Computational Logic	SS 2017	LVA 703607
EXAM 1		May 8, 2017

This exam consists of five exercises. The available points for each item are written in the margin. You need at least 50 points to pass.

[20] 1 Complete the following table:

formula	$\alpha/\beta/\gamma/\delta$	rank	satisfiable
$A \supset \neg B$		1	
$(\forall x)[P(x) \vee Q(x)] \supset [(\exists x)Q(x) \vee (\forall y)P(y)]$			
$\neg(A \supset \neg(B \vee \neg A))$			✓
$(\exists x)[(\forall y)R(f(x), y) \supset R(x, c)]$			

[15] 2 Answer **three** of the following five questions.

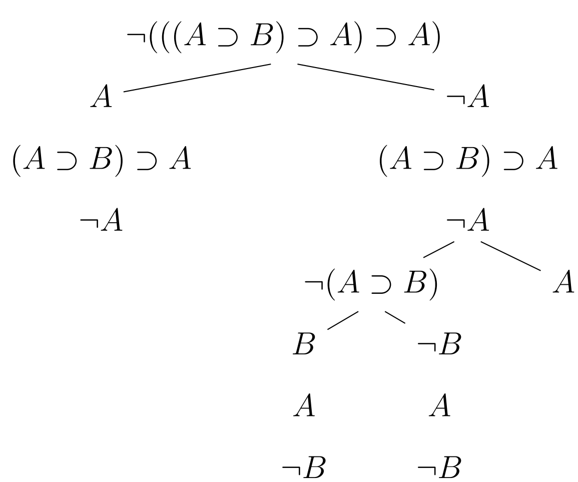
(a) What is an *alternate first-order consistency property*?

(b) Compute an *interpolant* of the valid sentence $(P(c) \wedge (\forall x)[P(x) \supset \neg Q(x)]) \supset \neg Q(c)$ using the procedure based on biased tableaux.

(c) Prove the following statement about *Kripke models*: If $\Vdash \varphi \vee \psi$ then $\Vdash \varphi$ or $\Vdash \psi$.

(d) Define the *Herbrand expansion* of an arbitrary sentence X over the Herbrand domain $D = \{t_1, t_2, t_3\}$.

(e) Transform the following tableau into a *cut-free* tableau using the cut-elimination procedure from the lecture:



3 Consider the propositional formula $\varphi = P \supset \neg(P \supset \bot)$.

[5] (a) Give a tableau proof of φ.

[10] (b) Give a proof of φ in the Hilbert system with the axioms

1	$X \supset (Y \supset X)$	2	$(X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z))$
3	$\bot \supset X$	4	$X \supset \top$
5	$\neg\neg X \supset X$	6	$X \supset (\neg X \supset Y)$
7	$\alpha \supset \alpha_1$	8	$\alpha \supset \alpha_2$
9	$(\beta_1 \supset X) \supset ((\beta_2 \supset X) \supset (\beta \supset X))$		

and Modus Ponens as only rule of inference. (You may use the Deduction Theorem.)

[10] (c) Give a tableau proof of the sentence $(\exists x)(\forall y)[P(x) \supset P(y)]$.

4 This exercise is about the propositional model existence theorem.

[2] (a) What is a propositional consistency property?

[6] (b) Give three examples of propositional consistency properties.

[2] (c) State the propositional model existence theorem.

[15] (d) Complete the following proof of the statement that every subset closed propositional consistency property $\mathcal{C}$ can be extended to a propositional consistency property of finite character, by filling in the missing parts.

Let $\mathcal{C}^+ = \{S \mid$ ______ **1** $\}$. We prove the following three properties:

(a) $\mathcal{C}^+$ is ______ **2**,

(b) $\mathcal{C}^+$ is ______ **3**,

(c) $\mathcal{C}^+$ is ______ **4**.

We start with property (a). Let $S \in \mathcal{C}$ and let F be ______ **5**. Because $\mathcal{C}$ is subset closed, $F \in \mathcal{C}$. Hence $S \in \mathcal{C}^+$ by definition. Next we consider property (b). So let $S \in \mathcal{C}^+$.

i. If A and $\neg A$ belong to S for some propositional letter A, then $\{A, \neg A\}$ is a finite subset of S and thus $\{A, \neg A\}$ is an element of $\mathcal{C}$. This contradicts the assumption that ______ **6**. Hence S does not contain both A and $\neg A$.

ii. If $\bot \in S$ then $\{\bot\}$ is a finite subset of S and thus $\{\bot\} \in \mathcal{C}$, contradicting the assumption that $\mathcal{C}$ is a propositional consistency property. Hence $\bot \notin S$. The same reasoning shows that ______ **7**.

iii. Suppose $\neg\neg Z \in S$ and consider an arbitrary finite subset F of $S \cup \{Z\}$. We have to show $F \in \mathcal{C}$ to obtain ______ **8**. Clearly, $F \cap S$ is a finite subset of S. Hence also $(F \cap S) \cup \{\neg\neg Z\}$ is a finite subset of S. Since $S \in \mathcal{C}^+$, $(F \cap S) \cup \{\neg\neg Z\} \in \mathcal{C}$ by the definition of $\mathcal{C}^+$. Since $\square$ **9**, we have $(F \cap S) \cup \{\neg\neg Z, Z\} \in \mathcal{C}$. Since F is a subset of $(F \cap S) \cup \{\neg\neg Z, Z\}$ and $\mathcal{C}$ is subset closed, it follows that $F \in \mathcal{C}$.

iv. Suppose $\alpha \in S$ and consider an arbitrary finite subset F of $S \cup \{\alpha_1, \alpha_2\}$. We have to show $F \in \mathcal{C}$ to obtain $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}^+$. Clearly, $F \cap S$ is finite subset of S. Hence also $(F \cap S) \cup \{\alpha\}$ is a finite subset of S. Since $S \in \mathcal{C}^+$, $(F \cap S) \cup \{\alpha\} \in \mathcal{C}$ by the definition of $\mathcal{C}^+$. Since $\mathcal{C}$ is a propositional consistency property, we have $(F \cap S) \cup \{\alpha, \alpha_1, \alpha_2\} \in \mathcal{C}$. Since F is a subset of $(F \cap S) \cup \{\alpha, \alpha_1, \alpha_2\}$ and $\mathcal{C}$ is subset closed, it follows that $F \in \mathcal{C}$.

v. Suppose $\beta \in S$. We have to show that $\square$ **10**. For a proof by contradiction, suppose that neither $S \cup \{\beta_1\}$ nor $S \cup \{\beta_2\}$ belongs to $\mathcal{C}^+$. By definition of $\mathcal{C}^+$, there exist finite subsets F_1 of $S \cup \{\beta_1\}$ and F_2 of $S \cup \{\beta_2\}$ such that $F_1, F_2 \notin \mathcal{C}$. Let $F =$ $\square$ **11**. Clearly $F \cap S$ is a finite subset of S. Since $\beta \in S$, $(F \cap S) \cup \{\beta\}$ is a finite subset of S. Since $S \in \mathcal{C}^+$, $(F \cap S) \cup \{\beta\} \in \mathcal{C}$. Because $\mathcal{C}$ is a propositional consistency property, we have $(F \cap S) \cup \{\beta, \beta_1\} \in \mathcal{C}$ or $(F \cap S) \cup \{\beta, \beta_2\} \in \mathcal{C}$. Note that $F_1 \subseteq (F \cap S) \cup \{\beta, \beta_1\}$ and $F_2 \subseteq (F \cap S) \cup \{\beta, \beta_2\}$. Since $\square$ **12**, we have $F_1 \in \mathcal{C}$ or $F_2 \in C$, providing the desired contradiction.

It remains to show property (c). So we need to show that $S \in \mathcal{C}^+$ if and only if $\square$ **13**. For the "if" direction, suppose every finite subset F of S belongs to $\mathcal{C}^+$. By definition of $\mathcal{C}^+$, every finite subset of F belongs to $\mathcal{C}$. Since F is $\square$ **14**, F belongs to $\mathcal{C}$. Hence $S \in \mathcal{C}^+$. For the "only if" direction, suppose $S \in \mathcal{C}^+$. So every finite subset of S belongs to $\mathcal{C}$. Since $\mathcal{C} \subseteq \mathcal{C}^+$ according to $\square$ **15**, every finite subset of S belongs to $\mathcal{C}^+$.

[15] $\boxed{5}$ Determine whether the following statements are true or false. Every correct answer is worth 3 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

(a) The rank of the first-order sentence $(\forall x)(\neg P(x) \supset \neg(\exists y)\neg P(f(y)))$ is 5.

(b) A set S of first-order sentences is satisfiable if and only if every finite subset of S is satisfiable.

(c) The simple type $((\sigma \to \tau) \to \sigma) \to \sigma$ is inhabited by a combinatory term.

(d) If $\mathcal{C}$ is a first-order consistency property and $\delta \in S \in \mathcal{C}$ then $S \cup \{\delta(t)\} \in \mathcal{C}$ for every closed term t of $L^{\mathbf{par}}$.

(e) The formula $(\exists x)R(x, f(b))$ is an interpolant of

$$(R(a,b) \wedge (\forall x)(\exists y)(R(a,x) \supset R(y, f(x)))) \supset (R(a,c) \vee (\exists x)R(x, f(b)))$$