SAT and SMT Solving

Sarah Winkler

Computational Logic Group
Department of Computer Science
University of Innsbruck

lecture 11
SS 2019

Outline

- Summary of Last Week
- Collision Attacks
- Nelson-Oppen Combination Method

Definition (Bit Vector Theory)

- variable $\mathbf{x}_k$ is list of length k of propositional variables $x_{k-1} \dots x_2 x_1 x_0$
- constant n_k is bit list of length k
- formulas built according to grammar

$$\begin{aligned}
\mathit{formula} &:= (\mathit{formula} \vee \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\neg \mathit{formula}) \mid \mathit{atom} \\
\mathit{atom} &:= \mathit{term}\ \mathit{rel}\ \mathit{term} \mid \mathit{true} \mid \mathit{false} \\
\mathit{rel} &:= \ = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\mathit{term} &:= (\mathit{term}\ \mathit{binop}\ \mathit{term}) \mid (\mathit{unop}\ \mathit{term}) \mid \mathit{var} \mid \mathit{constant} \mid \mathit{term}[i{:}j] \mid \\
&\qquad (\mathit{formula}\ ?\ \mathit{term} : \mathit{term}) \\
\mathit{binop} &:= + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid | \mid \hat{}\ \mid :: \\
\mathit{unop} &:= \sim \mid -
\end{aligned}$$

- axioms are equality axioms plus rules for arithmetic/comparison/bitwise operations on bit vectors of length k
- solution assigns bit list of length k to variables $\mathbf{x}_k$

Remarks

- theory is decidable because carrier is finite
- common decision procedures use translation to SAT (bit blasting)
  - eager: no DPLL(T), bit-blast entire formula to SAT problem
  - lazy: second SAT solver as BV theory solver, bit-blast only BV atoms
- solvers heavily rely on preprocessing via rewriting

Definition (Bit Blasting: Formulas)

bit blasting transformation **B** transforms BV formula into propositional formula:

$$\mathbf{B}(\varphi \vee \psi) = \mathbf{B}(\varphi) \vee \mathbf{B}(\psi)$$

$$\mathbf{B}(\varphi \wedge \psi) = \mathbf{B}(\varphi) \wedge \mathbf{B}(\psi)$$

$$\mathbf{B}(\neg\varphi) = \neg\mathbf{B}(\varphi)$$

$$\mathbf{B}(t_1 \ \mathit{rel} \ t_2) = \mathbf{B}_r(u_1 \ \mathit{rel} \ u_2) \wedge \varphi_1 \wedge \varphi_2 \quad \text{if } \mathbf{B}_t(t_1) = (u_1, \varphi_1) \text{ and } \mathbf{B}_t(t_2) = (u_2, \varphi_2)$$

bit blasting $\mathbf{B}_t$ for term t returns (result u, side condition φ)

$\mathbf{B}_r$ transforms atom into propositional formula

Definition (Bit Blasting: Atoms)
for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ set

- equality

$$\mathbf{B}_r(\mathbf{x}_{k+1} = \mathbf{y}_{k+1}) = (x_k \leftrightarrow y_k) \wedge \cdots \wedge (x_1 \leftrightarrow y_1) \wedge (x_0 \leftrightarrow y_0)$$

- inequality

$$\mathbf{B}_r(\mathbf{x}_{k+1} \neq \mathbf{y}_{k+1}) = (x_k \oplus y_k) \vee \cdots \vee (x_1 \oplus y_1) \vee (x_0 \oplus y_0)$$

- unsigned greater-than or equal

$$\mathbf{B}_r(\mathbf{x}_1 \geqslant_u \mathbf{y}_1) = y_0 \rightarrow x_0$$

$$\mathbf{B}_r(\mathbf{x}_{k+1} \geqslant_u \mathbf{y}_{k+1}) = (x_k \wedge \neg y_k) \vee ((x_k \leftrightarrow y_k) \wedge \mathbf{B}(\mathbf{x}[k-1{:}0] \geqslant \mathbf{y}[k-1{:}0]))$$

- unsigned greater-than

$$\mathbf{B}(\mathbf{x}_k >_u \mathbf{y}_k) = \mathbf{B}(\mathbf{x}_k \geqslant \mathbf{y}_k) \wedge \mathbf{B}(\mathbf{x}_k \neq \mathbf{y}_k)$$

Definition (Bit Blasting: Bitwise Operations)
for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ use fresh variable $\mathbf{z}_k$ and set

- bitwise and

$$\mathbf{B}_t(\mathbf{x}_k \& \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \wedge y_i)$$

- bitwise or

$$\mathbf{B}_t(\mathbf{x}_k | \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \vee y_i)$$

- bitwise exclusive or

$$\mathbf{B}_t(\mathbf{x}_k \ \hat{}\ \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \oplus y_i)$$

- bitwise negation

$$\mathbf{B}_t(-\mathbf{x}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow \neg x_i$$

Definition (Bit Blasting: Addition and Subtraction)

- addition

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = (\mathbf{s}_k, \varphi)$$

where

$$\varphi = (c_0 \leftrightarrow x_0 \wedge y_0) \wedge (s_0 \leftrightarrow x_0 \oplus y_0) \wedge$$
$$\bigwedge_{i=1}^{k-1} (c_i \leftrightarrow \mathrm{min2}(x_i, y_i, c_{i-1})) \wedge (s_i \leftrightarrow x_i \oplus y_i \oplus c_{i-1})$$

for fresh variables $\mathbf{s}_k$ and $\mathbf{c}_k$ and $\mathrm{min2}(a, b, d) = (a \wedge b) \vee (a \wedge d) \vee (b \wedge d)$

ripple-carry adder: $\mathbf{c}_k$ are carry bits

- unary minus

$$\mathbf{B}_t(-\mathbf{x}_k) = \mathbf{B}_t(\sim \mathbf{x}_k + \mathbf{1}_k)$$

- subtraction

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = \mathbf{B}_t(\mathbf{x}_k + (-\mathbf{y}_k))$$

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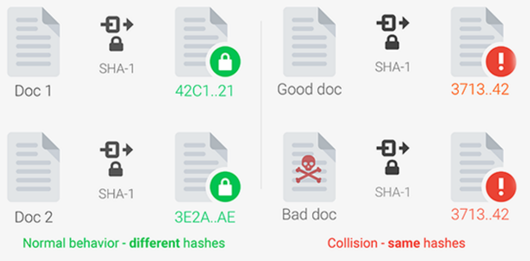

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Classical Collision Attack Scenario

Alice Malloy Bob

- Malloy aims to send malicious document to Bob pretending it be from Alice

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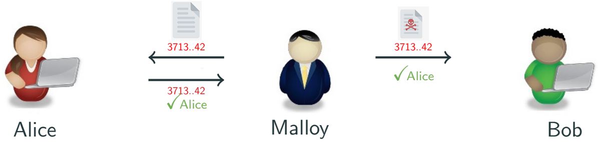

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Extension: Chosen-Prefix Collision Attack

find values x and y such that $\forall m_1\, m_2.\ f(x \cdot m_1) = f(y \cdot m_2)$

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Example

SHA-0, SHA-1, SHA-256, MD5, MD6, BLAKE2, RIPEMD-160, ...

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Collision Attack: Shift-Add-Xor Hash

- widely used non-cryptographic string hash function
- given string s, compute hash sax(s)

```
unsigned sax(char *s, int len){
    unsigned h = 0;
    for (int i = 0; i < len; i++)
        h = h ^ ((h << 5) + (h >> 2) + s[i]);
    return h;
}
```

- collision attack: sax_collision.py

More Cryptanalysis using SAT/SMT

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Tools for SAT/SMT-Based Cryptanalysis

- CryptoMiniSat
- CryptoSMT
- Transalg
- ...

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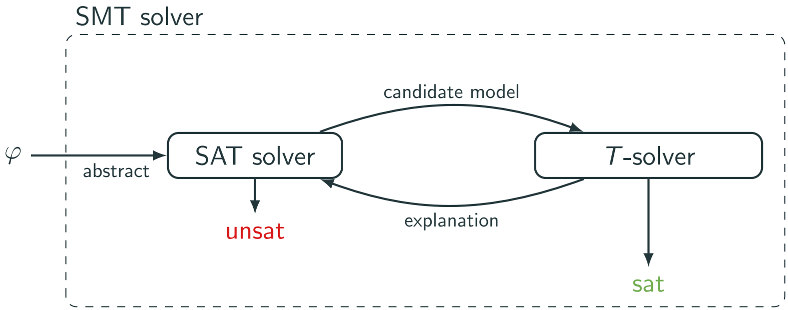

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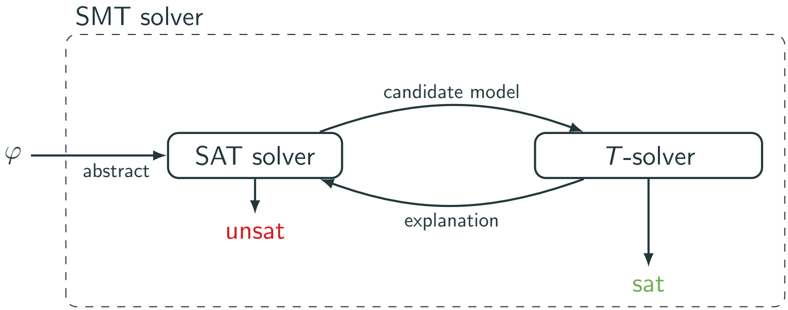

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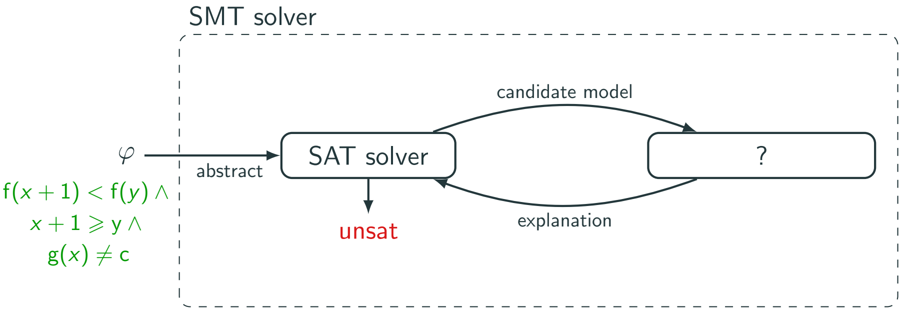

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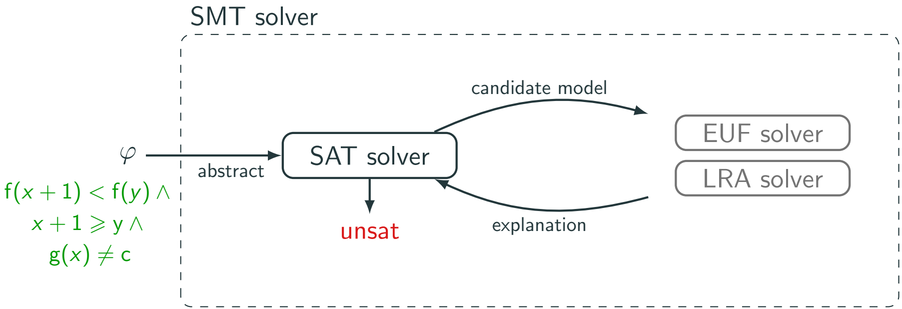

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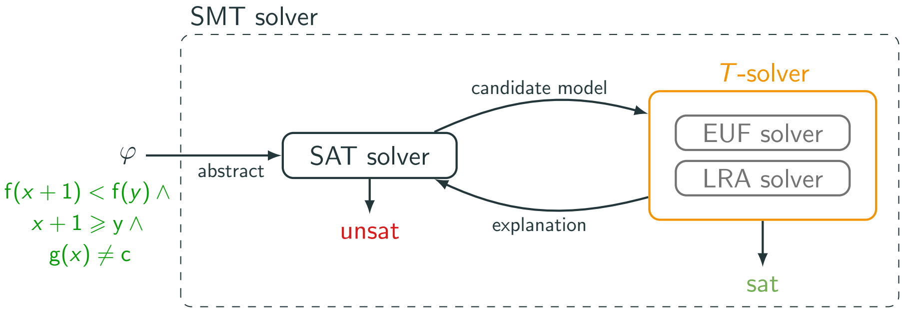

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Theory combinations — Nelson-Oppen method

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Definition

theory combination $T_1 \oplus T_2$ of two theories

- T_1 over signature Σ_1
- T_2 over signature Σ_2

has signature $\Sigma_1 \cup \Sigma_2$ and axioms $T_1 \cup T_2$

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combination of linear arithmetic and uninterpreted functions:

$$x \geqslant y \;\wedge\; y - z \geqslant x \;\wedge\; \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \;\wedge\; z \geqslant 0$$

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   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{x\,E\,y} x = y \quad \wedge \quad \bigwedge_{\neg(x\,E\,y)} x \neq y$$

Example

formula φ in combination of LIA and EUF:

$$\underbrace{1 \leqslant x \ \wedge \ x \leqslant 2 \ \wedge \ y = 1 \ \wedge \ z = 2}_{\varphi_1} \wedge \underbrace{\mathsf{f}(x) \neq \mathsf{f}(y) \ \wedge \ \mathsf{f}(x) \neq \mathsf{f}(z)}_{\varphi_2}$$

- $V = \{x, y, z\}$

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- $V = \{x, y, z\}$
- 5 different equivalence relations E:
  1. $\{\{x, y, z\}\}$
  2. $\{\{x, y\}, \{z\}\}$
  3. $\{\{x, z\}, \{y\}\}$
  4. $\{\{x\}, \{y, z\}\}$
  5. $\{\{x\}, \{y\}, \{z\}\}$

Nelson-Oppen Method: Nondeterministic Version

input: quantifier-free conjunction φ in theory combination $T_1 \oplus T_2$

output: satisfiable or unsatisfiable

1. purification

$$\varphi \;\approx\; \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

2. guess and check
   - V is set of shared variables in φ_1 and φ_2
   - guess equivalence relation E on V
   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{x\,E\,y} x = y \quad \wedge \quad \bigwedge_{\neg(x\,E\,y)} x \neq y$$

   - if $\varphi_1 \wedge \alpha(V, E)$ is T_1-satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is T_2-satisfiable then return satisfiable else return unsatisfiable

Example

formula φ in combination of LIA and EUF:

$$\underbrace{1 \leqslant x \ \wedge\ x \leqslant 2 \ \wedge\ y = 1 \ \wedge\ z = 2}_{\varphi_1} \wedge \underbrace{f(x) \neq f(y) \ \wedge\ f(x) \neq f(z)}_{\varphi_2}$$

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- φ is unsatisfiable

Outline

- Summary of Last Week
- Collision Attacks
- Nelson-Oppen Combination Method
  - Nondeterministic Version
  - Deterministic Version

Definition

theory T is convex if

$$F \vDash_T \bigvee_{i=1}^{n} u_i = v_i \qquad \text{implies} \qquad (\ F \vDash_T u_i = v_i \quad \text{for some } 1 \leqslant i \leqslant n\ )$$

$\forall$ quantifier-free conjunctive formula F and variables $u_1, \dots, u_n, v_1, \dots, v_n$

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Example

- linear arithmetic over integers (LIA) is not convex:

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- linear arithmetic over rationals and reals (LRA) is convex

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- linear arithmetic over rationals and reals (LRA) is convex
- equality logic with uninterpreted functions (EUF) is convex

Example

consider φ over combination of LRA and EUF:

$$x \geqslant y \;\wedge\; y - z \geqslant x \;\wedge\; \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \;\wedge\; z \geqslant 0$$

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$$\varphi_2 \ \wedge \ E \ \implies \ w_2 = w_3$$

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$$\varphi_1 \;\wedge\; E \quad\Longrightarrow\quad z = w_1$$

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test all (finitely many) equations,
or T-propagation

- compute implied equalities between shared variables:

$$E: \ x = y \ \wedge\ w_2 = w_3 \ \wedge\ z = w_1$$

- test satisfiability of $\varphi_2 \wedge E$ in EUF

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Nelson-Oppen Method: Deterministic Version

Input quantifier-free conjunction φ in combination $T_1 \oplus T_2$ of convex theories T_1 and T_2

Output satisfiable or unsatisfiable

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5. if E has been extended in steps 3 or 4 then go to step 2

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Remark

Nelson-Oppen decision procedure can be extended to non-convex theories: case-splitting for implied disjunction of equalities

Example

consider φ over combination of LIA and EUF:

$$1 \leqslant x \ \wedge\ x \leqslant 2 \ \wedge\ \mathsf{f}(x) \neq \mathsf{f}(1) \ \wedge\ \mathsf{f}(x) \neq \mathsf{f}(2)$$

- first purify φ:

$$\varphi_1\colon\ 1 \leqslant x \ \wedge\ x \leqslant 2 \ \wedge\ w_1 = 1 \ \wedge\ w_2 = 2$$

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- check satisfiability and compute (disjunction of) implied equalities:

$$E:$$

- test satisfiability of $\varphi_1 \wedge E$ in LIA

$$\varphi_1 \ \wedge \ E$$

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case-splitting for implied disjunction of equalities

Example

consider φ over combination of LIA and EUF:

$$1 \leqslant x \;\wedge\; x \leqslant 2 \;\wedge\; \mathsf{f}(x) \neq \mathsf{f}(1) \;\wedge\; \mathsf{f}(x) \neq \mathsf{f}(2)$$

- first purify φ:

$$\varphi_1: \; 1 \leqslant x \;\wedge\; x \leqslant 2 \;\wedge\; w_1 = 1 \;\wedge\; w_2 = 2$$

$$\varphi_2: \; f(x) \neq f(w_1) \;\wedge\; f(x) \neq f(w_2)$$

- check satisfiability and compute (disjunction of) implied equalities:

$$E:$$

- test satisfiability of $\varphi_1 \wedge E$ in LIA

$$\varphi_1 \;\wedge\; E \implies x = w_1 \;\vee\; x = w_2$$

Remark

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- φ is unsatisfiable

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Relevance of Program Equivalence

correctness of compiler optimizations, software verification

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int one(int x){
  unsigned z = x & (-1);
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Assert non-equivalence by SMT encoding:

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Remarks

- useful to combine BV and EUF theories
- checking equivalence of programs with loops is more challenging

Bibliography

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