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SAT and SMT Solving

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Outline

- Summary of Last Week
- Collision Attacks
- Nelson-Oppen Combination Method

Definition (Bit Vector Theory)

- variable $\mathbf{x}_k$ is list of length k of propositional variables $x_{k-1} \dots x_2 x_1 x_0$
- constant n_k is bit list of length k
- formulas built according to grammar

$$
\begin{aligned}
\mathit{formula} &:= (\mathit{formula} \vee \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\neg \mathit{formula}) \mid \mathit{atom} \\
\mathit{atom} &:= \mathit{term}\ \mathit{rel}\ \mathit{term} \mid \mathit{true} \mid \mathit{false} \\
\mathit{rel} &:= \ = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\mathit{term} &:= (\mathit{term}\ \mathit{binop}\ \mathit{term}) \mid (\mathit{unop}\ \mathit{term}) \mid \mathit{var} \mid \mathit{constant} \mid \mathit{term}[i{:}j] \mid \\
&\qquad (\mathit{formula}\ ?\ \mathit{term} : \mathit{term}) \\
\mathit{binop} &:= + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid | \mid \hat{}\ \mid :: \\
\mathit{unop} &:= \sim \mid -
\end{aligned}
$$

- axioms are equality axioms plus rules for arithmetic/comparison/bitwise operations on bit vectors of length k
- solution assigns bit list of length k to variables $\mathbf{x}_k$

Remarks

- theory is decidable because carrier is finite
- common decision procedures use translation to SAT (bit blasting)
  - eager: no DPLL(T), bit-blast entire formula to SAT problem
  - lazy: second SAT solver as BV theory solver, bit-blast only BV atoms
- solvers heavily rely on preprocessing via rewriting

Definition (Bit Blasting: Formulas)

bit blasting transformation $\mathbf{B}$ transforms BV formula into propositional formula:

$$\mathbf{B}(\varphi \vee \psi) = \mathbf{B}(\varphi) \vee \mathbf{B}(\psi)$$

$$\mathbf{B}(\varphi \wedge \psi) = \mathbf{B}(\varphi) \wedge \mathbf{B}(\psi)$$

$$\mathbf{B}(\neg\varphi) = \neg\mathbf{B}(\varphi)$$

$$\mathbf{B}(t_1 \ \mathit{rel} \ t_2) = \mathbf{B}_r(u_1 \ \mathit{rel} \ u_2) \wedge \varphi_1 \wedge \varphi_2 \quad \text{if } \mathbf{B}_t(t_1) = (u_1, \varphi_1) \text{ and } \mathbf{B}_t(t_2) = (u_2, \varphi_2)$$

bit blasting $\mathbf{B}_t$ for term t returns (result u, side condition φ)

$\mathbf{B}_r$ transforms atom into propositional formula

Definition (Bit Blasting: Atoms)
for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ set

- equality

$$\mathbf{B}_r(\mathbf{x}_{k+1} = \mathbf{y}_{k+1}) = (x_k \leftrightarrow y_k) \wedge \cdots \wedge (x_1 \leftrightarrow y_1) \wedge (x_0 \leftrightarrow y_0)$$

- inequality

$$\mathbf{B}_r(\mathbf{x}_{k+1} \neq \mathbf{y}_{k+1}) = (x_k \oplus y_k) \vee \cdots \vee (x_1 \oplus y_1) \vee (x_0 \oplus y_0)$$

- unsigned greater-than or equal

$$\mathbf{B}_r(\mathbf{x}_1 \geqslant_u \mathbf{y}_1) = y_0 \rightarrow x_0$$

$$\mathbf{B}_r(\mathbf{x}_{k+1} \geqslant_u \mathbf{y}_{k+1}) = (x_k \wedge \neg y_k) \vee ((x_k \leftrightarrow y_k) \wedge \mathbf{B}(\mathbf{x}[k-1{:}0] \geqslant \mathbf{y}[k-1{:}0]))$$

- unsigned greater-than

$$\mathbf{B}(\mathbf{x}_k >_u \mathbf{y}_k) = \mathbf{B}(\mathbf{x}_k \geqslant \mathbf{y}_k) \wedge \mathbf{B}(\mathbf{x}_k \neq \mathbf{y}_k)$$

Definition (Bit Blasting: Bitwise Operations)
for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ use fresh variable $\mathbf{z}_k$ and set

- bitwise and

$$\mathbf{B}_t(\mathbf{x}_k \& \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \wedge y_i)$$

- bitwise or

$$\mathbf{B}_t(\mathbf{x}_k | \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \vee y_i)$$

- bitwise exclusive or

$$\mathbf{B}_t(\mathbf{x}_k \;\hat{}\; \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \oplus y_i)$$

- bitwise negation

$$\mathbf{B}_t(-\mathbf{x}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow \neg x_i$$

Definition (Bit Blasting: Addition and Subtraction)

- addition

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = (\mathbf{s}_k, \varphi)$$

where

$$\varphi = (c_0 \leftrightarrow x_0 \wedge y_0) \wedge (s_0 \leftrightarrow x_0 \oplus y_0) \wedge$$
$$\bigwedge_{i=1}^{k-1} (c_i \leftrightarrow \min2(x_i, y_i, c_{i-1})) \wedge (s_i \leftrightarrow x_i \oplus y_i \oplus c_{i-1})$$

for fresh variables $\mathbf{s}_k$ and $\mathbf{c}_k$ and $\min2(a, b, d) = (a \wedge b) \vee (a \wedge d) \vee (b \wedge d)$

ripple-carry adder: $\mathbf{c}_k$ are carry bits

- unary minus

$$\mathbf{B}_t(-\mathbf{x}_k) = \mathbf{B}_t(\sim \mathbf{x}_k + \mathbf{1}_k)$$

- subtraction

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = \mathbf{B}_t(\mathbf{x}_k + (-\mathbf{y}_k))$$

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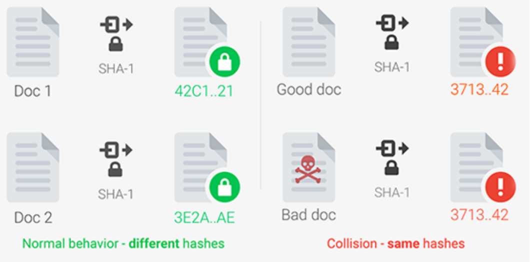
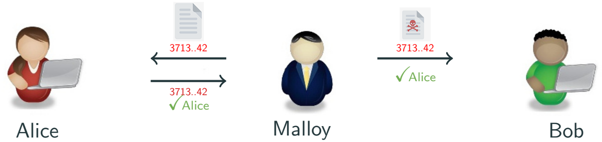
Cryptographic Hash Functions

- cryptographic hash function f is one-way hash function (SHA-1, MD5, ...)
- considered infeasible to invert, and to find messages with same hash
- problem: hash collisions



Classical Collision Attack Scenario



- Malloy aims to send malicious document to Bob pretending it be from Alice

Cryptographic Hash Function

(currently) practically infeasible to invert

- maps data of arbitrary size to bit string of fixed size (hash value)
- is one-way function

Example

SHA-0, SHA-1, SHA-256, MD5, MD6, BLAKE2, RIPEMD-160, ...

Collision Attack: Shift-Add-Xor Hash

- widely used non-cryptographic string hash function
- given string s, compute hash `sax`(s)

```
unsigned sax(char *s, int len){
    unsigned h = 0;
    for (int i = 0; i < len; i++)
        h = h ^ ((h << 5) + (h >> 2) + s[i]);
    return h;
}
```

- collision attack: `sax_collision.py`

More Cryptanalysis using SAT/SMT

- collision attacks (preimage attacks) for current hash functions such as MD4, MD5, SHA-256, CryptoHash, Keccak, ...
- exhibit classes of weak keys (or prove their absence) for block ciphers such as IDEA, WIDEA-*n*, or MESH-8
- solve inversion problems, e.g. for 20 bit DES key
- reason about crypto primitives
- help prove complexity bounds of certain operations

Tools for SAT/SMT-Based Cryptanalysis

- CryptoMiniSat
- CryptoSMT
- Transalg
- ...

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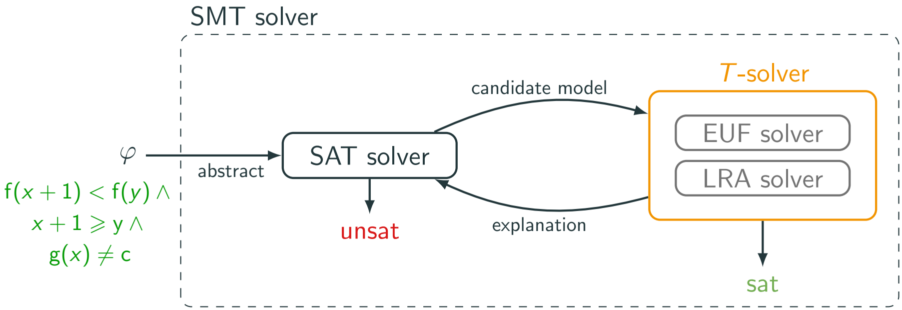
How to Be Lazy



Theory T	T**-solving method**	
▶ equality logic	equality graphs	✓
▶ equality + uninterpreted functions (EUF)	congruence closure	✓
▶ linear arithmetic (LRA and LIA)	DPLL(T) Simplex (+ cuts)	✓
▶ bitvectors (BV)	bit-blasting	✓

Theory combinations Nelson-Oppen method

Definitions

- (first-order) theory T consists of
  - signature Σ: set of function and predicate symbols
  - axioms $\mathcal{A}$: set of sentences in first-order logic in which only function and predicate symbols of Σ appear
- theory is stably infinite if every satisfiable quantifier-free formula has model with infinite carrier set

Definition

theory combination $T_1 \oplus T_2$ of two theories

- T_1 over signature Σ_1
- T_2 over signature Σ_2

has signature $\Sigma_1 \cup \Sigma_2$ and axioms $T_1 \cup T_2$

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Example

combination of linear arithmetic and uninterpreted functions:

$$x \geqslant y \ \wedge \ y - z \geqslant x \ \wedge \ \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \ \wedge \ z \geqslant 0$$

Assumptions

two stably infinite theories

- T_1 over signature Σ_1
- T_2 over signature Σ_2

such that

- $\Sigma_1 \cap \Sigma_2 = \{=\}$
- T_1-satisfiability of quantifier-free Σ_1-formulas is decidable
- T_2-satisfiability of quantifier-free Σ_2-formulas is decidable

Nelson-Oppen Method: Nondeterministic Version

input: quantifier-free conjunction φ in theory combination $T_1 \oplus T_2$

output: satisfiable or unsatisfiable

1. purification

$$\varphi \;\approx\; \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

2. guess and check
   - V is set of shared variables in φ_1 and φ_2
   - guess equivalence relation E on V
   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{x \mathrel{E} y} x = y \quad \wedge \quad \bigwedge_{\neg(x \mathrel{E} y)} x \neq y$$

   - if $\varphi_1 \wedge \alpha(V, E)$ is T_1-satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is T_2-satisfiable then return satisfiable else return unsatisfiable

Example

formula φ in combination of LIA and EUF:

$$\underbrace{1 \leqslant x \ \wedge\ x \leqslant 2 \ \wedge\ y = 1 \ \wedge\ z = 2}_{\varphi_1} \wedge \underbrace{\mathsf{f}(x) \neq \mathsf{f}(y) \ \wedge\ \mathsf{f}(x) \neq \mathsf{f}(z)}_{\varphi_2}$$

- $V = \{x, y, z\}$
- 5 different equivalence relations E:
  1. $\{\{x, y, z\}\}$ — $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
  2. $\{\{x, y\}, \{z\}\}$ — $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
  3. $\{\{x, z\}, \{y\}\}$ — $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
  4. $\{\{x\}, \{y, z\}\}$ — $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
  5. $\{\{x\}, \{y\}, \{z\}\}$ — $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
- φ is unsatisfiable

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Definition

theory T is convex if

$$F \vDash_T \bigvee_{i=1}^{n} u_i = v_i \qquad \text{implies} \qquad (F \vDash_T u_i = v_i \quad \text{for some } 1 \leqslant i \leqslant n)$$

$\forall$ quantifier-free conjunctive formula F and variables $u_1, \ldots, u_n, v_1, \ldots, v_n$

Example

- linear arithmetic over integers (LIA) is not convex:

  $$1 \leqslant x \leqslant 2 \ \wedge \ y = 1 \ \wedge \ z = 2 \quad \vDash_T \quad x = y \ \vee \ x = z$$

  holds but none of

  $$1 \leqslant x \leqslant 2 \ \wedge \ y = 1 \ \wedge \ z = 2 \quad \vDash_T \quad x = y$$
  $$1 \leqslant x \leqslant 2 \ \wedge \ y = 1 \ \wedge \ z = 2 \quad \vDash_T \quad x = z$$

- linear arithmetic over rationals and reals (LRA) is convex
- equality logic with uninterpreted functions (EUF) is convex

Example

consider φ over combination of LRA and EUF:

$$x \geqslant y \;\wedge\; y - z \geqslant x \;\wedge\; \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \;\wedge\; z \geqslant 0$$

- first purify φ:

$$\varphi_1: \; x \geqslant y \;\wedge\; y - z \geqslant x \;\wedge$$

$$\varphi_2: \; \mathsf{f}(w_1) \neq \mathsf{f}(z) \;\wedge\; w_2 =$$

test all (finitely many) equations,
or T-propagation

- compute implied equalities between shared variables:

$$E: \; x = y \;\wedge\; w_2 = w_3 \;\wedge\; z = w_1$$

- test satisfiability of $\varphi_2 \wedge E$ in EUF and compute implied equalities

$$\varphi_2 \;\wedge\; E \;\Longrightarrow\; \bot$$

- φ is unsatisfiable

Nelson-Oppen Method: Deterministic Version

Input quantifier-free conjunction φ in combination $T_1 \oplus T_2$ of convex theories T_1 and T_2

Output satisfiable or unsatisfiable

1. purification $\varphi \approx \varphi_1 \wedge \varphi_2$ for Σ_1-formula φ_1 and Σ_2-formula φ_2
2. V: set of shared variables in φ_1 and φ_2
   E: already discovered equalities between variables in V
3. test satisfiability of $\varphi_1 \wedge E$ (and add implied equations)
   - if $\varphi_1 \wedge E$ is T_1-unsatisfiable then return unsatisfiable
   - else add new implied equalities to E
4. test satisfiability of $\varphi_2 \wedge E$ (and add implied equations)
   - if $\varphi_2 \wedge E$ is T_2-unsatisfiable then return unsatisfiable
   - else add new implied equalities to E
5. if E has been extended in steps 3 or 4 then go to step 2
   else return satisfiable

Remark

Nelson-Oppen decision procedure can be extended to non-convex theories: case-splitting for implied disjunction of equalities

Example

consider φ over combination of LIA and EUF:

$$1 \leqslant x \ \wedge \ x \leqslant 2 \ \wedge \ \mathsf{f}(x) \neq \mathsf{f}(1) \ \wedge \ \mathsf{f}(x) \neq \mathsf{f}(2)$$

- first purify φ:

$$\varphi_1 : \ 1 \leqslant x \ \wedge \ x \leqslant 2 \ \wedge \ w_1 = 1 \ \wedge \ w_2 = 2$$
$$\varphi_2 : \ f(x) \neq f(w_1) \ \wedge \ f(x) \neq f(w_2)$$

- check satisfiability and compute (disjunction of) implied equalities:

$$E : \ x = w_2$$

- test satisfiability of $\varphi_2 \wedge E$ in EUF

$$\varphi_2 \ \wedge \ E \ \implies \ \bot$$

- case split: $x = w_1$ or $x = w_2$
- φ is unsatisfiable

Application: Checking Program Equivalence

Relevance of Program Equivalence

correctness of compiler optimizations, software verification

Example (Are the following two programs equivalent?)

```
int one(int x){
  unsigned z = x & (-1);
  unsigned y = z * 2;
  return foo(z) + y;
}
```

```
int two(int x){
  return foo(x) + (x << 1);
}
```

uninterpreted functions

bit vectors

Assert non-equivalence by SMT encoding:

$$\mathbf{one}_{32} = \mathrm{foo}(\mathbf{z}_{32}) + \mathbf{y}_{32} \land \mathbf{z}_{32} = \mathbf{x}_{32} \,\&\, (-\mathbf{1})_{32} \land \mathbf{y}_{32} = \mathbf{z}_{32} \times \mathbf{2}_{32} \land$$
$$\mathbf{two}_{32} = \mathrm{foo}(\mathbf{x}_{32}) + (\mathbf{x}_{32} \ll \mathbf{1}_{32}) \land$$
$$\mathbf{one}_{32} \neq \mathbf{two}_{32}$$

Remarks

- useful to combine BV and EUF theories
- checking equivalence of programs with loops is more challenging

Bibliography

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