Computational Logic — SS 2017 — LVA 703607

EXAM 1 – SOLUTIONS — May 8, 2017

1

formula	$\alpha/\beta/\gamma/\delta$	rank	satisfiable
$A \supset \neg B$	β	1	✓
$(\forall x)[P(x) \vee Q(x)] \supset [(\exists x)Q(x) \vee (\forall y)P(y)]$	β	6	✓
$\neg(A \supset \neg(B \vee \neg A))$	α	3	✓
$(\exists x)[(\forall y)R(f(x), y) \supset R(x, c)]$	δ	3	✓

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(a) An alternate first-order consistency property is collection $\mathcal{C}$ of sets of sentences of $L^{\mathbf{par}}$ such that for each $S \in \mathcal{C}$ the following properties hold:

(1) for any propositional letter A, not both $A \in S$ and $\neg A \in S$,

(2) $\bot \notin S$, $\neg\top \notin S$,

(3) if $\neg\neg Z \in S$ then $S \cup \{Z\} \in \mathcal{C}$,

(4) if $\alpha \in S$ then $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$,

(5) if $\beta \in S$ then $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$.

(6) if $\gamma \in S$ then $S \cup \{\gamma(t)\} \in \mathcal{C}$ for every closed term t of $L^{\mathbf{par}}$

(7) if $\delta \in S$ then $S \cup \{\delta(p)\} \in \mathcal{C}$ for every parameter p that is new to S

(b) The closed tableau

$$\neg((P(c) \wedge (\forall x)[P(x) \supset \neg Q(x)]) \supset \neg Q(c))$$
$$P(c) \wedge (\forall x)[P(x) \supset \neg Q(x)]$$
$$\neg\neg Q(c)$$
$$Q(c)$$
$$P(c)$$
$$(\forall x)[P(x) \supset \neg Q(x)]$$
$$P(c) \supset \neg Q(c)$$
$$\begin{array}{cc} \swarrow & \searrow \\ \neg P(c) & \neg Q(c) \end{array}$$

is transformed into the following closed biased tableau:

$$\begin{array}{c}
L(P(c) \land (\forall x)[P(x) \supset \neg Q(x)]) \\
R(\neg\neg Q(c)) \\
R(Q(c)) \\
L(P(c)) \\
L((\forall x)[P(x) \supset \neg Q(x)]) \\
L(P(c) \supset \neg Q(c)) \\
\begin{array}{cc} \diagup & \diagdown \\ L(\neg P(c)) & L(\neg Q(c)) \end{array}
\end{array}$$

Next we use the calculation rules for interpolants:

$$\begin{array}{c}
L(P(c) \land (\forall x)[P(x) \supset \neg Q(x)]) \\
R(\neg\neg Q(c)) \\
R(Q(c)) \\
L(P(c)) \\
L((\forall x)[P(x) \supset \neg Q(x)]) \\
L(P(c) \supset \neg Q(c)) \\
\begin{array}{cc} \diagup & \diagdown \\ L(\neg P(c)) & L(\neg Q(c)) \\ [\bot] & [\neg Q(c)] \end{array}
\end{array}$$

$$\begin{array}{c}
L(P(c) \land (\forall x)[P(x) \supset \neg Q(x)]) \\
R(\neg\neg Q(c)) \\
R(Q(c)) \\
L(P(c)) \\
L((\forall x)[P(x) \supset \neg Q(x)]) \\
L(P(c) \supset \neg Q(c)) \\
[\bot \lor \neg Q(c)]
\end{array}$$

$$\begin{array}{c}
L(P(c) \land (\forall x)[P(x) \supset \neg Q(x)]) \\
R(\neg\neg Q(c)) \\
R(Q(c)) \\
L(P(c)) \\
L((\forall x)[P(x) \supset \neg Q(x)]) \\
[\bot \lor \neg Q(c)]
\end{array}$$

$$\begin{array}{c}
L(P(c) \land (\forall x)[P(x) \supset \neg Q(x)]) \\
R(\neg\neg Q(c)) \\
R(Q(c)) \\
[\bot \lor \neg Q(c)]
\end{array}$$

$$\begin{array}{c}
L(P(c) \land (\forall x)[P(x) \supset \neg Q(x)]) \\
R(\neg\neg Q(c)) \\
[\bot \lor \neg Q(c)]
\end{array}$$

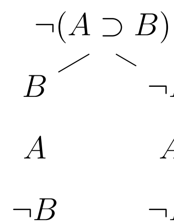
The resulting interpolant is $\bot \lor \neg Q(c)$.

(c) Suppose neither $\Vdash \varphi$ nor $\Vdash \psi$ holds. Then there exist Kripke models $\mathcal{C} = \langle C, \leqslant_C, \Vdash_C \rangle$ and $\mathcal{D} = \langle D, \leqslant_D, \Vdash_D \rangle$ such that $c \nVdash_C \varphi$ and $d \nVdash_D \psi$ for states $c \in C$ and $d \in D$. We assume without loss of generality that $C \cap D = \varnothing$. Define the Kripke model $\mathcal{E} = \langle E, \leqslant, \Vdash \rangle$ with $E = C \cup D \cup \{e\}$ where e is a new state, $\leqslant \; = \; \leqslant_C \cup \leqslant_D \cup \{(e,c),(e,d)\}$, and $\Vdash \; = \; \Vdash_C \cup \Vdash_D$. We claim that $e \nVdash \varphi \lor \psi$. Suppose to the contrary $e \Vdash \varphi \lor \psi$. Hence $e \Vdash \varphi$ or $e \Vdash \psi$. Using monotonicity we obtain $c \Vdash \varphi$ or $d \Vdash \psi$ and thus $c \Vdash_C \varphi$ or $d \Vdash_D \psi$ by the definition of $\leqslant$. This is a contradiction. Therefore $\varphi \vee \psi$ does not hold in state e of the Kripke model $\mathcal{E}$ and thus $\Vdash \varphi \vee \psi$ does not hold.

(d) The Herbrand expansion $\mathcal{E}(X, D)$ with $D = \{t_1, t_2, t_3\}$) is defined recursively:

- if L is literal then $\mathcal{E}(L, D) = L$,
- $\mathcal{E}(\neg\neg Z, D) = \mathcal{E}(Z, D)$,
- $\mathcal{E}(\alpha, D) = \mathcal{E}(\alpha_1, D) \wedge \mathcal{E}(\alpha_2, D)$,
- $\mathcal{E}(\beta, D) = \mathcal{E}(\beta_1, D) \vee \mathcal{E}(\beta_2, D)$,
- $\mathcal{E}(\gamma, D) = \mathcal{E}(\gamma(t_1), D) \wedge \mathcal{E}(\gamma(t_2), D) \wedge \mathcal{E}(\gamma(t_3), D)$,
- $\mathcal{E}(\delta, D) = \mathcal{E}(\delta(t_1), D) \vee \mathcal{E}(\delta(t_2), D) \vee \mathcal{E}(\delta(t_3), D)$.

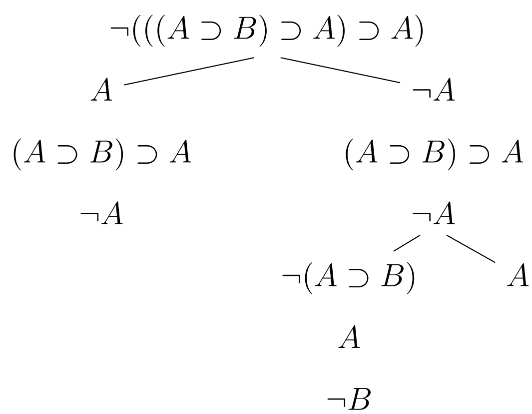
(e) The minimal cut



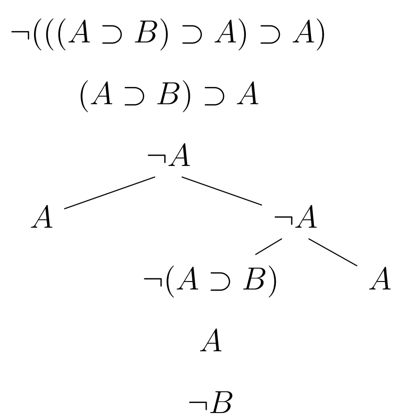
is transformed into

$$\neg(A \supset B)$$
$$A$$
$$\neg B$$

and hence we obtain the following tableau:



In the next cut-elimination step we obtain the following tableau:



Now there is a cut at a branch end, and eliminating it results in the following final cut-free tableau:

$$\neg(((A \supset B) \supset A) \supset A)$$

$$(A \supset B) \supset A$$

$$\neg A$$

$$\neg(A \supset B) \qquad\qquad A$$

$$A$$

$$\neg B$$

3 (a) $\neg(P \supset \neg(P \supset \bot))$

$$P$$

$$\neg\neg(P \supset \bot)$$

$$P \supset \bot$$

$$\neg P \qquad\qquad \bot$$

(b) First of all, using Modus Ponens, we have $P, P \supset \bot \vdash_{ph} \bot$ and thus $P \vdash_{ph} (P \supset \bot) \supset \bot$ using the Deduction Theorem. Second, the formula

$$(\neg(P \supset \bot) \supset \neg(P \supset \bot)) \supset ((\bot \supset \neg(P \supset \bot)) \supset (((P \supset \bot) \supset \bot) \supset \neg(P \supset \bot)))$$

is an instance of Axiom Scheme 9 (with $\beta = (P \supset \bot) \supset \bot$ and $X = \neg(P \supset \bot)$). From the lecture we know $\vdash_{ph} \neg(P \supset \bot) \supset \neg(P \supset \bot)$. Moreover $\bot \supset \neg(P \supset \bot)$ is an instance of Axiom Scheme 3. Hence we obtain $\vdash_{ph} ((P \supset \bot) \supset \bot) \supset \neg(P \supset \bot)$ by two applications of the Deduction Theorem. Combining this with $P \vdash_{ph} (P \supset \bot) \supset \bot$ and using Modus Ponens yields $P \vdash_{ph} \neg(P \supset \bot)$. A final application of the Deduction Theorem yields $\vdash_{ph} P \supset \neg(P \supset \bot)$.

(c) $\neg(\exists x)(\forall y)[P(x) \supset P(y)]$

$$\neg(\forall y)[P(c) \supset P(y)]$$

$$\neg[P(c) \supset P(d)]$$

$$P(c)$$

$$\neg P(d)$$

$$\neg(\forall y)[P(d) \supset P(y)]$$

$$\neg[P(d) \supset P(e)]$$

$$P(d)$$

$$\neg P(e)$$

4 (a) A propositional consistency property is a collection $\mathcal{C}$ of sets of propositional formulas such that for each $S \in \mathcal{C}$ the following properties hold:

(1) for any propositional letter A, not both $A \in S$ and $\neg A \in S$,
(2) $\bot \notin S$, $\neg\top \notin S$,
(3) if $\neg\neg Z \in S$ then $S \cup \{Z\} \in \mathcal{C}$,
(4) if $\alpha \in S$ then $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$,
(5) if $\beta \in S$ then $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$.

(b) For instance,

- the collection of all sets S such that every finite subset of S is satisfiable,
- the collection of all Craig consistent sets, where a finite set S is Craig consistent if $\langle S_1 \rangle \supset \neg \langle S_2 \rangle$ has no interpolant for some partition $S_1 \uplus S_2$ of S
- the collection of all tableau consistent sets,
- the collection of all X–tableau consistent sets, which are sets S such that $S \vdash_{pt} X$ does not hold,
- the collection of all X–Hilbert consistent sets, which are sets S such that $S \vdash_{ph} X$ does not hold.

(c) The propositional model existence theorem states that every member of a propositional consistency property is satisfiable.

(d)

1	all finite subsets of S belong to $\mathcal{C}$
2	an extension of $\mathcal{C}$
3	a propositional consistency property
4	of finite character
5	an arbitrary finite subset of S
6	$\mathcal{C}$ is a propositional consistency property
7	$\neg\top \notin S$
8	$S \cup \{Z\} \in \mathcal{C}^+$
9	$\mathcal{C}$ is a propositional consistency property
10	$S \cup \{\beta_1\} \in \mathcal{C}^+$ or $S \cup \{\beta_2\} \in \mathcal{C}^+$
11	$F_1 \cup F_2$

,

12	$\mathcal{C}$ is subset closed
13	every finite subset of S belongs to $\mathcal{C}^+$
14	a finite subset of F
15	property (a)

5

statement	true	false
(a)	X	
(b)	X	
(c)		X
(d)		X
(e)	X	