C$ is unsatisfiable is unsatisfiable core (UC) of φ
- minimal unsatisfiable core ψ is UC such that every subset of ψ is satisfiable
- SUC (minimum unsatisfiable core) is UC such that $|\psi|$ is minimal

Remark

SUC is always minimal unsatisfiable core

Definition (Resolution Graph)

directed acyclic graph $G = (V, E)$ is resolution graph for set of clauses φ

1. $V = V_i \uplus V_c$ is set of clauses and $V_i = \varphi$,
2. V_i nodes have no incoming edges,
3. there is exactly one node $\square$ without outgoing edges,
4. $\forall C \in V_c\ \exists$ edges $D \to C$, $D' \to C$ such that C is resolvent of D and D', and
5. there are no other edges.

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Definition (Resolution Graph)

directed acyclic graph $G = (V, E)$ is resolution graph for set of clauses φ if

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4. $\forall C \in V_c$ $\exists$ edges $D \to C$, $D' \to C$ such that C is resolvent of D and D', and
5. there are no other edges.

Algorithm minUnsatCore(φ)

Input: unsatisfiable formula φ
Output: minimal unsatisfiable core of φ

```
build resolution graph G = (V_i ⊎ V_c, E) for φ
while ∃ unmarked clause in V_i do
    C ← unmarked clause in V_i
    if SAT(‾Reach_G(C)‾) then                    ▷ subgraph without C satisfiable?
        mark C                                    ▷ C is UC member
    else
        build resolution graph G' = (V_i' ⊎ V_c', E') for ‾Reach_G(C)‾
        V_i ← V_i \ {C} and V_c ← V_c' ∪ (V_c \ Reach_G(C))
        E ← E' ∪ (E \ Reach_G^E(C))
        G ← (V_i ∪ V_c, E)
        G ← G|_{BReach_G(□)}                      ▷ restrict to nodes with path to □
return V_i
```

Theorem

if φ unsatisfiable then minUnsatCore(φ) is minimal unsatisfiable core of φ

Definition (Partial minUNSAT)

pminUNSAT(χ, φ) is minimal $|\psi|$ such that $\psi \subseteq \varphi$ and $\chi \wedge \bigwedge_{C \in \psi} \neg C$ satisfiable

Lemma

$$|\varphi| = |\text{pminUNSAT}(\chi, \varphi)| + |\text{pmaxSAT}(\chi, \varphi)|$$

Theorem

FuMalik(χ, φ) = pminUNSAT(χ, φ)

Algorithm FuMalik(χ, φ)

Input: clause set φ and satisfiable clause set χ
Output: minUNSAT(χ, φ)

```
cost ← 0
while ¬SAT(χ ∪ φ) do
    UC ← unsatCore(χ ∪ φ)
    B ← ∅
    for C ∈ UC ∩ φ do                          ▷ loop over soft clauses in core
        b ← new blocking variable
        φ ← φ \ {C} ∪ {C ∨ b}
        B ← B ∪ {b}
    χ ← χ ∪ CNF(∑_{b∈B} b = 1)                 ▷ cardinality constraint is hard
    cost ← cost + 1
return cost
```

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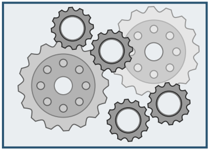
SMT Solving

input:	formula φ involving theory T	
output:	SAT + valuation v such that $v(\varphi) = T$	if φ satisfiable
	UNSAT	otherwise

$\varphi \longrightarrow$ 

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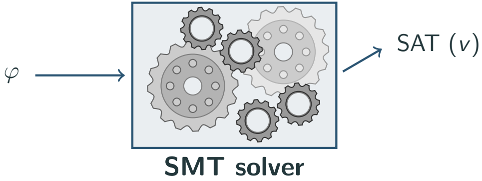

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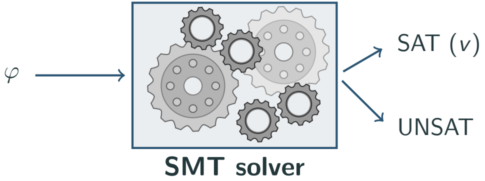

Example (Theories)

- arithmetic $\quad 2a + b \geqslant c \vee (a - b = c + 3 \wedge p)$

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φ

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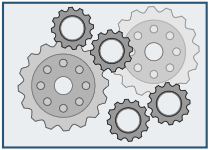

SMT solver

SAT (v) $\quad v(a) = 3 \quad v(b) = 0 \quad v(c) = 0 \quad v(p) = \mathrm{T}$

UNSAT

Example (Theories)

- arithmetic $\quad 2a + b \geqslant c \vee (a - b = c + 3 \wedge p)$
- uninterpreted functions $\quad \mathrm{f}(x, y) \neq \mathrm{f}(y, x) \wedge \mathrm{g}(\mathrm{a}) \to \mathrm{g}(\mathrm{f}(x, x)) = \mathrm{g}(y)$

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φ → SMT solver

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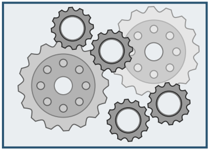

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→ SAT (v) $v(a) = 3$ $v(b) = 0$ $v(c) = 0$ $v(p) = \mathrm{T}$

→ UNSAT

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- uninterpreted functions $\mathrm{f}(x, y) \neq \mathrm{f}(y, x) \wedge \mathrm{g}(\mathrm{a}) \rightarrow \mathrm{g}(\mathrm{f}(x, x)) = \mathrm{g}(y)$
- bit vectors $((\mathrm{zext}_{32}\ a_8) + b_{32}) \times c_{32} >_u 0_{32}$

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for formulas F and G and list of literals M:

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Definition (Theory of Equality)

theory of equality (EQ) uses binary predicate $\approx$ and consists of axioms

$$\forall x.\, (x \approx x) \quad \forall x\, y.\, (x \approx y \;\rightarrow\; y \approx x) \quad \forall x\, y\, z.\, (x \approx y \wedge y \approx z \;\rightarrow\; x \approx z)$$

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Example

- $u \approx v \wedge \neg(v \approx w)$ is EQ-consistent

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- $u \approx v \wedge \neg(v \approx w)$ is EQ-consistent
- $u \approx v \wedge \neg(v \approx w) \wedge (w \approx u \vee u \approx w)$ is EQ-inconsistent

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EUF over set of function symbols $\mathcal{F}$ consists of equality axioms:

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function symbol f takes $n > 0$ arguments

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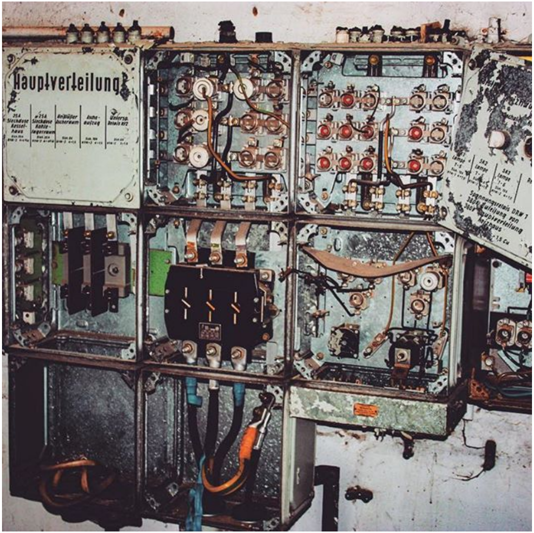
$$\forall x_1 y_1\ \dots\ x_n y_n\ (x_1 \approx y_1 \wedge \cdots \wedge x_n \approx y_n \ \rightarrow\ f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n))$$

Uninterpreted Functions in Real Life



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EUF over set of function symbols $\mathcal{F}$ consists of equality axioms:

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EUF over $\mathcal{F} = \{\mathsf{a}/0,\ \mathsf{b}/0,\ \mathsf{f}/1,\ \mathsf{add}/2\}$ consists of axioms

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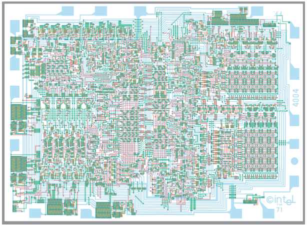
Application: Verification of Microprocessors

- verify that 3-stage pipelined MIPS processor satisfies intended instruction set architecture



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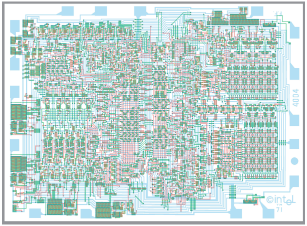

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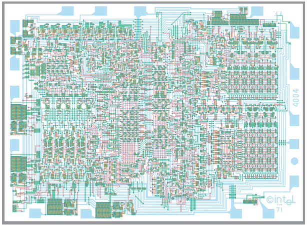

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consider formula φ mixing propositional logic with theory T

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- drawbacks:
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  - even more complicated with multiple theories

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     - otherwise T-solver generates T-consequence C of φ excluding v, repeat from 1 with $\varphi \wedge C$

Example

$$g(a) \approx c \wedge (\neg(f(g(a)) \approx f(c)) \vee g(a) \approx d) \wedge \neg(c \approx d)$$

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- T-solver gets $g(a) \approx c \wedge f(g(a)) \not\approx f(c) \wedge c \not\approx d$
  T-unsatisfiable: $g(a) \approx c$ implies $f(g(a)) \approx f(c)$
- block valuation v_1 in future: add $\neg x_1 \vee x_2 \vee x_4$

- $\psi_2 = x_1 \wedge (\neg x_2 \vee x_3) \wedge \neg x_4 \wedge (\neg x_1 \vee x_2 \vee x_4)$
  satisfiable: $v_2(x_1) = v_2(x_2) = v_2(x_3) = \mathrm{T}$ and $v_2(x_4) = \mathrm{F}$
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  T-unsatisfiable

Example

$$\underbrace{\mathsf{g(a) \approx c}}_{x_1} \land (\neg(\underbrace{\mathsf{f(g(a)) \approx f(c)}}_{x_2}) \lor \underbrace{\mathsf{g(a) \approx d}}_{x_3}) \land \neg(\underbrace{\mathsf{c \approx d}}_{x_4})$$

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$$\underbrace{\mathsf{g(a)} \approx \mathsf{c}}_{x_1} \wedge (\neg\underbrace{(\mathsf{f(g(a))} \approx \mathsf{f(c)})}_{x_2} \vee \underbrace{\mathsf{g(a)} \approx \mathsf{d}}_{x_3}) \wedge \neg\underbrace{(\mathsf{c} \approx \mathsf{d})}_{x_4}$$

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- block valuation v_2 in future: add $\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee x_4$

- $\psi_3 = x_1 \wedge (\neg x_2 \vee x_3) \wedge \neg x_4 \wedge (\neg x_1 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee x_4)$

Example

$$\underbrace{g(a) \approx c}_{x_1} \land (\neg\underbrace{(f(g(a)) \approx f(c))}_{x_2} \lor \underbrace{g(a) \approx d}_{x_3}) \land \neg\underbrace{(c \approx d)}_{x_4}$$

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- unsatisfiable

Outline

- Summary of Last Week
- Satisfiability Modulo Theories
- DPLL(T)
- Equality and Uninterpreted Functions in Practice

Approach

- most state-of-the-art SMT solvers use DPLL(T):
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- T-backjump $\quad M\ l^d\ N \parallel F, C \quad \Longrightarrow \quad M\ l' \parallel F, C$
  if $M\ l^d\ N \vDash \neg C$ and $\exists$ clause $C' \vee l'$ such that
  - $F, C \vDash_T C' \vee l'$
  - $M \vDash \neg C'$ and l' is undefined in M, and l' or l'^c occurs in F or in $M\ l^d\ N$

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  - $F, C \models_T C' \vee l'$
  - $M \models \neg C'$ and l' is undefined in M, and l' or l'^c occurs in F or in $M\, l^d\, N$
- T-learn $\qquad M \parallel F \quad \Longrightarrow \quad M \parallel F,\ C$
  if $F \models_T C$ and all atoms of C occur in M or F
- T-forget $\qquad M \parallel F,\ C \quad \Longrightarrow \quad M \parallel F$
  if $F \models_T C$

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- T-forget $\quad M \parallel F,\ C \implies M \parallel F$
  if $F \vDash_T C$
- T-propagate $\quad M \parallel F \implies M\ l \parallel F$
  if $M \vDash_T l$, literal l or l^c occurs in F, and l is undefined in M

Naive Lazy Approach in DPLL(T)

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Improvement 1: Incremental T-Solver

- T-solver checks T-consistency of model M whenever literal is added to M

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Remark

all three improvements can be combined

Example (Revisited with DPLL(T))

$$\underbrace{\mathsf{g}(\mathsf{a}) \approx \mathsf{c}}_{1} \wedge (\neg\underbrace{(\mathsf{f}(\mathsf{g}(\mathsf{a})) \approx \mathsf{f}(\mathsf{c}))}_{2} \vee \underbrace{\mathsf{g}(\mathsf{a}) \approx \mathsf{d}}_{3}) \wedge \neg\underbrace{(\mathsf{c} \approx \mathsf{d})}_{4}$$

$\| \; 1,\ (\overline{2} \vee 3),\ \overline{4}$

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$$\| \; 1,\ (\overline{2} \vee 3),\ \overline{4}$$

$$\Longrightarrow \quad 1 \; \| \; 1,\ (\overline{2} \vee 3),\ \overline{4} \qquad \text{unit propagate}$$

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	$\parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	
$\Longrightarrow$	$1 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	unit propagate
$\Longrightarrow$	$1\ \overline{4} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	unit propagate

Example (Revisited with DPLL(T))

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	$\parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$		
$\Longrightarrow$	$1 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$		unit propagate
$\Longrightarrow$	$1\ \overline{4} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$		unit propagate
$\Longrightarrow$	$1\ \overline{4}\ \overline{2}^{d} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$		decide

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$\Longrightarrow$	$1 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	unit propagate
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$\Longrightarrow$	$1\ \overline{4}\ \overline{2}^{d} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	decide
$\Longrightarrow$	$1\ \overline{4}\ \overline{2}^{d} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2 \vee 4)$	T-learn

Example (Revisited with DPLL(T))

$$\underbrace{\mathsf{g(a) \approx c}}_{1} \wedge (\neg(\underbrace{\mathsf{f(g(a)) \approx f(c)}}_{2}) \vee \underbrace{\mathsf{g(a) \approx d}}_{3}) \wedge \neg(\underbrace{\mathsf{c \approx d}}_{4})$$

$$
\begin{array}{rrl}
 & \parallel 1,\ (\overline{2} \vee 3),\ \overline{4} & \\
\Longrightarrow & 1 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4} & \text{unit propagate} \\
\Longrightarrow & 1\ \overline{4} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4} & \text{unit propagate} \\
\Longrightarrow & 1\ \overline{4}\ \overline{2}^{d} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4} & \text{decide} \\
\Longrightarrow & 1\ \overline{4}\ \overline{2}^{d} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2 \vee 4) & T\text{-learn} \\
\Longrightarrow & 1\ \overline{4}\ 2 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2 \vee 4) & T\text{-backjump}
\end{array}
$$

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$\Longrightarrow$	$1\, \bar{4}\, \bar{2}^d \parallel 1,\ (\bar{2} \vee 3),\ \bar{4},\ (\bar{1} \vee 2 \vee 4)$	T-learn
$\Longrightarrow$	$1\, \bar{4}\, 2 \parallel 1,\ (\bar{2} \vee 3),\ \bar{4},\ (\bar{1} \vee 2 \vee 4)$	T-backjump
$\Longrightarrow$	$1\, \bar{4}\, 2\, 3 \parallel 1,\ (\bar{2} \vee 3),\ \bar{4},\ (\bar{1} \vee 2 \vee 4)$	unit propagate

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$\Longrightarrow$	$1\,\overline{4}\,\overline{2}^{d} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2 \vee 4)$	T-learn
$\Longrightarrow$	$1\,\overline{4}\,2 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2 \vee 4)$	T-backjump
$\Longrightarrow$	$1\,\overline{4}\,2\,3 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2 \vee 4)$	unit propagate
$\Longrightarrow$	$1\,\overline{4}\,2\,3 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2 \vee 4),\ (\overline{1} \vee \overline{2} \vee \overline{3} \vee 4)$	T-learn
$\Longrightarrow$	FailState	fail

Lazyness in DPLL(T)

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T-solver SAT solver

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Example (SMT-LIB 2 for Propositional Logic)

formula $(x_1 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee \neg x_1) \wedge (\neg x_1 \vee x_2 \vee x_3)$ can be expressed by

```
(declare-const x1 Bool)
(declare-const x2 Bool)
(declare-const x3 Bool)
(assert (or x1 (not x3)))
(assert (or x2 x3 (not x1)))
(assert (or (not x1) x2 x3))
(check-sat)
(get-model)
```

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Propositional Logic in SMT-LIB 2

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Propositional Logic in SMT-LIB 2

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- `assert` demands given formula to be satisfied
- `check-sat` issues satisfiability check of conjunction of assertions

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(assert (or (not x1) x2 x3))
(check-sat)
(get-model)
```

Propositional Logic in SMT-LIB 2

- `declare-const` x `Bool` creates propositional variable named x
- prefix notation for `and`, `or`, `not`, `implies`
- `assert` demands given formula to be satisfied
- `check-sat` issues satisfiability check of conjunction of assertions
- `get-model` prints model (after satisfiability check)

Example (SMT-LIB 2 for EUF)

$f(f(a)) \approx a \wedge f(a) \approx b \wedge \neg(a \approx b)$ is expressed as

```
(declare-sort A)
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(declare-const b A)
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(assert (= (f (f a)) a))
(assert (= (f a) b))
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- `(distinct` x y`)` is equivalent to `not(=` x y`)`

Example

$2x \geqslant y + z \wedge \neg(x \approx y)$ is expressed as

```
(declare-const x Int)
(declare-const y Int)
(declare-const z Int)
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- `=` also covers equality on $\mathbb{Z}$
- `<`, `<=`, `>`, `>=` are $<_{\mathbb{Z}}$, $\leqslant_{\mathbb{Z}}$, $>_{\mathbb{Z}}$, $\geqslant_{\mathbb{Z}}$

EUF in python/z3

```
A = DeclareSort('A') # new uninterpreted sort named 'A'
a = Const('a', A) # create constant of sort A
b = Const('b', A) # create another constant of sort A
f = Function('f', A, A) # create function of sort A -> A

s = Solver()
s.add(f(f(a)) == a, f(a) == b, a != b)

print s.check() # sat
m = s.model()
print "interpretation assigned to A:"
print m[A] # [A!val!0, A!val!1]
print "interpretations:"
print m[f] # [A!val!0 -> A!val!1, A!val!1 -> A!val!0, ...]
print m[a] # A!val!0
print m[b] # A!val!1
```

Example (Quantifiers and Monkeys)

In a village of monkeys every monkey owns at least two bananas: 

```
(declare-sort monkey)
(declare-sort banana)
(declare-fun owns (monkey banana) Bool)
(declare-fun b1 (monkey) banana)
(declare-fun b2 (monkey) banana)

(assert (forall ((M monkey)) (not (= (b1 M) (b2 M)))))
(assert (forall ((M monkey)) (owns M (b1 M))))
(assert (forall ((M monkey)) (owns M (b2 M))))
(assert (forall ((M1 monkey) (M2 monkey) (B banana))
  (implies (and (owns M1 B) (owns M2 B)) (= M1 M2))))
```

DPLL(T)

Robert Nieuwenhuis, Albert Oliveras, and Cesare Tinelli.
Solving SAT and SAT Modulo Theories: From an Abstract Davis-Putnam-Logemann-Loveland Procedure to DPLL(T).
Journal of the ACM 53(6), pp. 937–977, 2006.

Application

Miroslav N. Velev and Randal E. Bryant.
Bit-level abstraction in the verification of pipelined microprocessors by correspondence checking.
In Proc. of Formal Methods in Computer-Aided Design, pp. 18–35, 1998.