



SAT and SMT Solving

Sarah Winkler

Computational Logic Group
Department of Computer Science
University of Innsbruck

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Definition (Bit Vector Theory)

- ▶ variable \mathbf{x}_k is list of length k of propositional variables $x_{k-1} \dots x_2 x_1 x_0$
- ▶ constant n_k is bit list of length k
- ▶ formulas built according to grammar

$$\text{formula} := (\text{formula} \vee \text{formula}) \mid (\text{formula} \wedge \text{formula}) \mid (\neg \text{formula}) \mid \text{atom}$$

$$\text{atom} := \text{term} \text{ rel } \text{term} \mid \text{true} \mid \text{false}$$

$$\text{rel} := = \mid \neq \mid \geq_u \mid \geq_s \mid >_u \mid >_s$$

$$\text{term} := (\text{term} \text{ binop } \text{term}) \mid (\text{unop } \text{term}) \mid \text{var} \mid \text{constant} \mid \text{term}[i:j] \mid (\text{formula} ? \text{term} : \text{term})$$

$$\text{binop} := + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid \mid \mid \hat{\mid} \mid ::$$

$$\text{unop} := \sim \mid -$$

- ▶ axioms are equality axioms plus rules for arithmetic/comparison/bitwise operations on bit vectors of length k
- ▶ solution assigns bit list of length k to variables \mathbf{x}_k

Outline

- Summary of Last Week
- Collision Attacks
- Nelson-Oppen Combination Method

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Remarks

- ▶ theory is decidable because carrier is finite
- ▶ common decision procedures use translation to SAT (bit blasting)
 - ▶ eager: no DPLL(T), bit-blast entire formula to SAT problem
 - ▶ lazy: second SAT solver as BV theory solver, bit-blast only BV atoms
- ▶ solvers heavily rely on preprocessing via rewriting

Definition (Bit Blasting: Formulas)

bit blasting transformation \mathbf{B} transforms BV formula into propositional formula:

$$\mathbf{B}(\varphi \vee \psi) = \mathbf{B}(\varphi) \vee \mathbf{B}(\psi)$$

$$\mathbf{B}(\varphi \wedge \psi) = \mathbf{B}(\varphi) \wedge \mathbf{B}(\psi)$$

$$\mathbf{B}(\neg \varphi) = \neg \mathbf{B}(\varphi)$$

$$\mathbf{B}(t_1 \text{ rel } t_2) = \mathbf{B}_r(u_1 \text{ rel } u_2) \wedge \varphi_1 \wedge \varphi_2 \quad \text{if } \mathbf{B}_t(t_1) = (u_1, \varphi_1) \text{ and } \mathbf{B}_t(t_2) = (u_2, \varphi_2)$$

bit blasting \mathbf{B}_t for term t
returns (result u , side condition φ)

\mathbf{B}_r transforms atom into propositional formula

Definition (Bit Blasting: Atoms)

for bit vectors \mathbf{x}_k and \mathbf{y}_k set

- ▶ equality

$$\mathbf{B}_r(\mathbf{x}_{k+1} = \mathbf{y}_{k+1}) = (x_k \leftrightarrow y_k) \wedge \dots \wedge (x_1 \leftrightarrow y_1) \wedge (x_0 \leftrightarrow y_0)$$

- ▶ inequality

$$\mathbf{B}_r(\mathbf{x}_{k+1} \neq \mathbf{y}_{k+1}) = (x_k \oplus y_k) \vee \dots \vee (x_1 \oplus y_1) \vee (x_0 \oplus y_0)$$

- ▶ unsigned greater-than or equal

$$\mathbf{B}_r(\mathbf{x}_1 \geq_u \mathbf{y}_1) = y_0 \rightarrow x_0$$

$$\mathbf{B}_r(\mathbf{x}_{k+1} \geq_u \mathbf{y}_{k+1}) = (x_k \wedge \neg y_k) \vee ((x_k \leftrightarrow y_k) \wedge \mathbf{B}(\mathbf{x}[k-1:0] \geq \mathbf{y}[k-1:0]))$$

- ▶ unsigned greater-than

$$\mathbf{B}(\mathbf{x}_k >_u \mathbf{y}_k) = \mathbf{B}(\mathbf{x}_k \geq \mathbf{y}_k) \wedge \mathbf{B}(\mathbf{x}_k \neq \mathbf{y}_k)$$

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Definition (Bit Blasting: Addition and Subtraction)

- ▶ addition

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = (\mathbf{s}_k, \varphi)$$

where

$$\varphi = (c_0 \leftrightarrow x_0 \wedge y_0) \wedge (s_0 \leftrightarrow x_0 \oplus y_0) \wedge \bigwedge_{i=1}^{k-1} (c_i \leftrightarrow \min2(x_i, y_i, c_{i-1})) \wedge (s_i \leftrightarrow x_i \oplus y_i \oplus c_{i-1})$$

ripple-carry adder:
 c_k are carry bits

for fresh variables \mathbf{s}_k and \mathbf{c}_k and $\min2(a, b, d) = (a \wedge b) \vee (a \wedge d) \vee (b \wedge d)$

- ▶ unary minus

$$\mathbf{B}_t(-\mathbf{x}_k) = \mathbf{B}_t(\sim \mathbf{x}_k + \mathbf{1}_k)$$

- ▶ subtraction

$$\mathbf{B}_t(\mathbf{x}_k - \mathbf{y}_k) = \mathbf{B}_t(\mathbf{x}_k + (-\mathbf{y}_k))$$

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Definition (Bit Blasting: Bitwise Operations)

for bit vectors \mathbf{x}_k and \mathbf{y}_k use fresh variable \mathbf{z}_k and set

- ▶ bitwise and

$$\mathbf{B}_t(\mathbf{x}_k \& \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \quad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \wedge y_i)$$

- ▶ bitwise or

$$\mathbf{B}_t(\mathbf{x}_k | \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \quad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \vee y_i)$$

- ▶ bitwise exclusive or

$$\mathbf{B}_t(\mathbf{x}_k \wedge \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \quad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \oplus y_i)$$

- ▶ bitwise negation

$$\mathbf{B}_t(-\mathbf{x}_k) = (\mathbf{z}_k, \varphi) \quad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow \neg x_i$$

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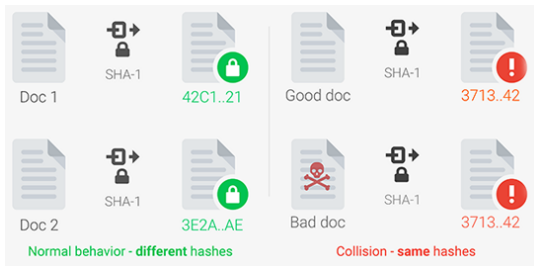
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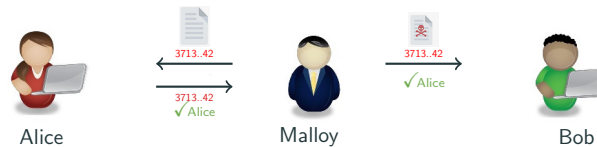
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Cryptographic Hash Functions

- ▶ cryptographic hash function f is **one-way** hash function (SHA-1, MD5, ...)
- ▶ considered infeasible to invert, and to find messages with same hash
- ▶ problem: **hash collisions**



Classical Collision Attack Scenario



- ▶ Malloy aims to send malicious document to Bob pretending it be from Alice

SMT-Based Collision Finding

- ▶ encode f as operation on bit vectors x, v representing strings

More Cryptanalysis using SAT/SMT

- ▶ collision attacks (preimage attacks) for current hash functions such as MD4, MD5, SHA-256, CryptoHash, Keccak, ...
- ▶ exhibit classes of weak keys (or prove their absence) for block ciphers such as IDEA, WIDEA- n , or MESH-8
- ▶ solve inversion problems, e.g. for 20 bit DES key
- ▶ reason about crypto primitives
- ▶ help prove complexity bounds of certain operations

Tools for SAT/SMT-Based Cryptanalysis

- ▶ CryptoMiniSat
- ▶ CryptoSMT
- ▶ Transalg
- ▶ ...

Cryptographic Hash Function

(currently) practically infeasible to invert

- ▶ maps data of arbitrary size to bit string of fixed size (~~hash~~ **value**)
- ▶ is **one-way function**

Example

SHA-0, SHA-1, SHA-256, MD5, MD6, BLAKE2, RIPEMD-160, ...

Collision Attack: Shift-Add-Xor Hash

- ▶ widely used non-cryptographic string hash function
- ▶ given string s , compute hash $sax(s)$

```
unsigned sax(char *s, int len){
    unsigned h = 0;
    for (int i = 0; i < len; i++){
        h = h ^ ((h << 5) + (h >> 2) + s[i]);
    }
    return h;
}
```

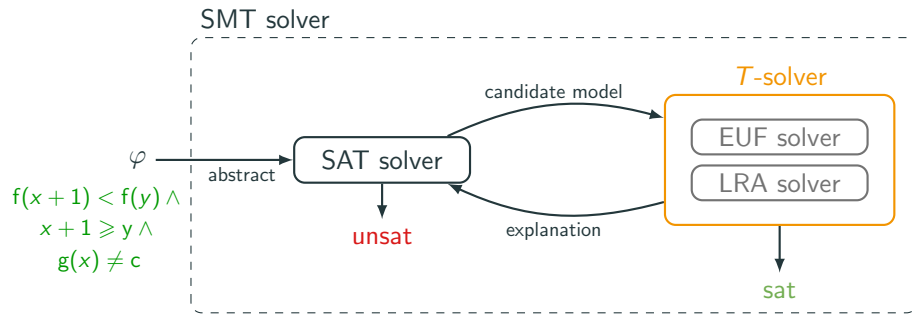
- ▶ collision attack: `sax_collision.py`

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 - Deterministic Version

How to Be Lazy



Theory T

- ▶ equality logic
- ▶ equality + uninterpreted functions (EUF)
- ▶ linear arithmetic (LRA and LIA)
- ▶ bitvectors (BV)

T -solving method

- equality graphs ✓
- congruence closure ✓
- DPLL(T) Simplex (+ cuts) ✓
- bit-blasting ✓

Theory combinations

Nelson-Oppen method

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Definitions

- ▶ (first-order) **theory** T consists of
 - ▶ signature Σ : set of function and predicate symbols
 - ▶ axioms \mathcal{A} : set of sentences in first-order logic in which only function and predicate symbols of Σ appear
- ▶ theory is **stably infinite** if every satisfiable quantifier-free formula has model with infinite carrier set

Definition

theory combination $T_1 \oplus T_2$ of two theories

- ▶ T_1 over signature Σ_1
- ▶ T_2 over signature Σ_2

has signature $\Sigma_1 \cup \Sigma_2$ and axioms $T_1 \cup T_2$

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Example

combination of linear arithmetic and uninterpreted functions:

$$x \geq y \wedge y - z \geq x \wedge f(f(y) - f(x)) \neq f(z) \wedge z \geq 0$$

Assumptions

two stably infinite theories

- ▶ T_1 over signature Σ_1
- ▶ T_2 over signature Σ_2

such that

- ▶ $\Sigma_1 \cap \Sigma_2 = \{=\}$
- ▶ T_1 -satisfiability of quantifier-free Σ_1 -formulas is decidable
- ▶ T_2 -satisfiability of quantifier-free Σ_2 -formulas is decidable

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Nelson-Oppen Method: Nondeterministic Version

input: quantifier-free conjunction φ in theory combination $T_1 \oplus T_2$

output: satisfiable or unsatisfiable

1 purification

$\varphi \approx \varphi_1 \wedge \varphi_2$ for Σ_1 -formula φ_1 and Σ_2 -formula φ_2

2 guess and check

- ▶ V is set of shared variables in φ_1 and φ_2
- ▶ guess equivalence relation E on V
- ▶ **arrangement** $\alpha(V, E)$ is formula

$$\bigwedge_{x E y} x = y \quad \wedge \quad \bigwedge_{\neg(x E y)} x \neq y$$

- ▶ if $\varphi_1 \wedge \alpha(V, E)$ is T_1 -satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is T_2 -satisfiable then return satisfiable else return unsatisfiable

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Example

formula φ in combination of LIA and EUF:

$$\underbrace{1 \leq x \wedge x \leq 2 \wedge y = 1 \wedge z = 2}_{\varphi_1} \wedge \underbrace{f(x) \neq f(y) \wedge f(x) \neq f(z)}_{\varphi_2}$$

- ▶ $V = \{x, y, z\}$
- ▶ 5 different equivalence relations E :
 - 1 $\{\{x, y, z\}\}$ $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
 - 2 $\{\{x, y\}, \{z\}\}$ $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
 - 3 $\{\{x, z\}, \{y\}\}$ $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
 - 4 $\{\{x\}, \{y, z\}\}$ $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
 - 5 $\{\{x\}, \{y\}, \{z\}\}$ $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
- ▶ φ is unsatisfiable

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Definition

theory T is **convex** if

$$F \models_T \bigvee_{i=1}^n u_i = v_i \quad \text{implies} \quad (F \models_T u_i = v_i \quad \text{for some } 1 \leq i \leq n)$$

\forall quantifier-free conjunctive formula F and variables $u_1, \dots, u_n, v_1, \dots, v_n$

Example

- ▶ linear arithmetic over integers (LIA) is **not convex**:

$$1 \leq x \leq 2 \wedge y = 1 \wedge z = 2 \quad \models_T \quad x = y \vee x = z$$

holds but none of

$$1 \leq x \leq 2 \wedge y = 1 \wedge z = 2 \quad \not\models_T \quad x = y$$

$$1 \leq x \leq 2 \wedge y = 1 \wedge z = 2 \quad \not\models_T \quad x = z$$

- ▶ linear arithmetic over rationals and reals (LRA) is convex
- ▶ equality logic with uninterpreted functions (EUF) is convex

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Example

consider φ over combination of LRA and EUF:

$$x \geq y \wedge y - z \geq x \wedge f(f(y) - f(x)) \neq f(z) \wedge z \geq 0$$

- ▶ first purify φ :

$$\begin{aligned} \varphi_1: & x \geq y \wedge y - z \geq x \\ \varphi_2: & f(w_1) \neq f(z) \wedge w_2 = \end{aligned}$$

test all (finitely many) equations,
or T -propagation

- ▶ compute implied equalities between shared variables:

$$E: x = y \wedge w_2 = w_3 \wedge z = w_1$$

- ▶ test satisfiability of $\varphi_2 \wedge E$ in EUF and compute implied equalities

$$\varphi_2 \wedge E \implies \perp$$

- ▶ φ is **unsatisfiable**

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Remark

Nelson-Oppen decision procedure can be extended to non-convex theories:
case-splitting for implied disjunction of equalities

Example

consider φ over combination of LIA and EUF:

$$1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$$

- ▶ first purify φ :

$$\begin{aligned} \varphi_1: & 1 \leq x \wedge x \leq 2 \wedge w_1 = 1 \wedge w_2 = 2 \\ \varphi_2: & f(x) \neq f(w_1) \wedge f(x) \neq f(w_2) \end{aligned}$$

- ▶ check satisfiability and compute (disjunction of) implied equalities:

$$E: x = w_2$$

- ▶ test satisfiability of $\varphi_2 \wedge E$ in EUF

$$\varphi_2 \wedge E \implies \perp$$

- ▶ case split: $x = w_1$ or $x = w_2$
- ▶ φ is **unsatisfiable**

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Nelson-Oppen Method: Deterministic Version

Input: quantifier-free conjunction φ in combination $T_1 \oplus T_2$ of convex theories T_1 and T_2

Output: satisfiable or unsatisfiable

- 1 **purification** $\varphi \approx \varphi_1 \wedge \varphi_2$ for Σ_1 -formula φ_1 and Σ_2 -formula φ_2
- 2 V : set of shared variables in φ_1 and φ_2
 E : already discovered equalities between variables in V
- 3 test satisfiability of $\varphi_1 \wedge E$ (and add implied equalities)
 - ▶ if $\varphi_1 \wedge E$ is **T_1 -unsatisfiable** then return unsatisfiable
 - ▶ else **add** new implied equalities to E
- 4 test satisfiability of $\varphi_2 \wedge E$ (and add implied equalities)
 - ▶ if $\varphi_2 \wedge E$ is **T_2 -unsatisfiable** then return unsatisfiable
 - ▶ else **add** new implied equalities to E
- 5 if E has been extended in steps 3 or 4 then go to step 2 else return satisfiable

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Application: Checking Program Equivalence

Relevance of Program Equivalence

correctness of compiler optimizations, software verification

Example (Are the following two programs equivalent?)

```

int one(int x){
  unsigned z = x & (-1);
  unsigned y = z * 2;
  return foo(z) + y;
}

int two(int x){
  return foo(x) + (x << 1);
}
  
```

uninterpreted functions

bit vectors

Assert non-equivalence by SMT encoding:

$$\begin{aligned} \mathbf{one}_{32} &= \mathbf{foo}(\mathbf{z}_{32}) + \mathbf{y}_{32} \wedge \mathbf{z}_{32} = \mathbf{x}_{32} \& (-1)_{32} \wedge \mathbf{y}_{32} = \mathbf{z}_{32} \times \mathbf{2}_{32} \wedge \\ \mathbf{two}_{32} &= \mathbf{foo}(\mathbf{x}_{32}) + (\mathbf{x}_{32} \ll \mathbf{1}_{32}) \wedge \\ \mathbf{one}_{32} &\neq \mathbf{two}_{32} \end{aligned}$$

Remarks

- ▶ useful to combine BV and EUF theories
- ▶ checking equivalence of programs with loops is more challenging

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Bibliography



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