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SAT and SMT Solving

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lecture 12
SS 2019

Outline

- Summary of Last Week
- Quantifiers for SMT
- Instantiation Techniques

Definitions

- theory consists of
  - signature Σ: set of function and predicate symbols
  - axioms T: set of sentences in first-order logic in which only function and predicate symbols of Σ appear
- theory is stably infinite if every satisfiable quantifier-free formula has model with infinite carrier set
- theory T is convex if $F \models_T \bigvee_{i=1}^{n} u_i = v_i$ implies $F \models_T u_i = v_i$ for some $1 \leqslant i \leqslant n$ $\forall$ quantifier-free conjunction F and variables u_i, v_i

Definition

theory combination $T_1 \oplus T_2$ of two theories

- T_1 over signature Σ_1
- T_2 over signature Σ_2

has signature $\Sigma_1 \cup \Sigma_2$ and axioms $T_1 \cup T_2$

Nelson-Oppen Method: Nondeterministic Version

Input quantifier-free conjunction φ in theory combination $T_1 \oplus T_2$
Output satisfiable or unsatisfiable

1. purification

$$\varphi \approx \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

2. guess and check
   - V is set of shared variables in φ_1 and φ_2
   - guess equivalence relation E on V
   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{x\,E\,y} x = y \quad \wedge \quad \bigwedge_{\neg(x\,E\,y)} x \neq y$$

   - if $\varphi_1 \wedge \alpha(V, E)$ is T_1-satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is T_2-satisfiable then return satisfiable else return unsatisfiable

Nelson-Oppen Method: Deterministic Version

Input quantifier-free conjunction φ in combination $T_1 \oplus T_2$ of convex theories T_1 and T_2

Output satisfiable or unsatisfiable

1. purification $\varphi \approx \varphi_1 \wedge \varphi_2$ for Σ_1-formula φ_1 and Σ_2-formula φ_2
2. V: set of shared variables in φ_1 and φ_2
   E: already discovered equalities between variables in V
3. test satisfiability of $\varphi_1 \wedge E$ (and add implied equations)
   - if $\varphi_1 \wedge E$ is T_1-unsatisfiable then return unsatisfiable
   - else add new implied equalities to E
4. test satisfiability of $\varphi_2 \wedge E$ (and add implied equations)
   - if $\varphi_2 \wedge E$ is T_2-unsatisfiable then return unsatisfiable
   - else add new implied equalities to E
5. if E has been extended in steps 3 or 4 then go to step 2
   else return satisfiable

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  - Skolemization
- Instantiation Techniques

Applications of Quantifiers in SMT

Example (Homework 5)

Imagine a village of monkeys where each monkey owns at least two bananas. As the monkeys are well-organised, each tree contains exactly three monkeys. Monkeys are also very friendly, so every monkey has a partner.

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More important applications

- automated theorem proving
  $\forall x\ y\ z.\ \mathsf{inv}(x) \cdot x = 0 \wedge 0 \cdot x = x \wedge x \cdot (y \cdot z) = (x \cdot y) \cdot z$

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  $\forall x.\ \mathrm{pre}(x) \longrightarrow \mathrm{post}(x)$

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- function synthesis
  $\forall \mathit{input}.\ \exists \mathit{output}.\ \mathrm{F}(\mathit{input}, \mathit{output})$

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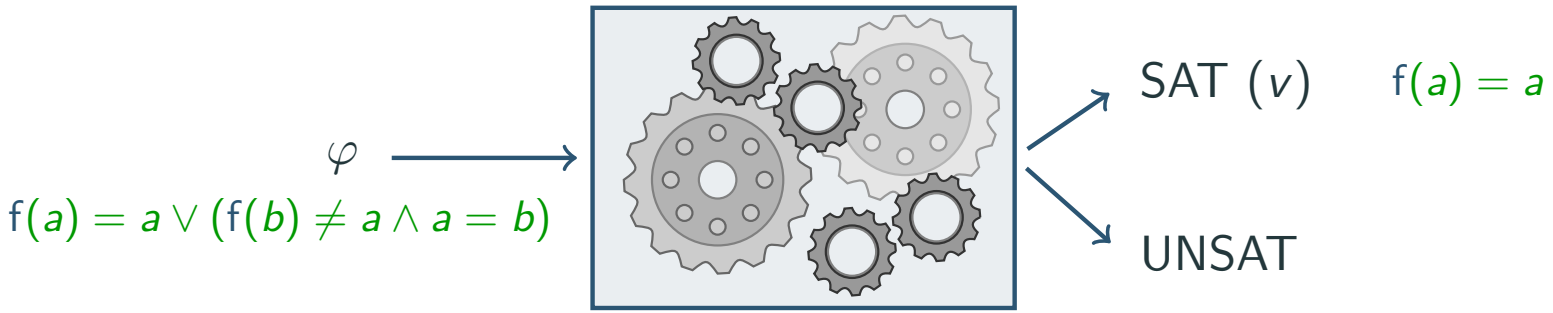
- function synthesis
  $\forall \mathit{input}.\ \exists \mathit{output}.\ \mathrm{F}(\mathit{input}, \mathit{output})$
- planning
  $\exists \mathit{plan}.\ \forall \mathit{time}.\ \mathrm{spec}(\mathit{plan}, \mathit{time})$

SMT Solving with Quantifiers

SMT solver



Decision Properties

- SMT solvers can decide propositional logic + LIA/LRA/EUF/BV/...

SMT Solving with Quantifiers

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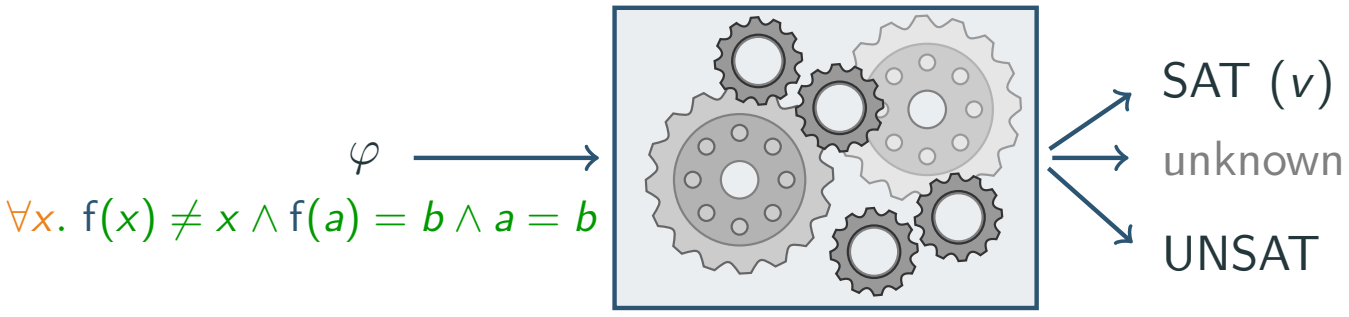

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- many SMT solvers also have support for quantifiers

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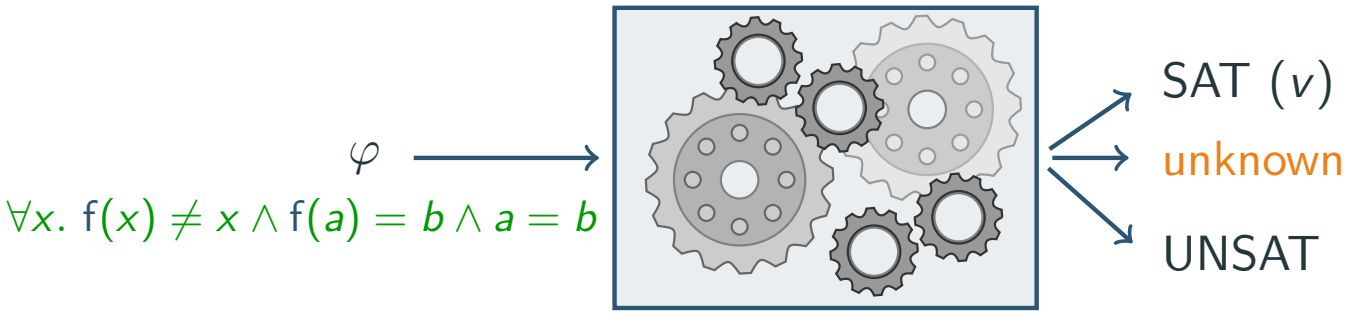

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  but have in general no decision procedure for theories + quantifiers

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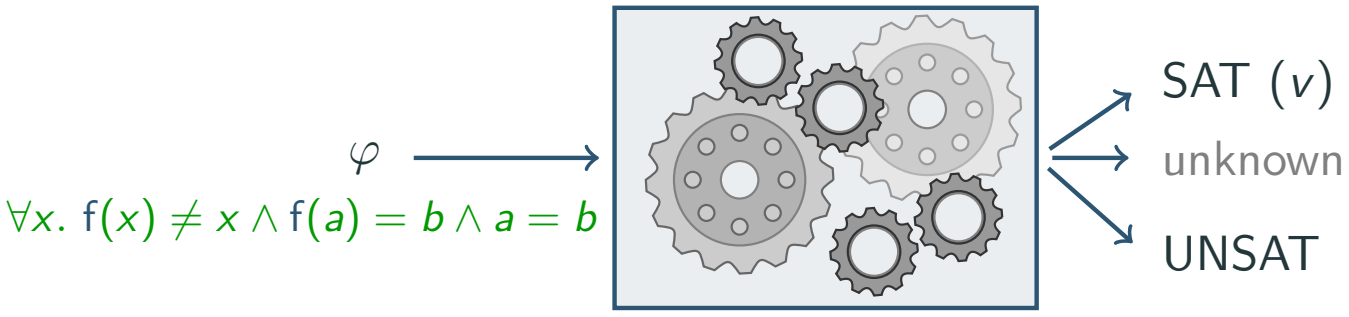

Decision Properties

first-order logic is undecidable!

- SMT solvers can decide propositional logic + LIA/LRA/EUF/BV/...
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  but have in general no decision procedure for theories + quantifiers

Skolemization

Getting rid of $\exists$ quantifiers

- replace $\exists x.\ \mathrm{P}(x)$ by $\mathrm{P}(\mathrm{a})$

Thoralf Skolem

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name witness for existential quantifier

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Skolemization

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2. replace $\forall x_1, \ldots, x_k \exists y\ \psi[y]$ by $\forall x_1, \ldots, x_k\ \psi[\mathsf{f}(x_1, \ldots, x_k)]$ for fresh f

   until no existential quantifiers left

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if φ' is skolemization of φ then φ and φ' are equisatisfiable

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can consider formulas of shape $\forall x_1, \ldots, x_n\ \varphi[x_1, \ldots, x_n]$

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Definition

Herbrand instance of Skolem formula $\forall x_1, \ldots, x_n \ \varphi[x_1, \ldots, x_n]$ is $\varphi[t_1, \ldots, t_n]$ where t_i is term over signature of φ

Definition

set of function symbols and constants

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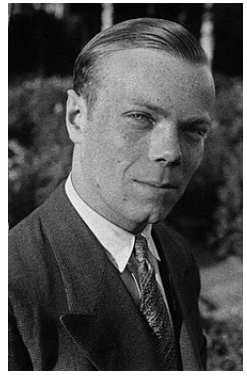
Remark

Herbrand instances are ground formulas, i.e., without (quantified) variables



Jacques Herbrand

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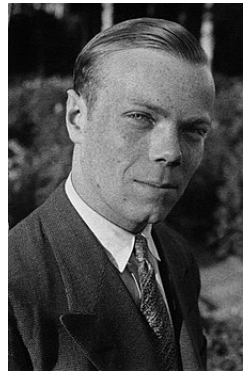
Theorem (Herbrand)

Skolem formula φ is unsatisfiable $\iff$
there exists finite unsatisfiable set of Herbrand instances of φ



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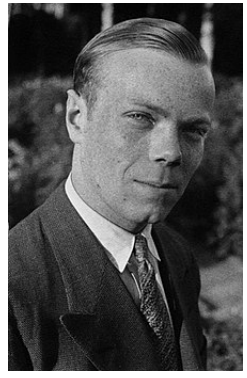

Jacques Herbrand

Caveats

- at least one constant required per sort
- holds for pure first order logic, not necessarily in presence of theories

Example: Is this syllogism correct?

All humans are mortal.

All Greeks are humans.

So all Greeks are mortal.

Aristotle

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All humans are mortal.	$\forall x.\ H(x) \longrightarrow M(x)$
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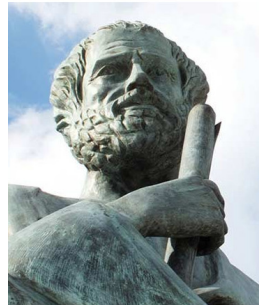

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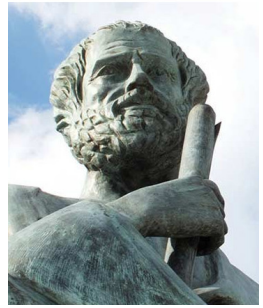

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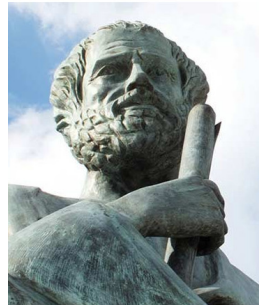

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cannot be answered by SMT solver

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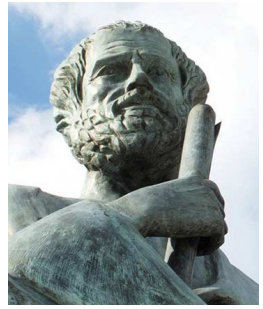

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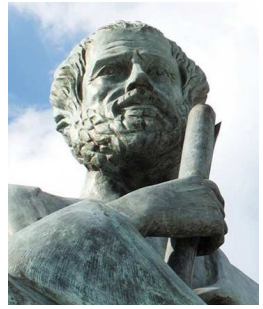

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$$\forall x.\ H(x) \longrightarrow M(x), \quad \forall x.\ G(x) \longrightarrow H(x), \quad G(a) \wedge \neg M(a)$$

- already unsatisfiable when replacing quantified formulas by Herbrand instances

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$$\forall x.\ H(x) \longrightarrow M(x),$$

when adding right Herbrand instances unsatisfiability can be detected by SMT solver

- already unsatisfiable when replacing quantified formulas by Herbrand instances

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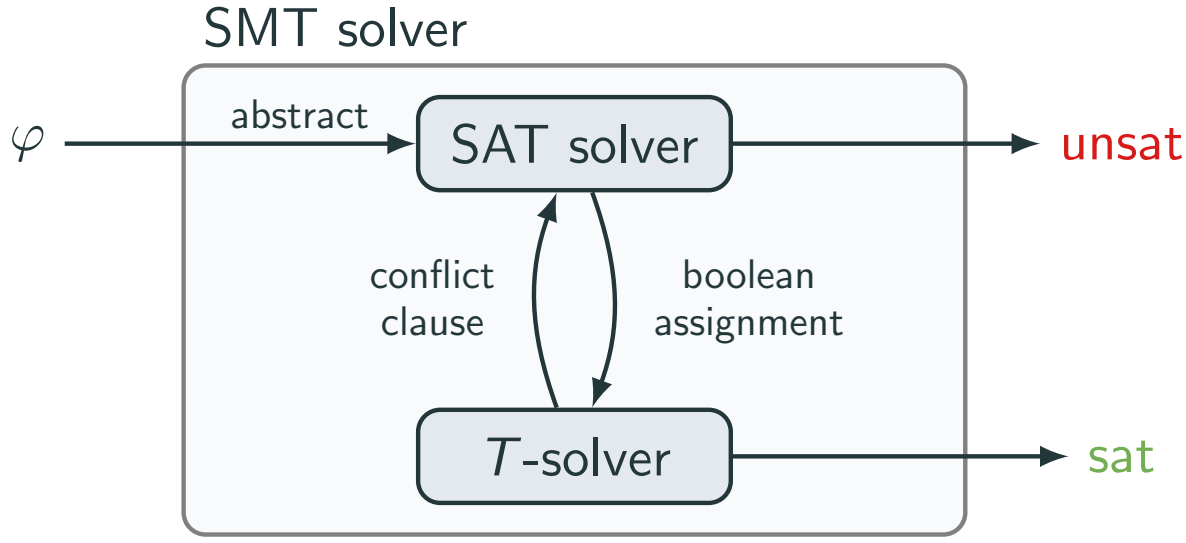
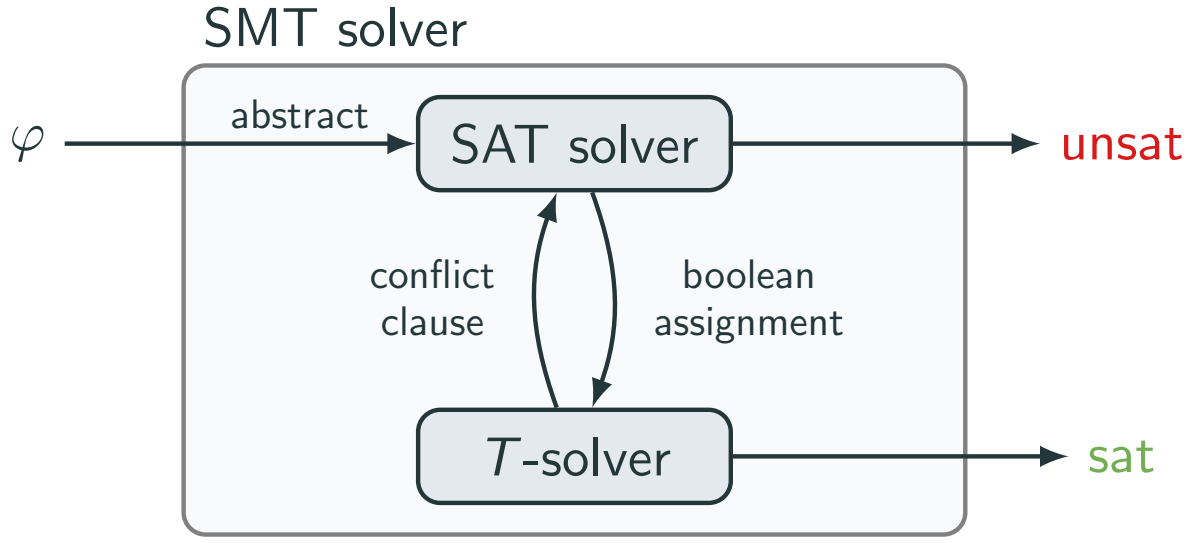
  - E-Matching
  - Enumerative Instantiation



Example

- $a = b \wedge \mathrm{g}(a) = a \wedge (\mathrm{f}(a) \neq \mathrm{f}(b) \vee b \neq \mathrm{g}(\mathrm{g}(a)))$



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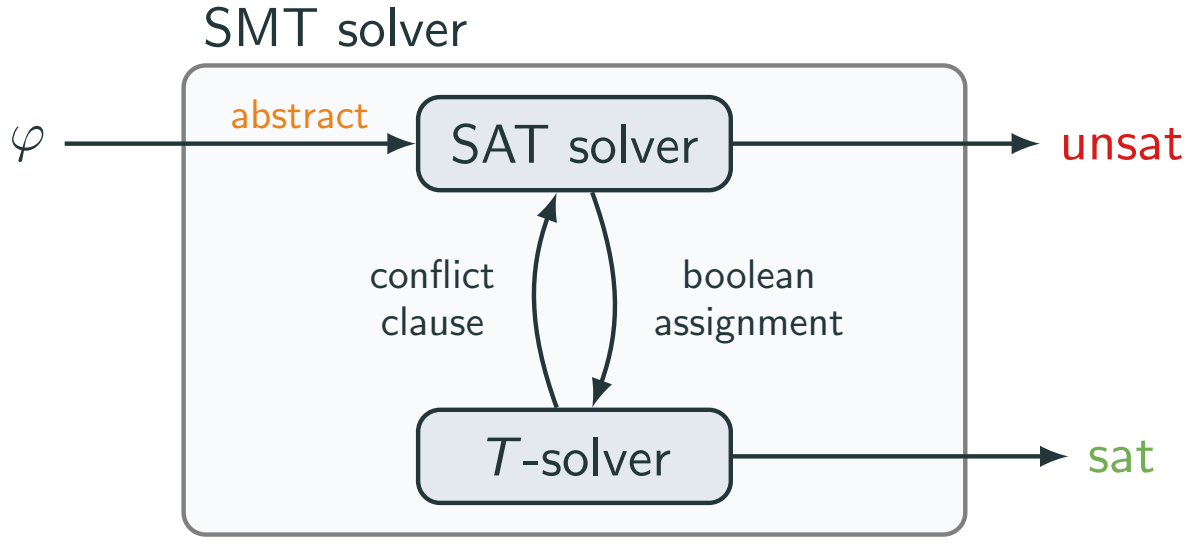

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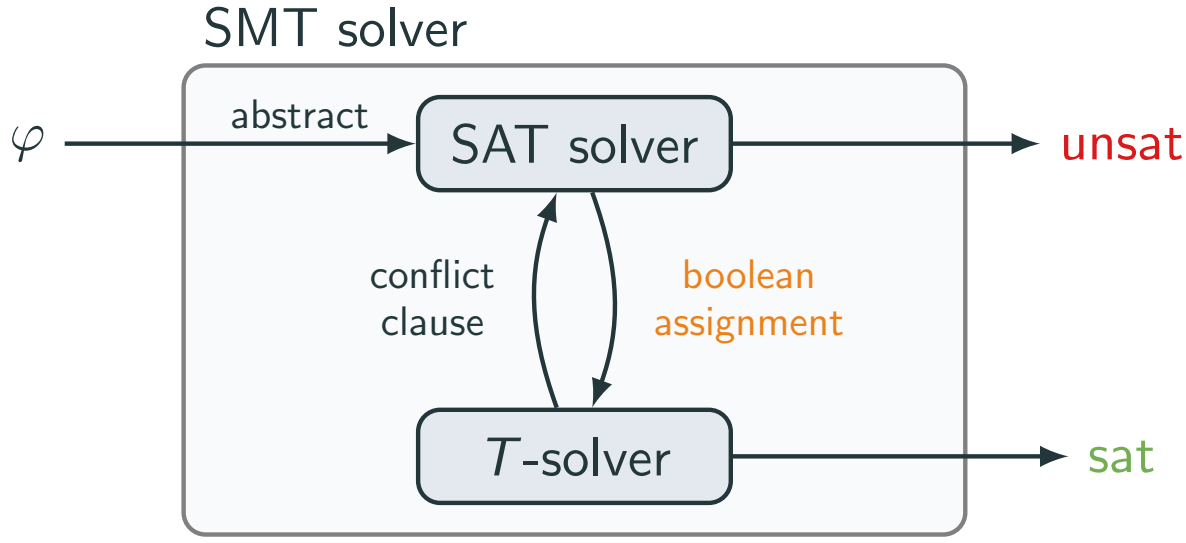
- abstract to $p_{a=b} \wedge p_{\mathrm{g}(a)=a} \wedge \left(p_{\mathrm{f}(a)\neq\mathrm{f}(b)} \vee p_{b\neq\mathrm{g}(\mathrm{g}(a))}\right)$



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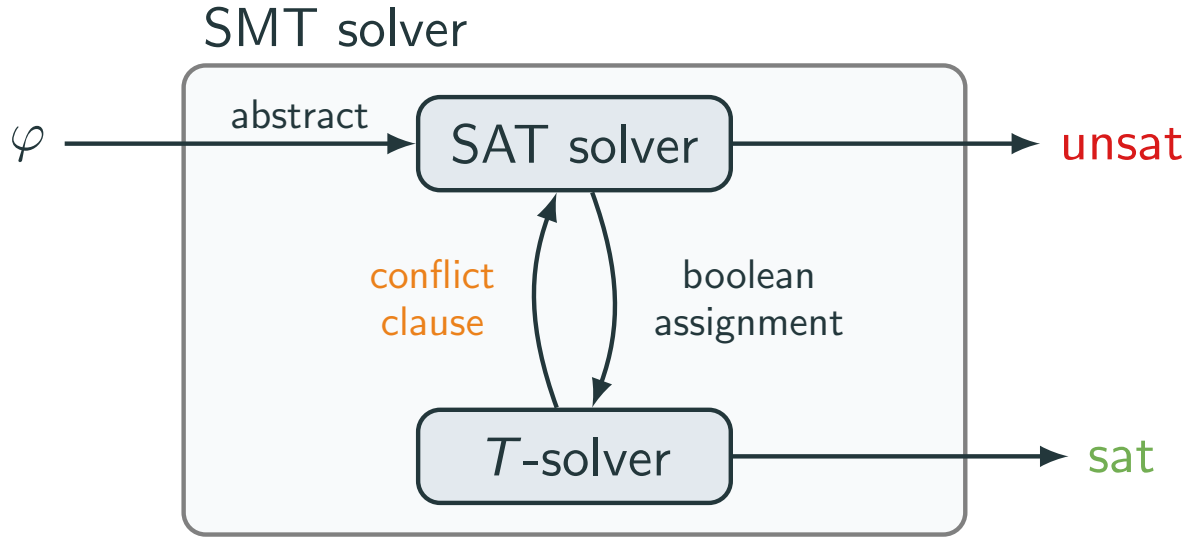

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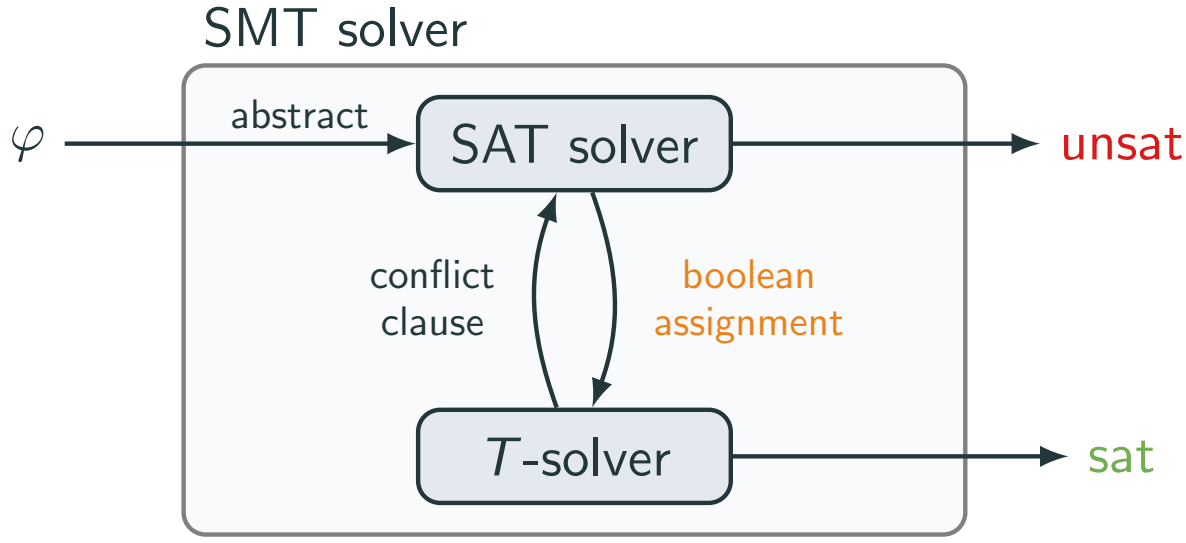

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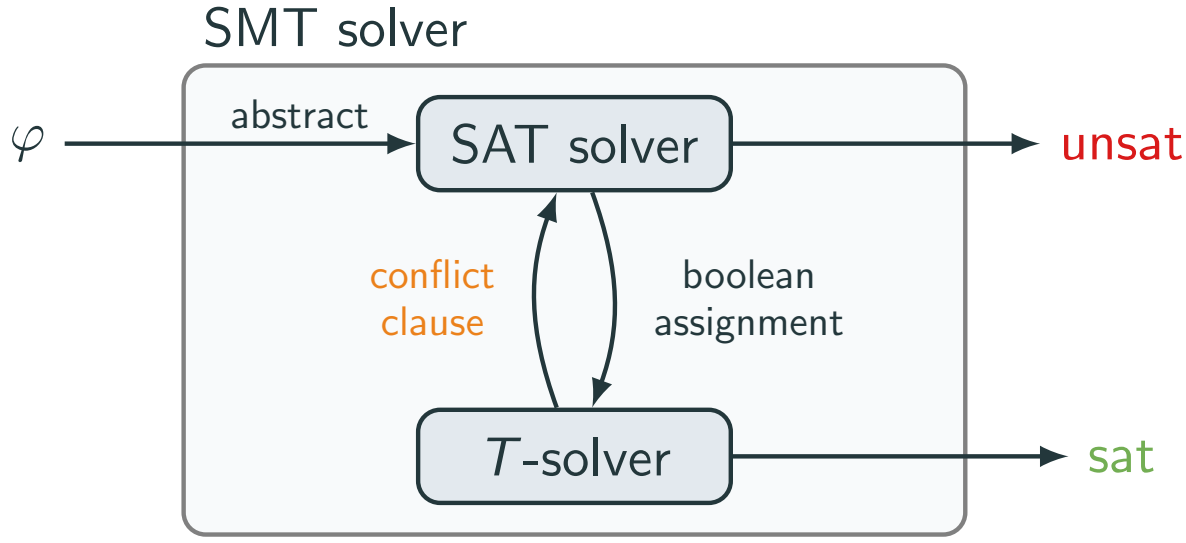

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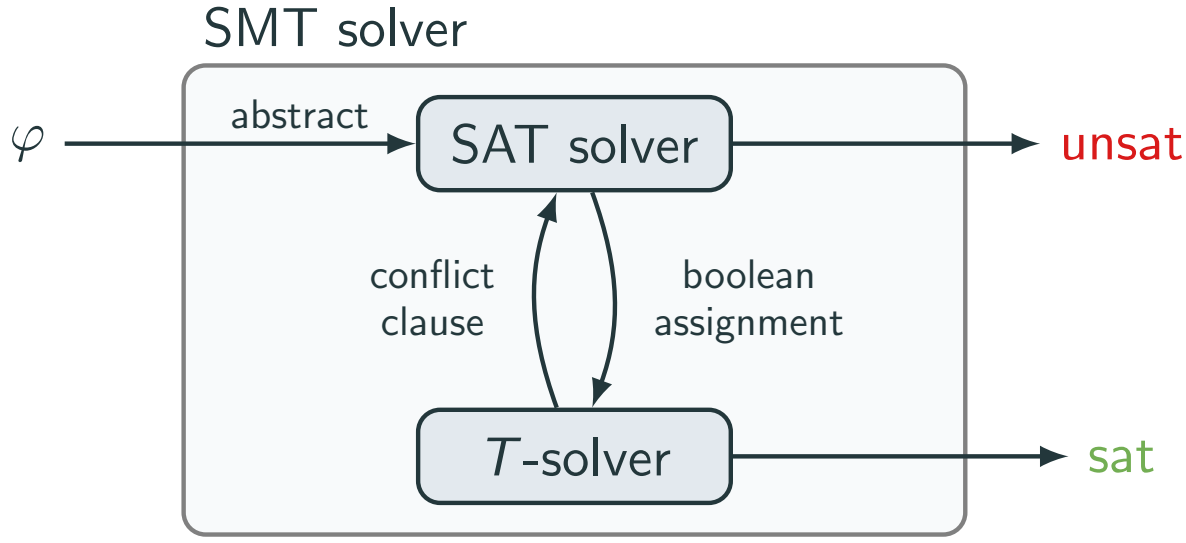

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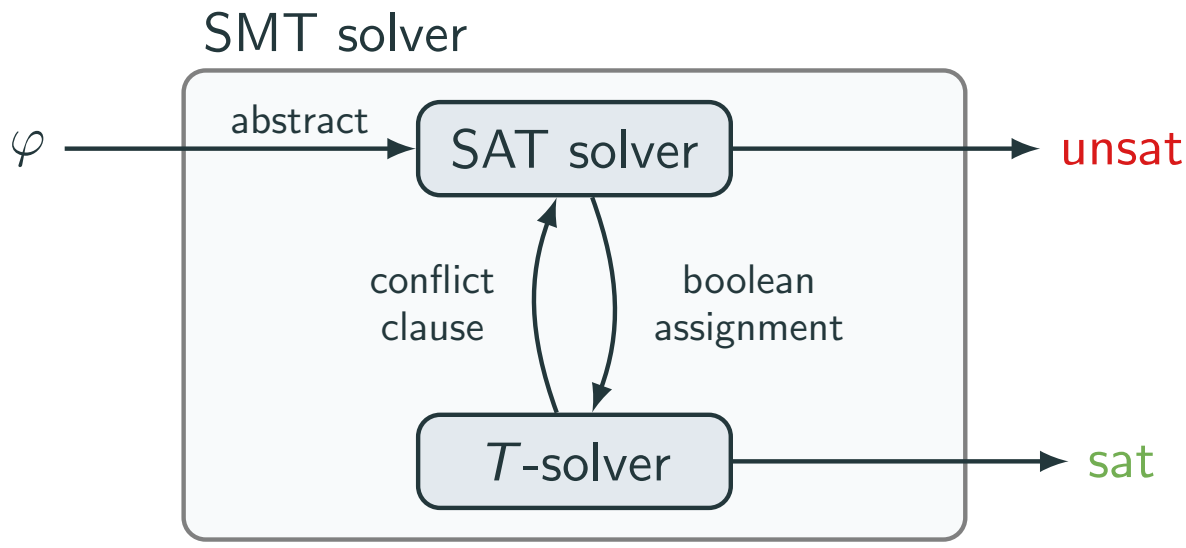
- SAT solver: unsat



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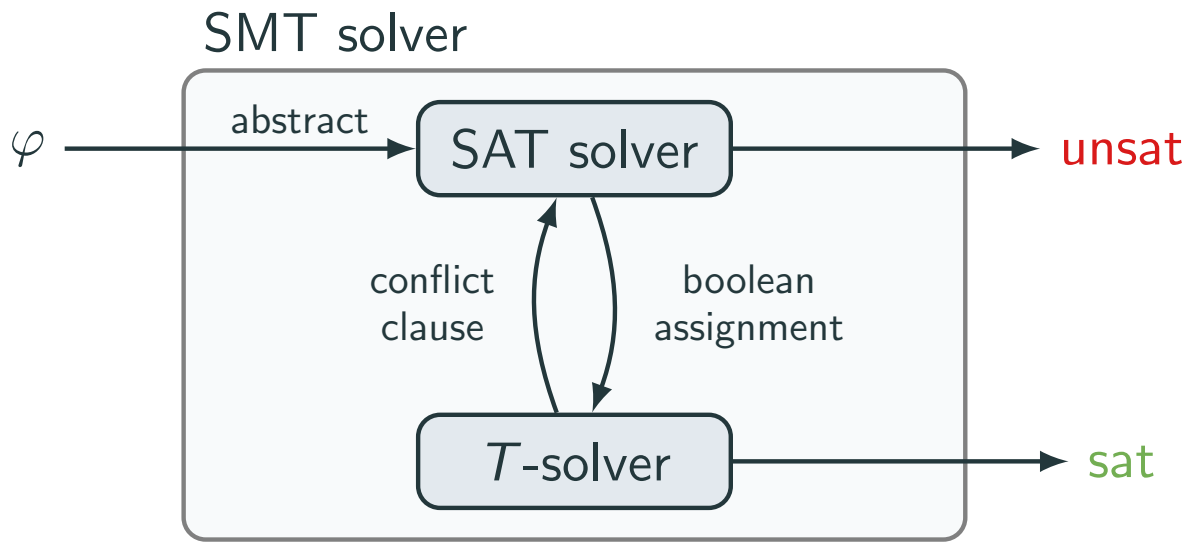

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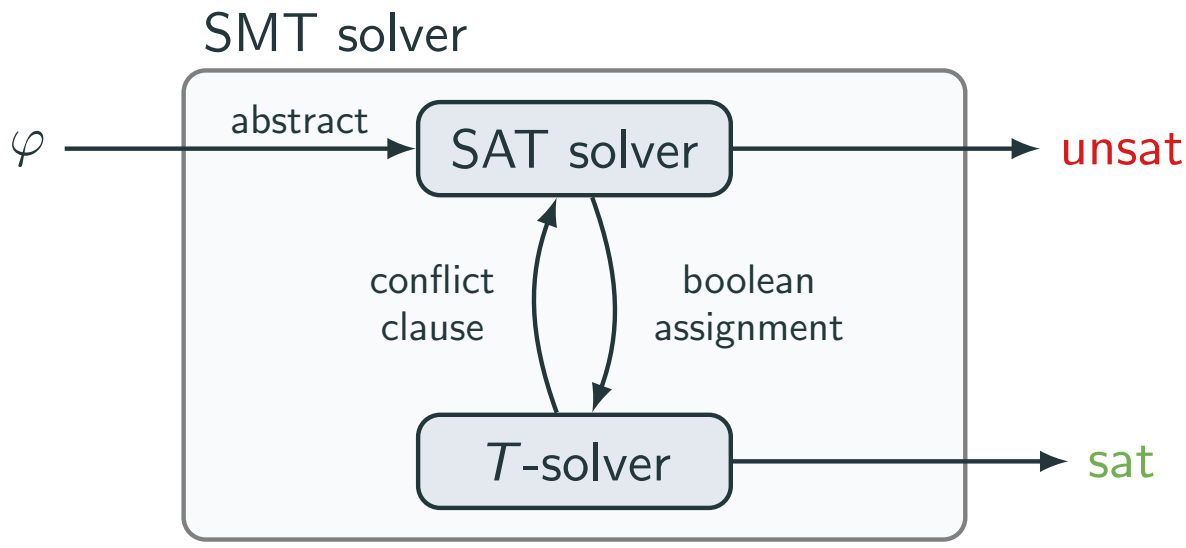

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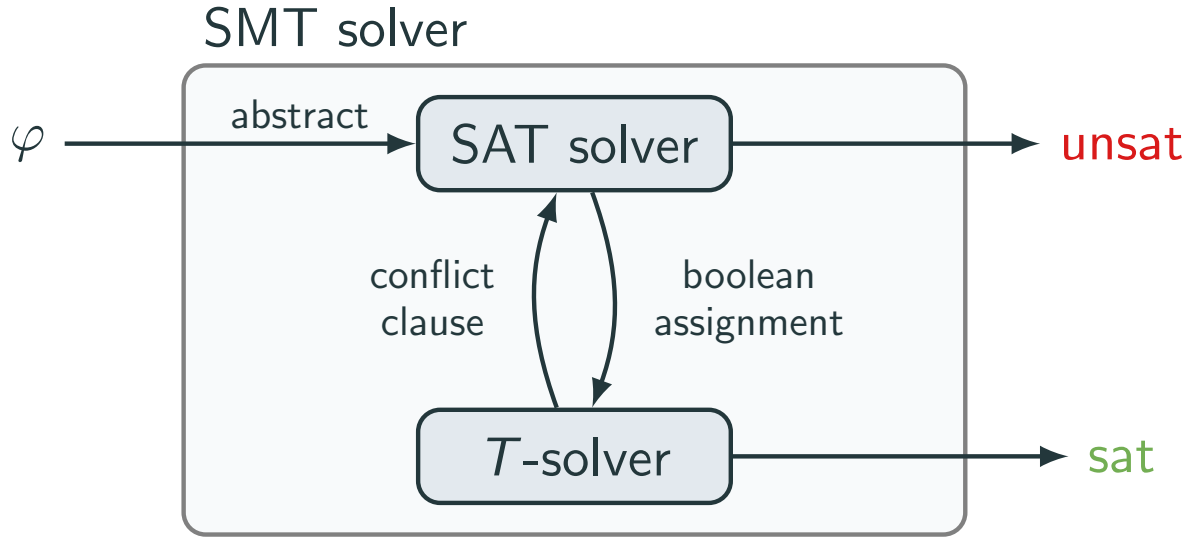

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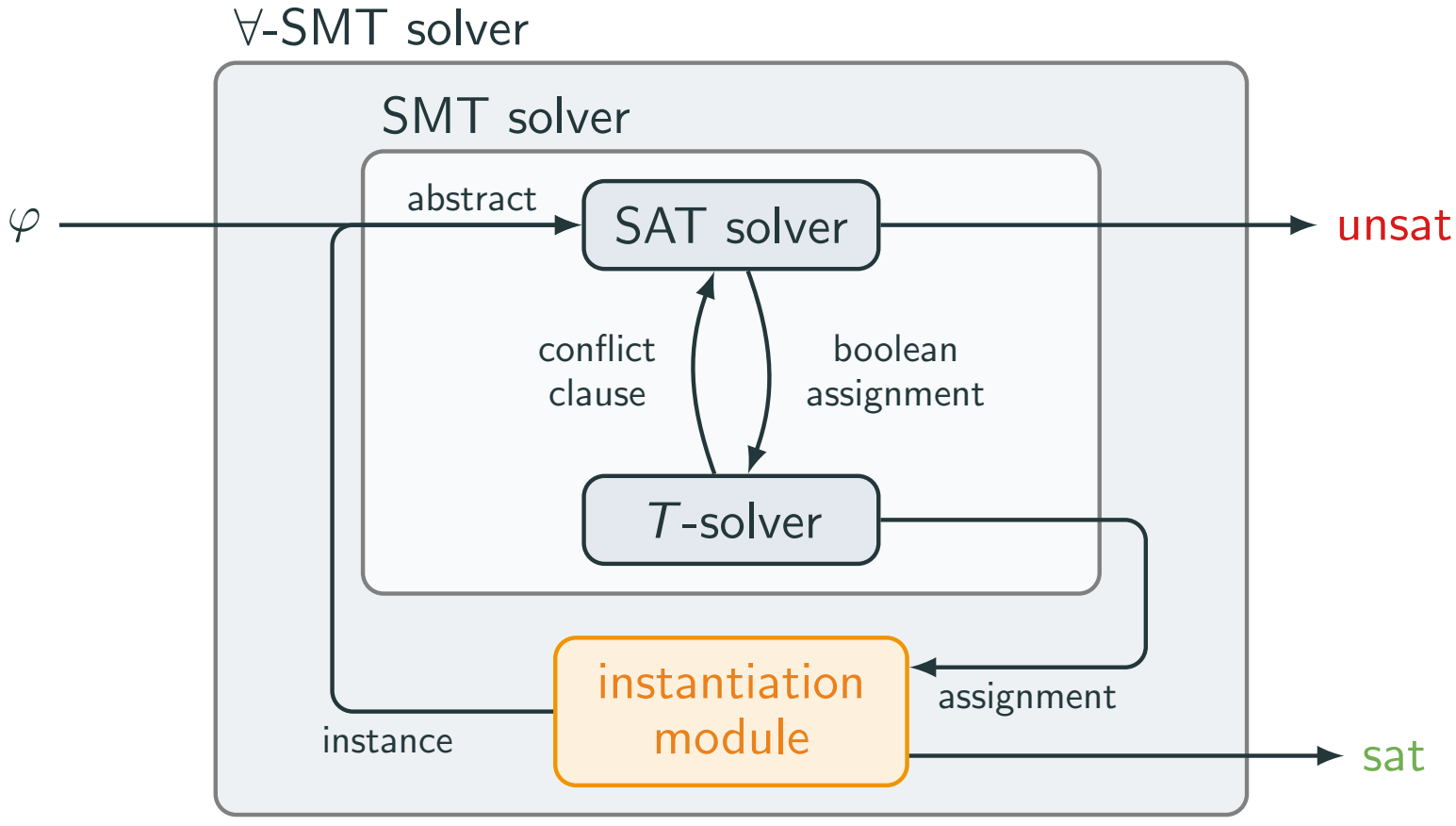

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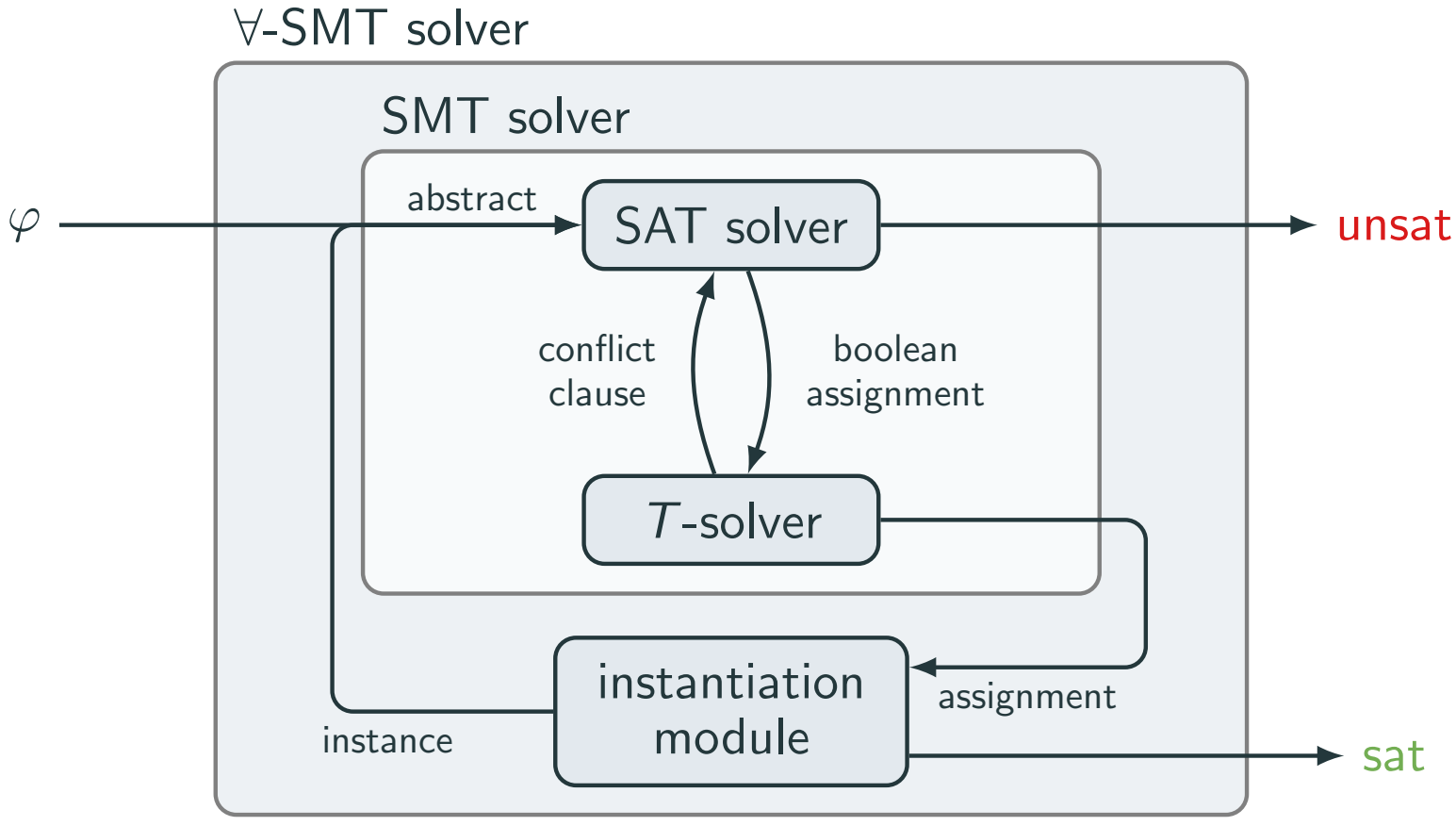

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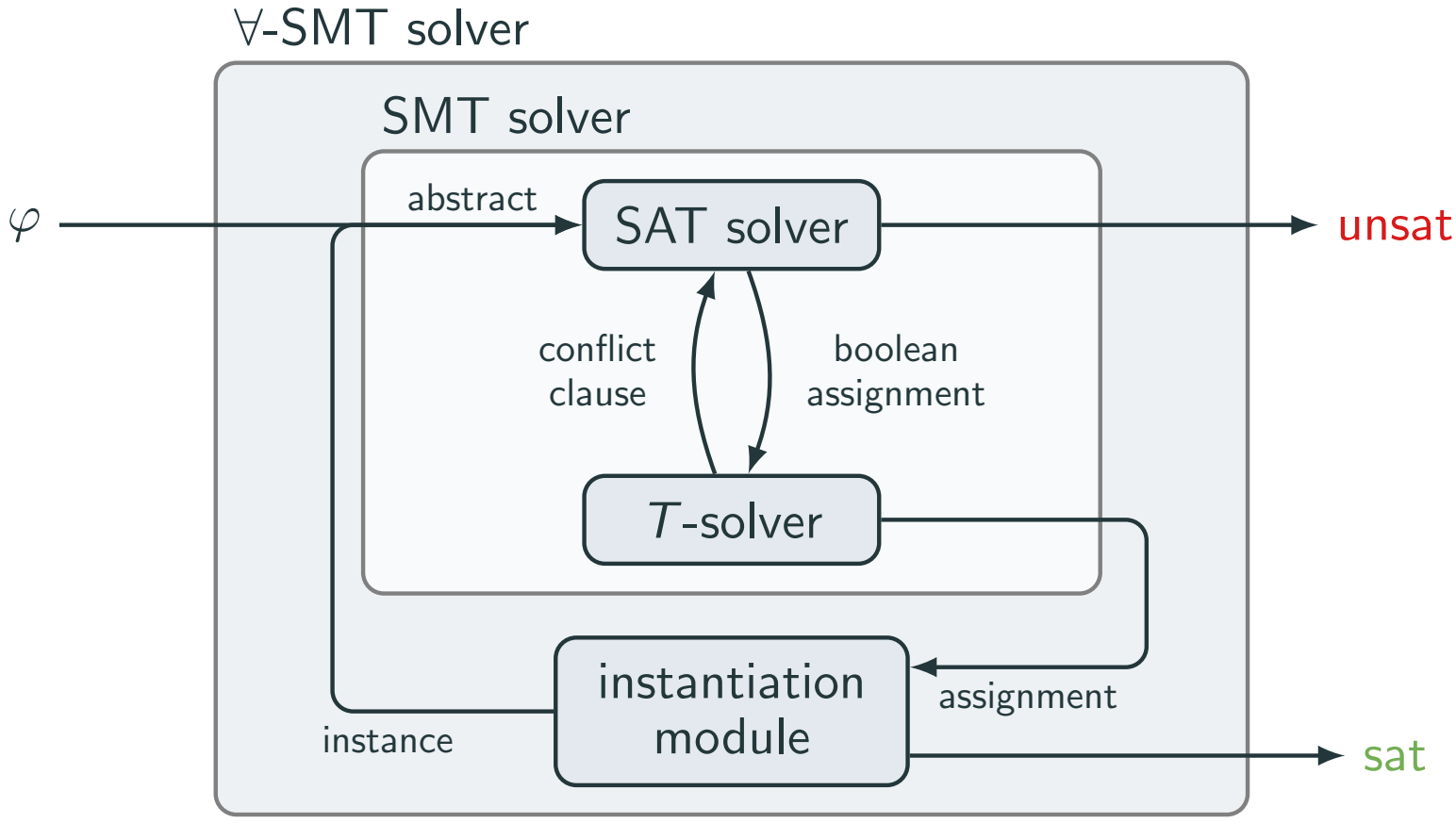

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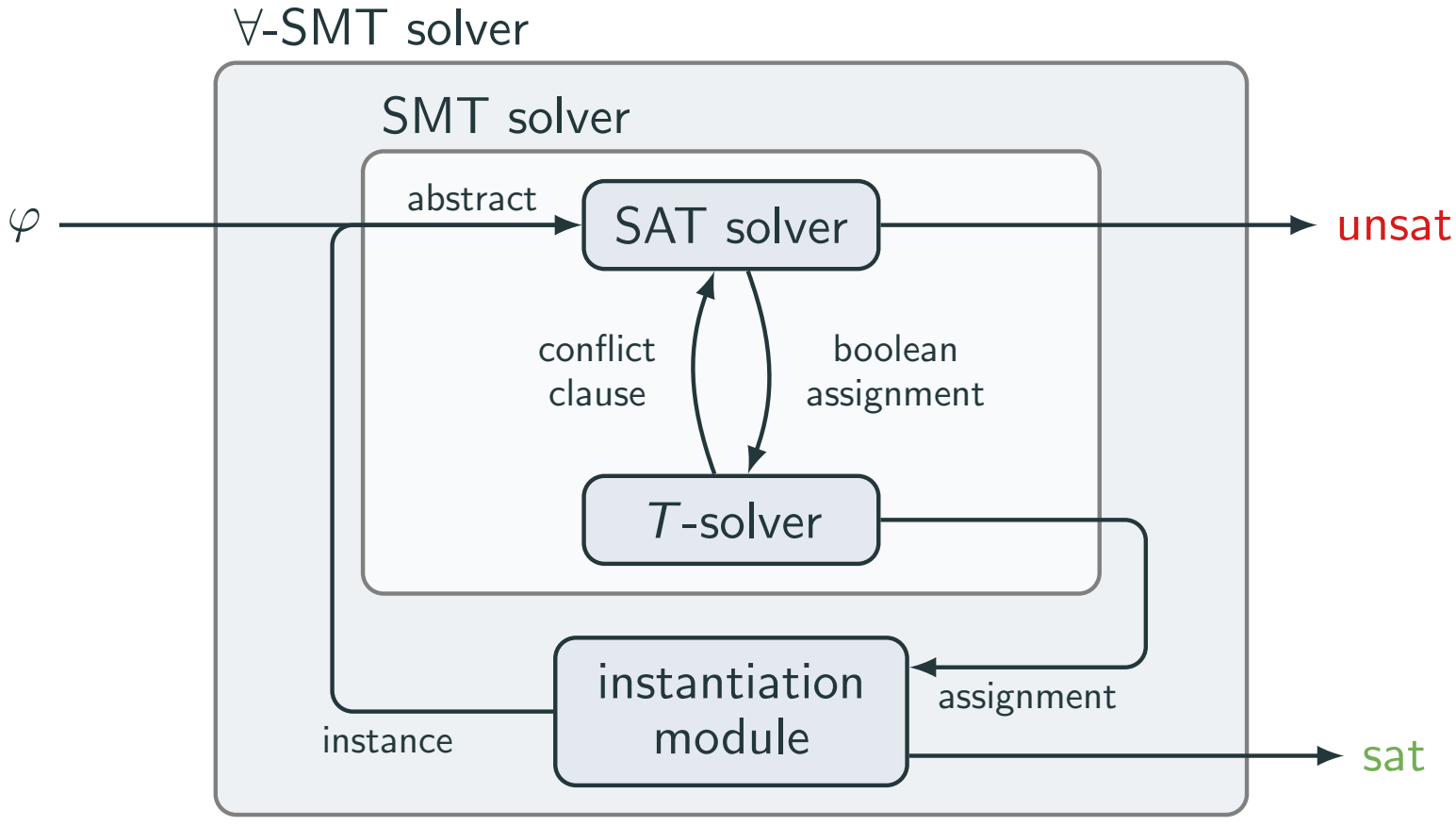
want $a = \mathrm{g}(a)$ whenever $p_{\forall x.x=\mathrm{g}(x)}$ true



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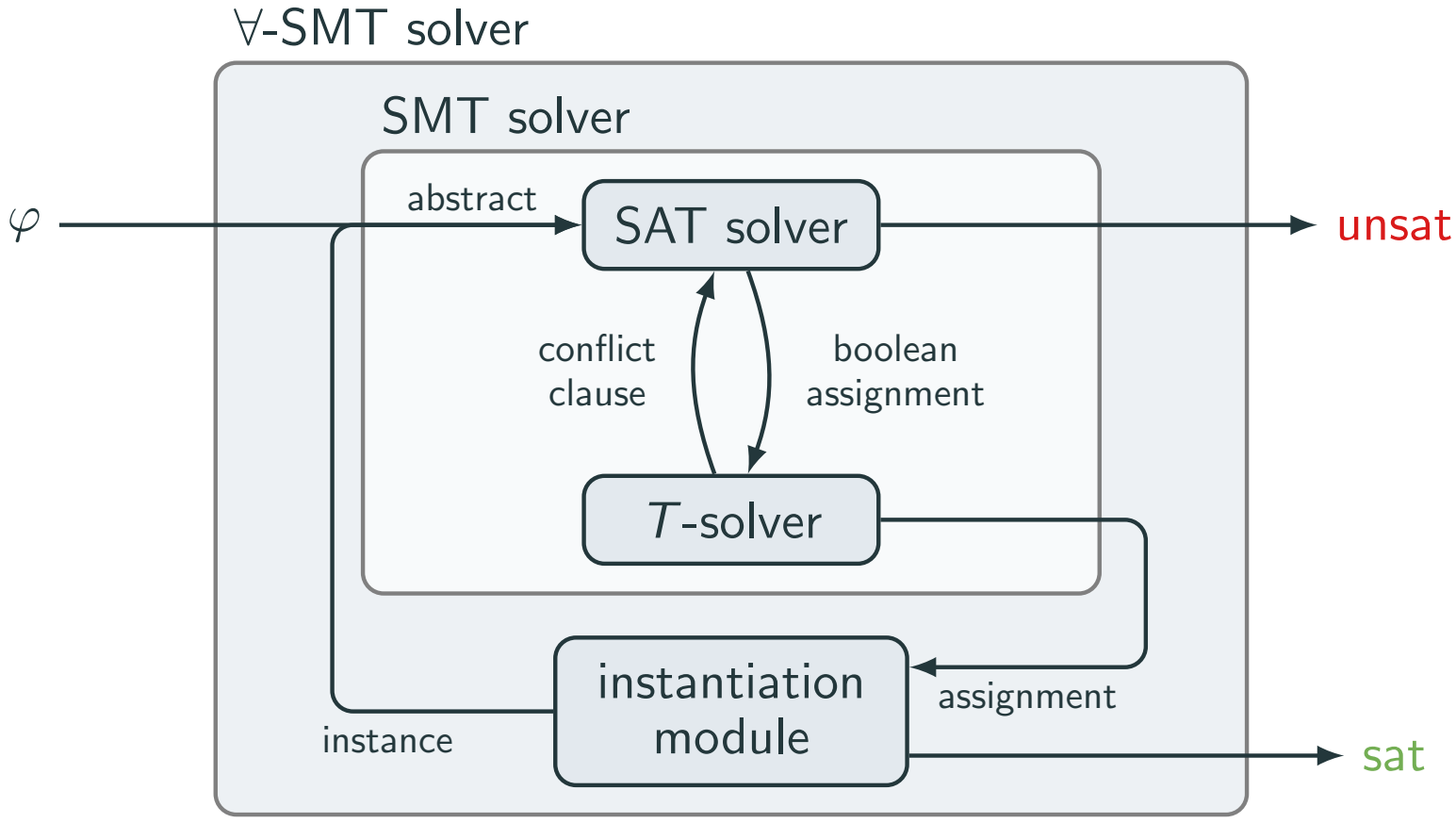

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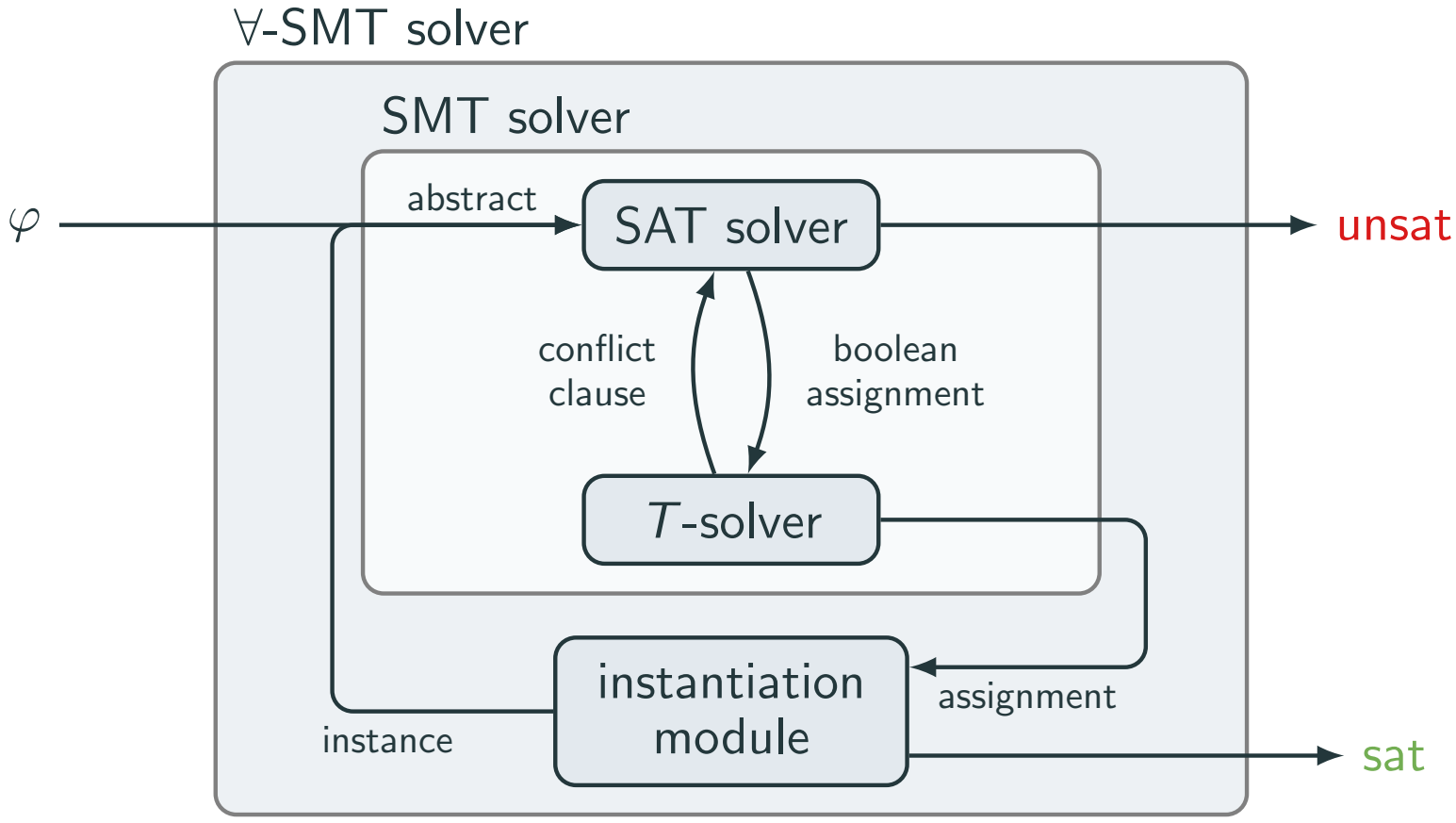

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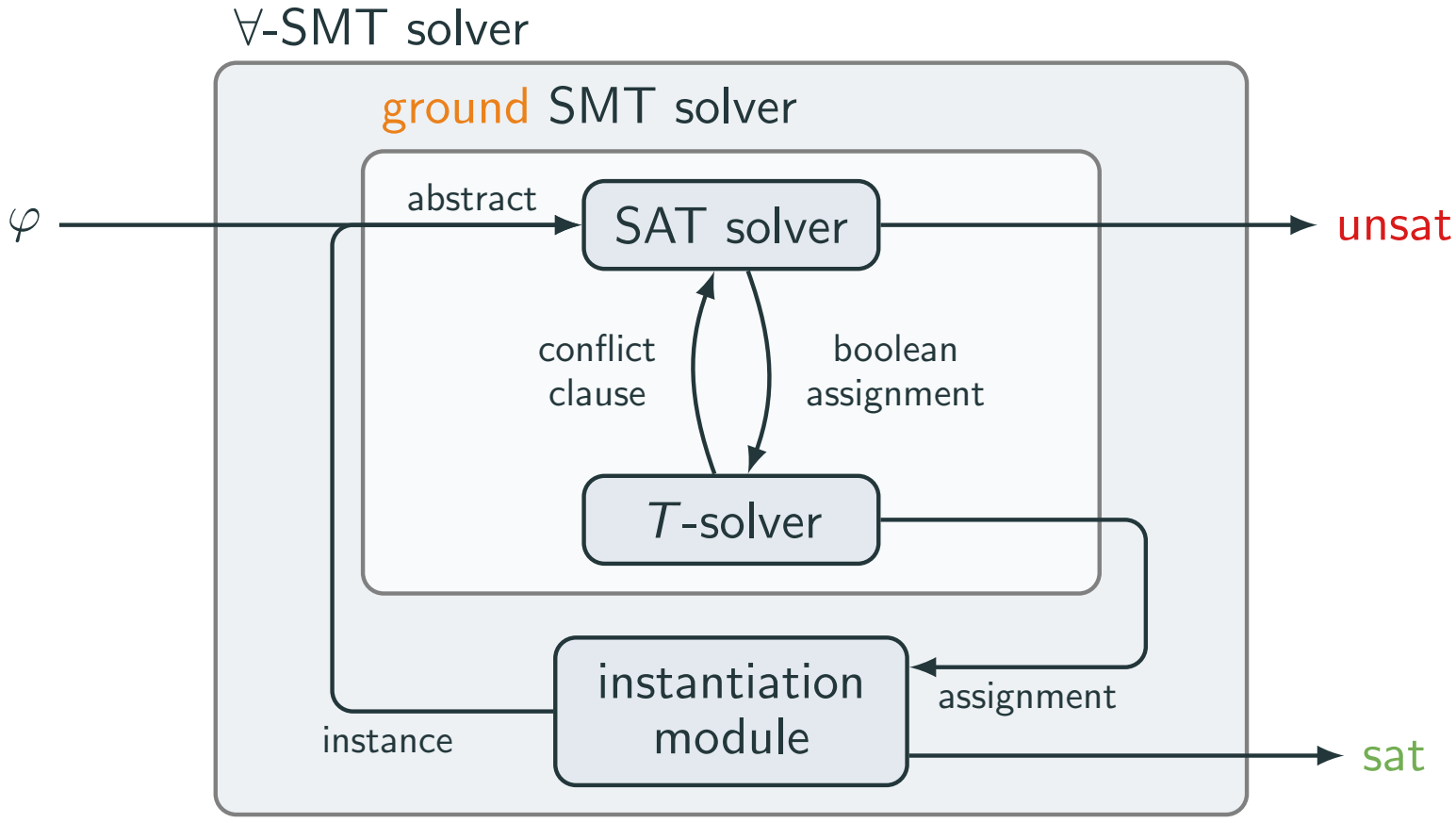

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Definition (Instance)

$$(\forall \overline{x}\ \varphi(\overline{x})) \longrightarrow \varphi\sigma$$

is instance where $\overline{x}\sigma$ does not contain variables $\overline{x}$

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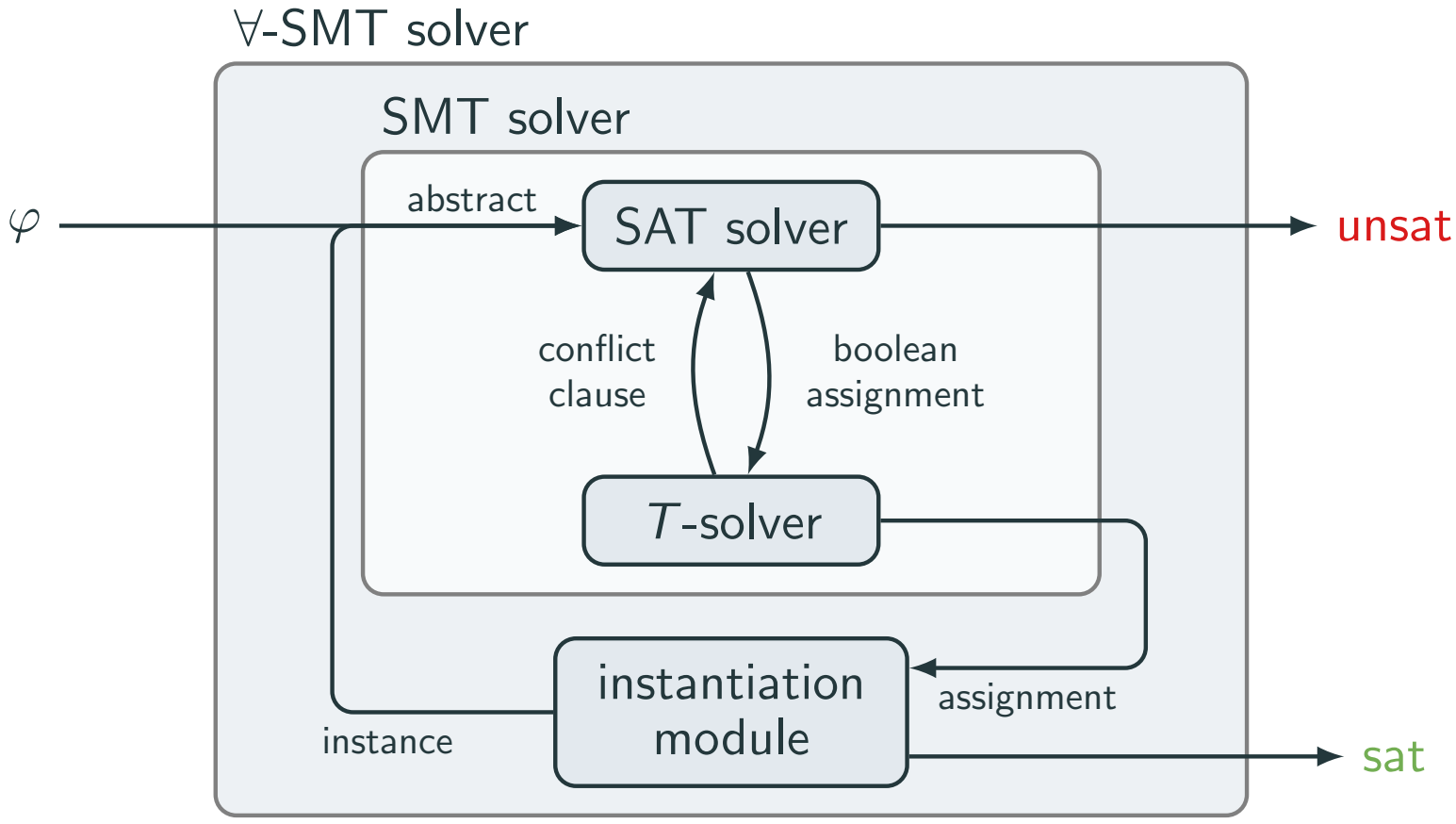
- $\varphi\sigma$ gets abstracted to propositional formula: involved variables have meaning for theory solver
- idea: $\varphi\sigma$ gets "activated" if propositional variable $p_{\forall \overline{x}\ \varphi(\overline{x})}$ is assigned true

Instantiation Framework



- split φ into
  - literals φ_Q with quantifiers
  - literals φ_E without quantifiers
- instantiation module generates instances of φ_Q to extend φ_E

E-Matching

Example

$$\varphi_E : \neg P(a),\ \neg P(b),\ \neg R(b)$$

$$\varphi_Q : \forall x.\ P(x) \vee R(x)$$

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trigger

- assume literal $P(x)$ is instantiation pattern
- find substitutions σ such that $P(x)\sigma$ occurs in φ_E — matching

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Instantiation via E-matching

for each $\forall \overline{x}.\psi$

- select set of instantiation patterns $\{t_1, \ldots, t_n\}$
- for each t_i let S_i be set of substitutions σ such that $t_i\sigma$ occurs in φ_E
- add $\{\psi\sigma \mid \sigma \in S_i\}$ to φ_E

Example

$$\forall x \forall y.\, \text{sibling}(x, y) \longleftrightarrow \text{mother}(x) = \text{mother}(y) \wedge \text{father}(x) = \text{father}(y)$$

$$\text{sibling}(\text{adam}, \text{bea})$$

$$\text{sibling}(\text{bea}, \text{chris})$$

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Remarks

- works as decision procedure for some theories (e.g., lists and arrays)
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- mostly efficient

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- suitable instantiation patterns?
  $\text{sibling}(x, y)$ sufficient

Remarks

- works as decision procedure for some theories (e.g., lists and arrays)
  but can easily omit necessary instances in other cases
- mostly efficient
- requires instantiation patterns (manually or heuristically determined)

Example

$$\forall x \forall y.\ \text{sibling}(x, y) \longleftrightarrow \text{mother}(x) = \text{mother}(y) \wedge \text{father}(x) = \text{father}(y)$$

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- instantiation patterns can be specified in SMT-LIB 🪄

Outline

- Summary of Last Week
- Quantifiers for SMT
- Instantiation Techniques
  - E-Matching
  - Enumerative Instantiation

Enumerative Instantiation

Why not use Herbrand's theorem directly?

Theorem (Herbrand)

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Enumerative instantiation

- instantiation module based on stronger version of Herbrand's theorem
- efficient implementation techniques

Theorem (Stronger Herbrand)

$\varphi_E \wedge \varphi_Q$ is unsatisfiable if and only if there exist infinite series

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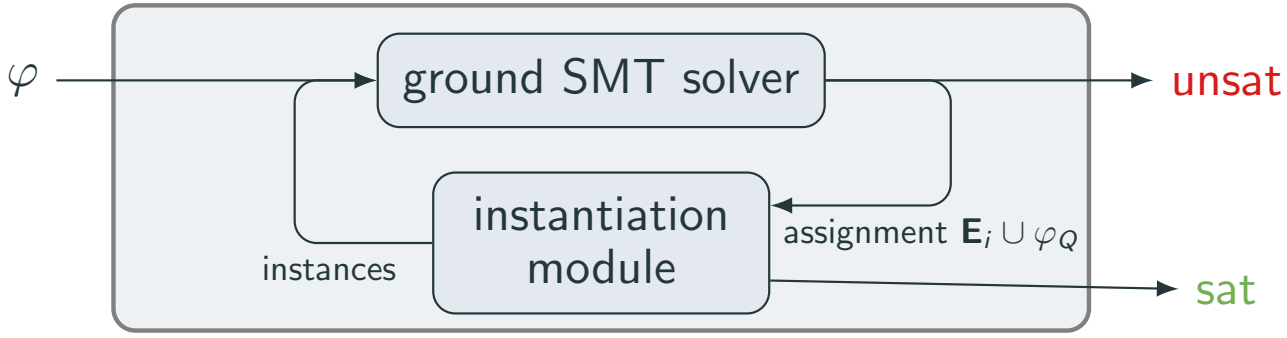
Direct application in ∀-SMT solver



- ground solver enumerates assignments $\mathbf{E}_i \cup \varphi_Q$
- instantiation returns $\forall \overline{x}\ \psi(\overline{x}) \longrightarrow Q$ for all $Q \in \mathbf{Q}_i$ generated from $\forall \overline{x}\ \psi(\overline{x})$

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then $\varphi_E \wedge \varphi_Q$ is satisfiable

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Fix ordering $>$ on tuples of terms without quantified variables.

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- ground solver: unsat

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Slide material partially taken from Pascal Fontaine's talk at SMT Summer School 2018.