universität innsbruck

SAT and SMT Solving

Sarah Winkler

Computational Logic Group
Department of Computer Science
University of Innsbruck

lecture 8
SS 2019

Outline

- Summary of Last Week
- Cyclic Simplex Example
- Deciding Equality Logic
- Branch and Bound

Definition (Theory of Linear Arithmetic over C)

- for variables $x_1, \dots, x_n$, formulas built according to grammar

$$\varphi ::= \varphi \wedge \varphi \mid t = t \mid t < t \mid t \leqslant t$$

$$t ::= a_1 x_1 + \cdots + a_n x_n + b \qquad \text{for } a_1, \dots, a_n, b \in \text{in carrier } C$$

- axioms are equality axioms plus calculation rules of arithmetic over C
- solution assigns values in C to $x_1, \dots, x_n$

Definitions

- carrier $\mathbb{R}$: linear real arithmetic (LRA),
  DPLL(T) simplex algorithm is decision procedure
- carrier $\mathbb{Z}$: linear integer arithmetic (LIA)

DPLL(T) Simplex Algorithm (1)

- linear arithmetic constraint solving over real or rational variables
- $x_1, \ldots, x_n$ are split into basic variables $\vec{x}_B$ and nonbasic variables $\vec{x}_N$

Input

constraints plus upper and lower bounds for $x_1, \ldots, x_n$:

$$A\,\vec{x}_N = \vec{x}_B \qquad \text{with tableau } A \in \mathbb{R}^{|B| \times |N|} \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Output

satisfying assignment or "unsatisfiable"

Invariant

(1) is satisfied and (2) holds for all nonbasic variables x_i

DPLL(T) Simplex Algorithm (2)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Method

- if (2) holds for all basic variables, return current assignment
- otherwise select basic variable x_i (so $i \in B$) which violates (2)
- select suitable nonbasic variable x_j (so $j \in N$) such that x_i and x_j can be swapped in a pivoting step, resulting in new tableau

  $$A'\, x_{N'} = x_{B'}$$

  with $N' = N \cup \{i\} - \{j\}$ and $B' = B \cup \{j\} - \{i\}$
- change value of x_i to l_i or u_i and update values of basic variables accordingly

DPLL(T) Simplex Algorithm (3)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Pivoting

- swap basic x_i and non-basic x_j

$$x_i = \sum_{k \in N} A_{ik} x_k \qquad \Longrightarrow \qquad x_j = \frac{1}{A_{ij}} \left(x_i - \sum_{k \in N - \{j\}} A_{ik} x_k \right) \tag{$\star$}$$

- new tableau A' consists of $(\star)$ and $A_{B-\{i\}}\vec{x}_N = \vec{x}_{B-\{i\}}$ with $(\star)$ substituted

Update

- assignment of x_i is updated to previously violated bound l_i or u_i,
- assignment of x_k is recomputed using $(\star)$ and A' for all $k \in B - \{i\} \cup \{j\}$

DPLL(T) Simplex Algorithm (4)

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \tag{2}$$

Suitability

- basic variable x_i violates lower and/or upper bound
- pick nonbasic variable x_j such that
  - if $x_i < l_i$: $A_{ij} > 0$ and $x_j < u_j$ or $A_{ij} < 0$ and $x_j > l_j$
  - if $x_i > u_i$: $A_{ij} > 0$ and $x_j > l_j$ or $A_{ij} < 0$ and $x_j < u_j$
- problem is unsatisfiable if no suitable pivot exists

Bland's Rule

- pick lexicographically smallest (i, j) that is suitable pivot
- guarantees termination

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Example (due to B. Felgenhauer)

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

$$\begin{array}{c} \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array}$$

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x_1	x_2	x_3	x_4
0	0	0	0

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$$\downarrow$$

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$$\longrightarrow$$

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Example (due to B. Felgenhauer)

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x_1	x_2	x_3	x_4
0	0	0	0

↓

$$\begin{array}{c|cc} & x_3 & x_2 \\ \hline x_1 & 1 & -2 \\ x_4 & 2 & -3 \end{array}$$

x_1	x_2	x_3	x_4
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x_1	x_2	x_3	x_4
3	-4	-5	2

↑

$$\begin{array}{c|cc} & x_3 & x_4 \\ \hline x_1 & -\frac{1}{3} & \frac{2}{3} \\ x_2 & \frac{2}{3} & -\frac{1}{3} \end{array}$$

x_1	x_2	x_3	x_4
$\frac{7}{3}$	$-\frac{11}{3}$	-5	1

↑

$$\begin{array}{c|cc} & x_1 & x_4 \\ \hline x_3 & -3 & 2 \\ x_2 & -2 & 1 \end{array}$$

x_1	x_2	x_3	x_4
0	1	2	1

Example (due to B. Felgenhauer)

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$$\downarrow \qquad\qquad\qquad\qquad \uparrow$$

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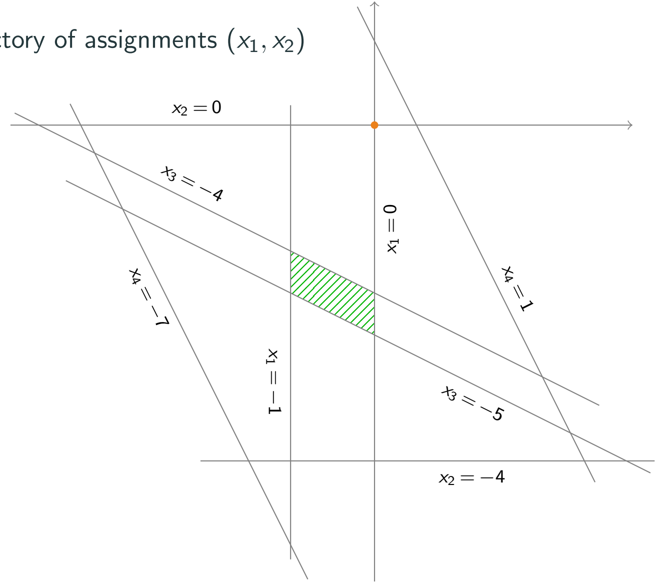
trajectory of assignments (x_1, x_2)



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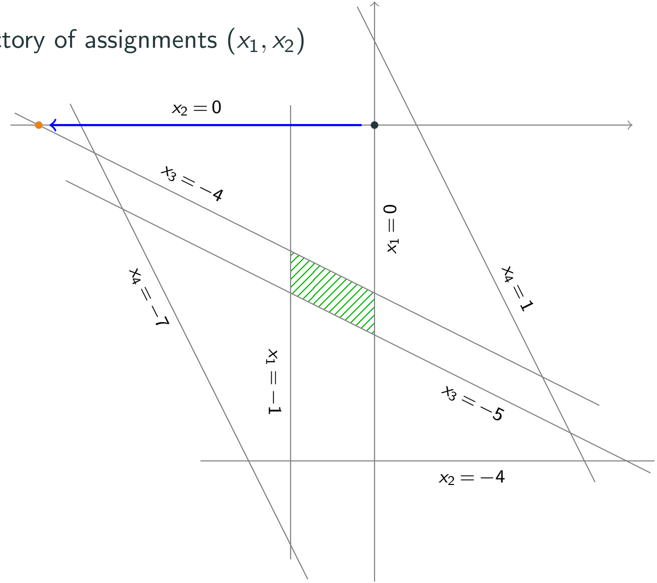

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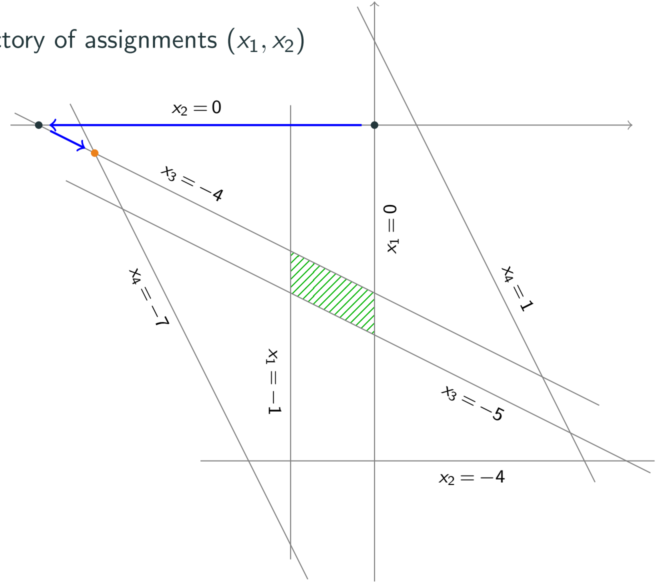
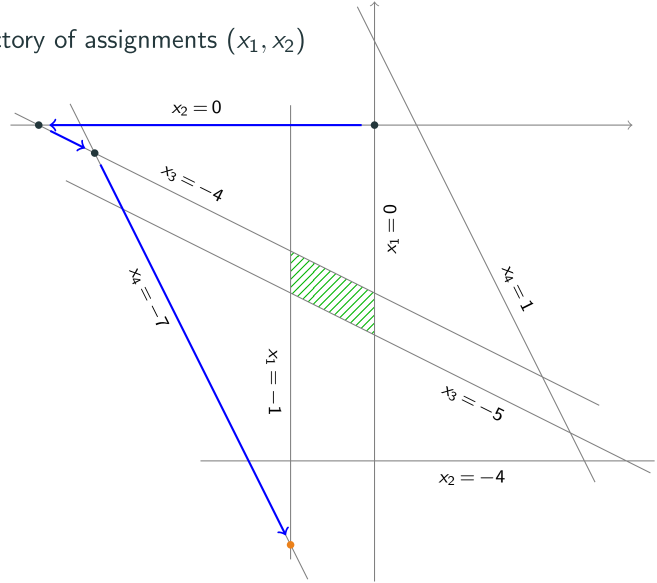


trajectory of assignments (x_1, x_2)



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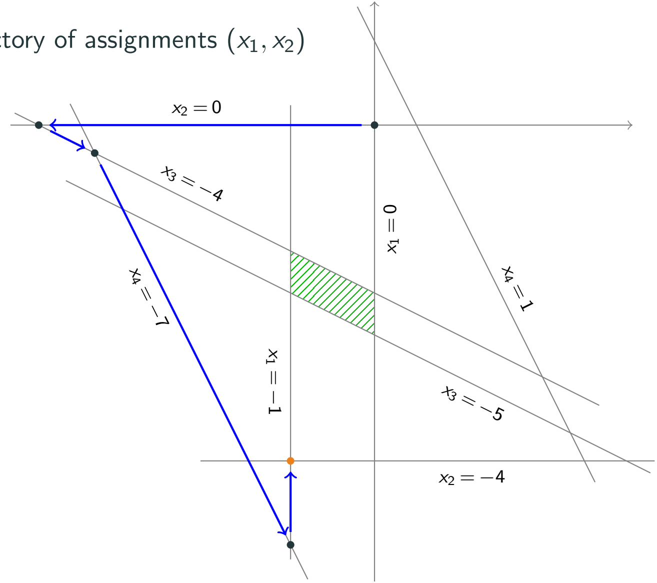

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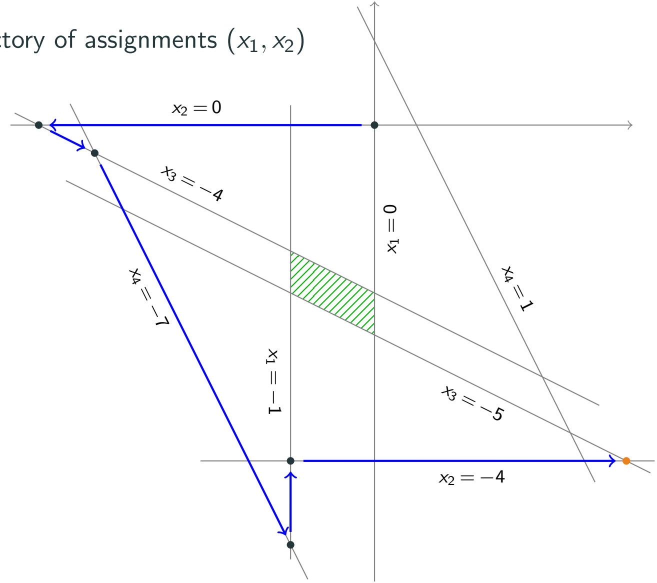

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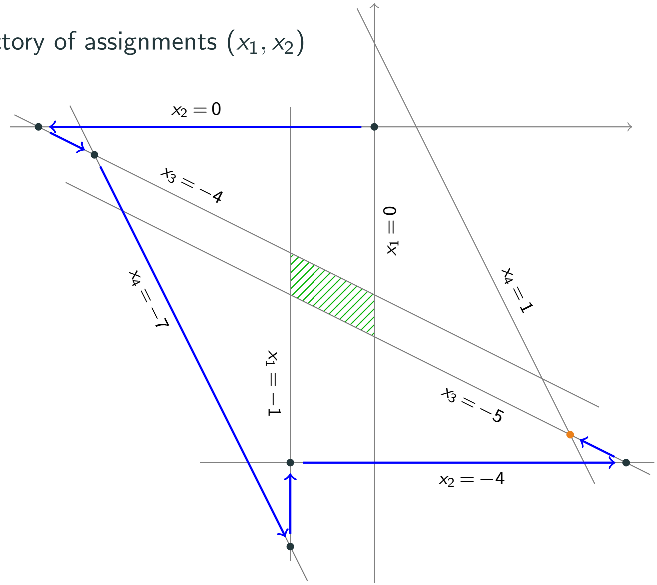

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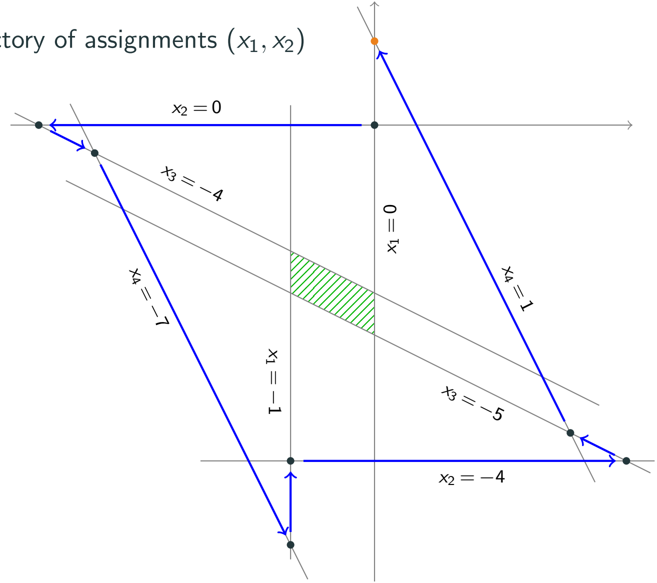

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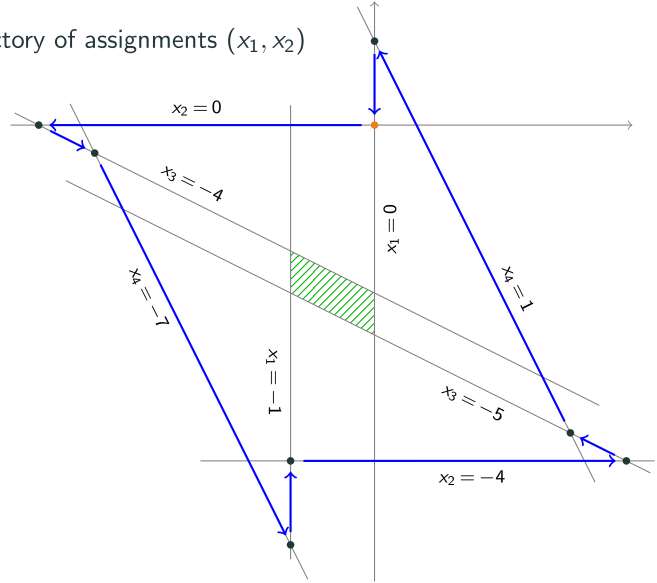

Example (due to B. Felgenhauer)

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

$$\begin{array}{c} \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{c} \\ x_1 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -4 & 0 & -4 & -8 \end{array}$$

violation of Bland's rule

$$\downarrow$$

$$\begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -\frac{10}{3} & -\frac{1}{3} & -4 & -7 \end{array}$$

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$$\downarrow$$

$$\begin{array}{c} \\ x_2 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} x_3 & x_1 \end{array} \\ \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -\frac{3}{2} & -4 & -\frac{7}{2} \end{array}$$

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x_1	x_2	x_3	x_4
0	0	0	0

↓

$$\begin{array}{c} \\ x_1 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array}$$

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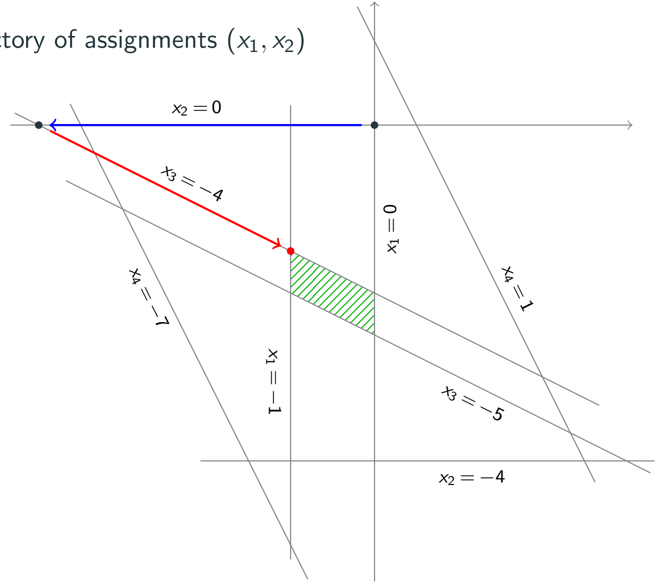
x_1	x_2	x_3	x_4
-1	$-\frac{3}{2}$	-4	$-\frac{7}{2}$

satisfying assignment

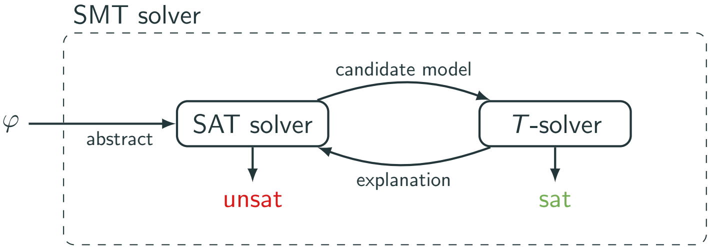
trajectory of assignments (x_1, x_2)



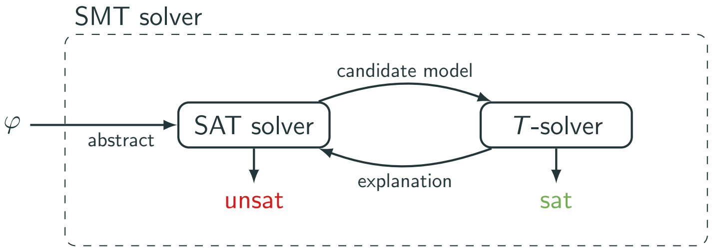
Outline

- Summary of Last Week
- Cyclic Simplex Example
- Deciding Equality Logic
- Branch and Bound

How to Be Lazy



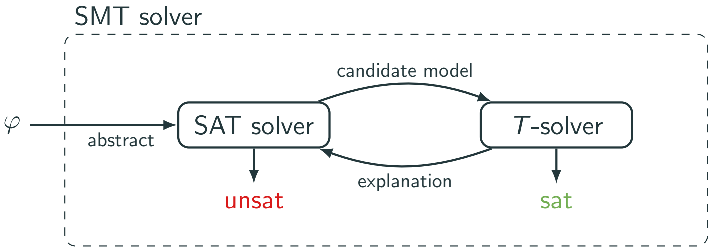
How to Be Lazy



Theory T

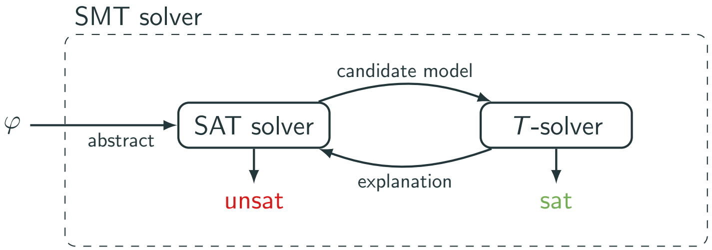
- equality logic
- equality + uninterpreted functions (EUF)
- linear real arithmetic (LRA)
- linear integer arithmetic (LIA)
- bitvectors (BV)
- arrays (A)

How to Be Lazy



Theory T	T-solving method	
▶ equality logic		
▶ equality + uninterpreted functions (EUF)	congruence closure	✓
▶ linear real arithmetic (LRA)	DPLL(T) Simplex	✓
▶ linear integer arithmetic (LIA)		
▶ bitvectors (BV)		
▶ arrays (A)		

How to Be Lazy



Theory T	T-solving method	
▶ equality logic	equality graphs	
▶ equality + uninterpreted functions (EUF)	congruence closure	✓
▶ linear real arithmetic (LRA)	DPLL(T) Simplex	✓
▶ linear integer arithmetic (LIA)	DPLL(T) Simplex + cuts	
▶ bitvectors (BV)		
▶ arrays (A)		

Input to Satisfiability Problem for Equality Logic

conjunction φ of equality logic literals over set of variables V

Definitions

- $\varphi_=$ is set of positive literals (equality literals) in φ
- $\varphi_{\neq}$ is set of negative literals (inequality literals) in φ

Input to Satisfiability Problem for Equality Logic

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Example

conjunction of equality literals

$$\varphi = x_1 = x_2 \wedge x_1 \neq x_3 \wedge x_3 = x_5 \wedge x_4 \neq x_6 \wedge x_6 \neq x_7 \wedge x_5 = x_9 \wedge$$
$$x_5 = x_7 \wedge x_8 \neq x_9 \wedge x_9 = x_{10} \wedge x_7 = x_9 \wedge x_5 \neq x_{10}$$

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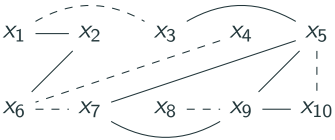
- equality graph



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Definitions

equality graph $G_=(\varphi) = (V,\ \varphi_=,\ \varphi_{\neq})$

- contradictory cycle is cycle with exactly one $\varphi_{\neq}$ edge

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- contradictory cycle is simple if it contains no node twice

Input to Satisfiability Problem for Equality Logic

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Definitions

equality graph $G_=(\varphi) = (V,\ \varphi_=,\ \varphi_{\neq})$

- contradictory cycle is cycle with exactly one $\varphi_{\neq}$ edge
- contradictory cycle is simple if it contains no node twice

Lemma

φ is satisfiable iff $G_=(\varphi)$ contains no simple contradictory cycles

Example

conjunction of equality literals

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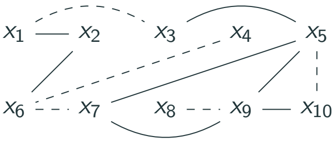

- contradictory cycles

Example

conjunction of equality literals

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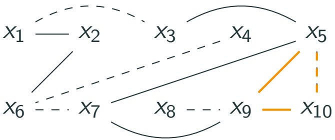

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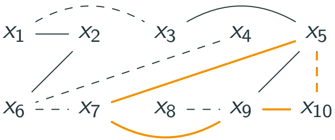

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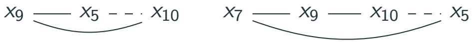

Example

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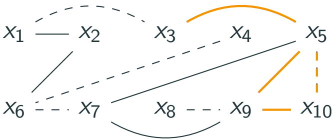

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Example

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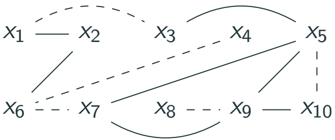
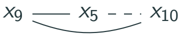
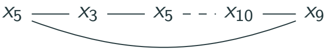


- contradictory cycles



simple

simple



not simple

Example

conjunction of equality literals

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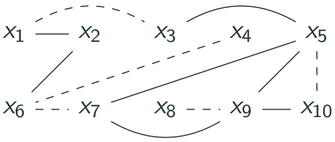
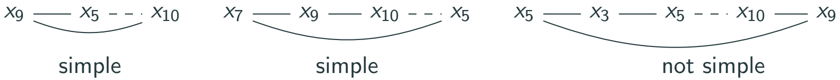


- contradictory cycles



- unsatisfiable

Outline

- Summary of Last Week
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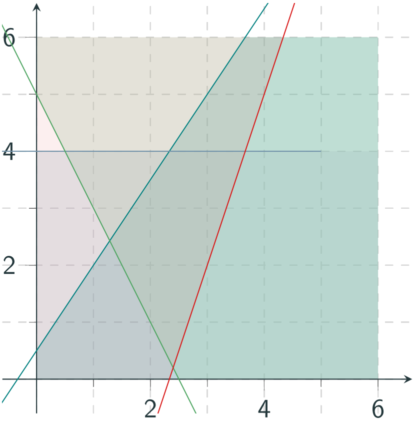
- Branch and Bound

Example

$$3x - 2y \geqslant -1$$
$$y \leqslant 4$$
$$2x + y \geqslant 5$$
$$3x - y \leqslant 7$$

- looking for solution in $\mathbb{Z}^2$



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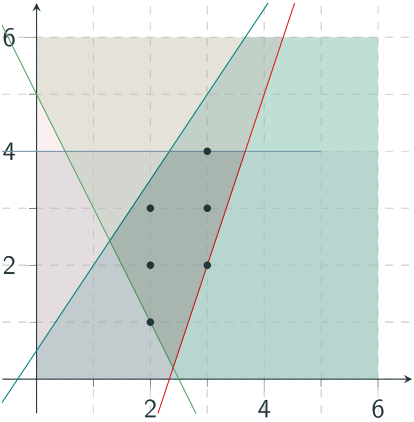
- infinite $\mathbb{R}^2$ solution space, six solutions in $\mathbb{Z}^2$



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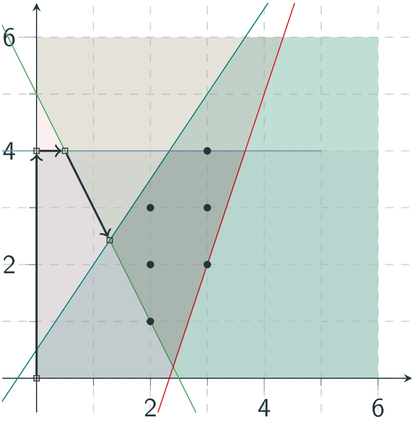
- Simplex returns $(\frac{9}{7}, \frac{17}{7})$



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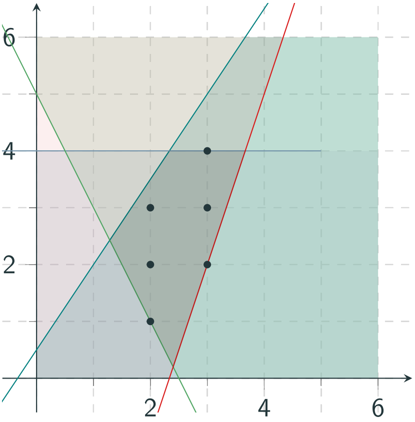

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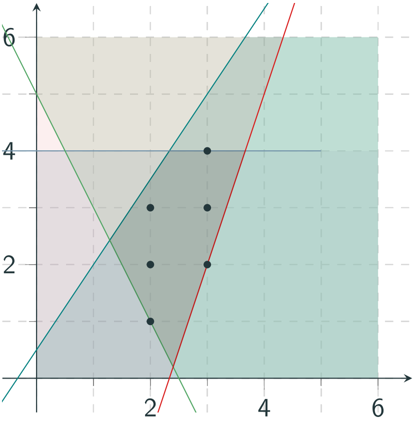

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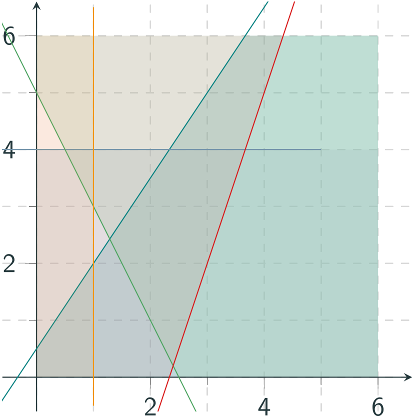

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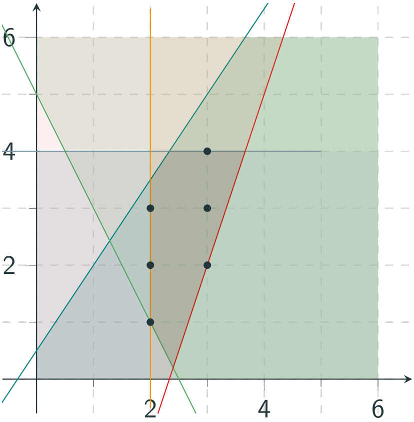

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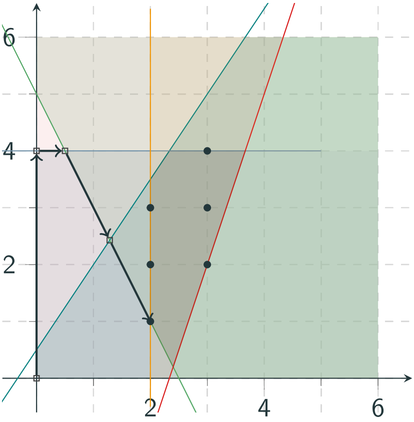

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Algorithm BranchAndBound(φ)

Input: LIA constraint φ
Output: unsatisfiable, or satisfying assignment

let *res* be result of deciding φ over $\mathbb{R}$ ▷ e.g. by Simplex
if *res* is unsatisfiable **then**
    return unsatisfiable
else if *res* is solution over $\mathbb{Z}$ **then**
    return *res*
else
    let x be variable assigned non-integer value q in *res*
    res $=$ BranchAndBound($\varphi \wedge x \leqslant \lfloor q \rfloor$)
    return *res* $\neq$ unsatisfiable **?** *res* **:** BranchAndBound($\varphi \wedge x \geqslant \lceil q \rceil$)

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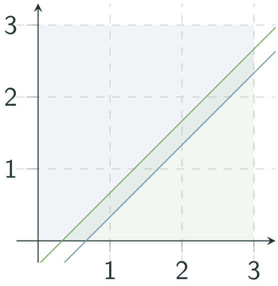

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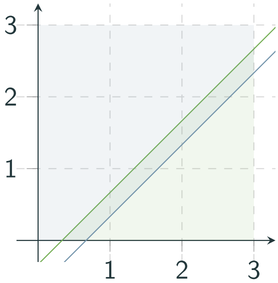

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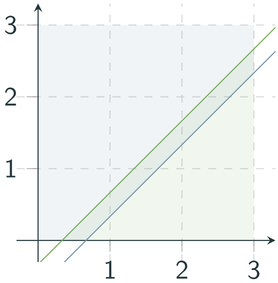

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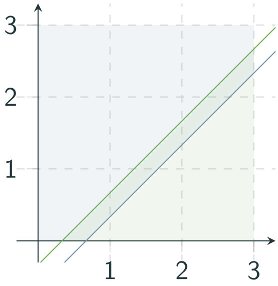

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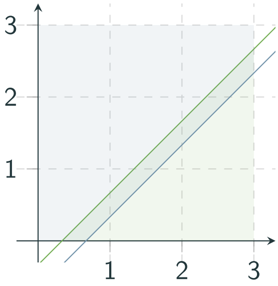

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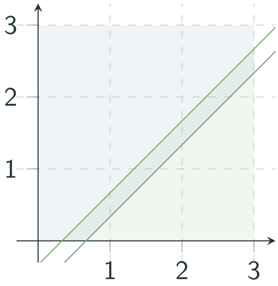

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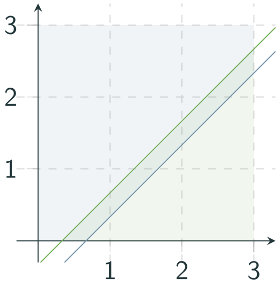

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- methods exist to derive solution bounds from tableau, but bounds are often too high for efficient practical procedures
- use cutting planes to restrict solution space more efficiently

Example (Scheduling Problem)

Is there a six-week cyclic work schedule for 22 employees who work 8-hour shifts, 24/7, 5 working days per week such that the following holds: In morning and afternoon shifts 6 employees are present, in night shifts 3. Joe does only morning shifts, Sally does not work on Sundays. Nobody works more than 6 days in a row.

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- number of employees n
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LIA Encoding

- integer variable corresponding to employee for each activity
- cardinality constraints for requirement matrix
- ...

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