universität innsbruck

Computational Logic | SS 2017 | LVA 703607

EXAM 2 | June 25, 2018

This exam consists of four exercises. The available points for each item are written in the margin. You need 50 points to pass.

[10] **1** Complete the following table:

formula	$\alpha/\beta/\gamma/\delta$	rank	valid
$\neg\top \supset \bot$			
$\neg[(A \supset B) \wedge (C \uparrow (\neg A \wedge B))]$			$\times$
$(\forall x)[(\exists y)R(f(x,y),c) \supset (\exists z)S(y,z)]$			
$(\forall x)[P(x) \vee Q(x)] \supset [(\exists x)P(x) \vee (\forall y)Q(y)]$	β		

2 This exercise is about Craig interpolation for propositional logic.

[4] (a) What is a propositional interpolant?

[9] (b) Compute an interpolant for the implication $(\neg(A \wedge B) \supset (\neg C \wedge B)) \supset ((D \supset A) \vee (D \supset \neg C))$ using the procedure based on biased tableaux.

[4] (c) What is a Craig consistent set of (propositional) formulas?

(d) Complete the following proof of the statement that the collection of all Craig consistent sets is a propositional consistency property, by filling in the missing parts.

[2] Let $\mathcal{C}$ be the collection of all Craig consistent sets and let $S \in \mathcal{C}$, so ______ **1** has no interpolant for some partition $S_1 \uplus S_2$ of S. In the sequel we simply say that $S_1 \uplus S_2$ has no interpolant.

We show that S fulfills the five conditions for being a propositional consistency property.

(1) Suppose $A, \neg A \in S$. If $A, \neg A \in S_1$ then $\bot$ is an interpolant of $S_1 \uplus S_2$. If $A, \neg A \in S_2$ then $\top$
[1] is an interpolant of $S_1 \uplus S_2$. If $A \in S_1$ and $\neg A \in S_2$ then ______ **2** is an interpolant of
[1] $S_1 \uplus S_2$. If $\neg A \in S_1$ and $A \in S_2$ then ______ **3** is an interpolant of $S_1 \uplus S_2$.

(2) Suppose $\bot \in S$. If $\bot \in S_1$ then $\bot$ is an interpolant of $S_1 \uplus S_2$. If $\bot \in S_2$ then $\top$ is an interpolant of $S_1 \uplus S_2$. Suppose $\neg\top \in S$. If $\neg\top \in S_1$ then $\bot$ is an interpolant of $S_1 \uplus S_2$. If $\neg\top \in S_2$ then $\top$ is an interpolant of $S_1 \uplus S_2$.

[1] (3) Suppose $\neg\neg Z \in S$. If $\neg\neg Z \in S_1$ then $(S_1 \cup \{Z\}) \uplus S_2$ has no interpolant. If ______ **4** then
[1] $(S_1 \uplus (S_2 \cup \{Z\})$ has no interpolant. It follows that ______ **5** $\in \mathcal{C}$.

(4) Suppose $\alpha \in S$. If $\alpha \in S_1$ then $(S_1 \cup \{\alpha_1, \alpha_2\}) \uplus S_2$ has no interpolant. If $\alpha \in S_2$ then $(S_1 \uplus (S_2 \cup \{\alpha, \alpha_2\})$ has no interpolant. It follows that $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$.

[2] (5) Suppose $\beta \in S$. For a proof by contradiction, further assume that neither ________ **6** nor

[1] ________ **7** belongs to $\mathcal{C}$. We distinguish two cases.

- [2], [1] If $\beta \in S_1$ then $\langle S_1 \rangle \equiv$ ________ **8**. Since ________ **9** is a partition of $S \cup \{\beta_1\}$, it
[1] must have an interpolant, say γ_1. Since ________ **10** is a partition of $S \cup \{\beta_2\}$, it must have an interpolant, say γ_2. We have

$$\langle S_1 \cup \{\beta_1\} \rangle \supset \gamma_1 \qquad \gamma_1 \supset \neg \langle S_2 \rangle$$

$$\langle S_1 \cup \{\beta_2\} \rangle \supset \gamma_2 \qquad \gamma_2 \supset \neg \langle S_2 \rangle$$

[2], [2] Hence $\langle S_1 \rangle$ ________ **11** $\supset \neg \langle S_2 \rangle$. It follows that ________ **12** an interpolant of $S_1 \uplus S_2$, contradicting the assumption.

- [6] If $\beta \in S_2$ then ________ **13**, contradicting the assumption.

It follows that $\mathcal{C}$ is a propositional consistency property.

[10] (e) Use the statement in the previous item to prove that every tautology $X \supset Y$ has an interpolant.

3 This exercise is about Herbrand's theorem for first-order logic.

[5] (a) Define the Herbrand expansion of a sentence X over a non-empty set $D = \{t_1, \ldots, t_n\}$ of closed terms.

[5] (b) What is a validity functional form? Define the Herbrand expansion of a sentence X.

[5] (c) State Herbrand's theorem.

[10] (d) Use Herbrand's theorem to determine the validity of the sentence $(\forall x)(\exists y)[(\forall z)R(x,z) \supset R(x,y)]$.

[15] **4** Determine whether the following statements are true or false. Every correct answer is worth 3 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

(a) The set of all propositional formulas of rank 0 is a Hintikka set.

(b) The atomic formula $P(x,y)$ occurs positively in $\neg(\forall x)[Q(x) \supset \neg(\exists y)(\neg P(x,y) \supset R(y))]$.

(c) The propositional formula $\neg\neg p \supset p$ is valid in intuitionistic logic (i.e., $\Vdash \neg\neg p \supset p$ holds).

(d) The rule

$$\frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}}$$

is a correct calculation rule for interpolants. Here $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$, c is a constant which does not occur in $\{X_1, \ldots, X_n\}$ / $\{Y_1, \ldots, Y_k\}$, and x is a fresh variable.

(e) The tableau

$$\neg(A \vee \neg A)$$
$$\neg A$$
$$\neg\neg A$$
$$A \qquad\qquad \neg A$$

is cut-free.