universität innsbruck

Computational Logic	SS 2017	LVA 703607

EXAM 2 June 25, 2018

1 *answer*

formula	$\alpha/\beta/\gamma/\delta$	rank	valid
$\neg\top \supset \bot$	β	2	✓
$\neg[(A \supset B) \wedge (C \uparrow (\neg A \wedge B))]$	β	4	×
$(\forall x)[(\exists y)R(f(x,y),c) \supset (\exists z)S(y,z)]$	γ	4	×
$(\forall x)[P(x) \vee Q(x)] \supset [(\exists x)P(x) \vee (\forall y)Q(y)]$	β	6	✓

2 (a) *definition*

An interpolant for a propositional implication $X \supset Y$ is a propositional formula Z such that

(1) every propositional letter of Z occurs in both X and Y, and

(2) both $X \supset Z$ and $Z \supset Y$ are tautologies.

(b) *computation*

Let $X = (\neg(A \wedge B) \supset (\neg C \wedge B)) \supset ((D \supset A) \vee (D \supset \neg C))$. Starting from a closed biased tableau for $\neg X$ (top left), we execute a few steps using the calculation rules for interpolants:

$$\begin{array}{c}
L(\neg(A \wedge B) \supset (\neg C \wedge B)) \\
R(\neg((D \supset A) \vee (D \supset \neg C))) \\
R(\neg(D \supset A)) \\
R(\neg(D \supset \neg C)) \\
R(D) \\
R(\neg A) \\
R(D) \\
R(\neg\neg C) \\
R(C) \\
\begin{array}{cc}
L(\neg\neg(A \wedge B)) & L(\neg C \wedge B) \\
L(A \wedge B) & L(\neg C) \\
L(A) & L(B) \\
L(B) &
\end{array}
\end{array}
\qquad
\begin{array}{c}
L(\neg(A \wedge B) \supset (\neg C \wedge B)) \\
R(\neg((D \supset A) \vee (D \supset \neg C))) \\
R(\neg(D \supset A)) \\
R(\neg(D \supset \neg C)) \\
R(D) \\
R(\neg A) \\
R(D) \\
R(\neg\neg C) \\
R(C) \\
\begin{array}{cc}
L(\neg\neg(A \wedge B)) & L(\neg C \wedge B) \\
L(A \wedge B) & L(\neg C) \\
L(A) & L(B) \\
L(B) & [\neg C] \\
[A] &
\end{array}
\end{array}$$

$$\begin{array}{c}
L(\neg(A \wedge B) \supset (\neg C \wedge B)) \\
R(\neg((D \supset A) \vee (D \supset \neg C))) \\
R(\neg(D \supset A)) \\
R(\neg(D \supset \neg C)) \\
R(D) \\
R(\neg A) \\
R(D) \\
R(\neg\neg C) \\
R(C) \\
\begin{array}{cc}
L(\neg\neg(A \wedge B)) & L(\neg C \wedge B) \\
L(A \wedge B) & [\neg C] \\
[A] &
\end{array}
\end{array}
\qquad
\begin{array}{c}
L(\neg(A \wedge B) \supset (\neg C \wedge B)) \\
R(\neg((D \supset A) \vee (D \supset \neg C))) \\
R(\neg(D \supset A)) \\
R(\neg(D \supset \neg C)) \\
R(D) \\
R(\neg A) \\
R(D) \\
R(\neg\neg C) \\
R(C) \\
\begin{array}{cc}
L(\neg\neg(A \wedge B)) & L(\neg C \wedge B) \\
[A] & [\neg C]
\end{array}
\end{array}$$

$$\begin{array}{c}
L(\neg(A \wedge B) \supset (\neg C \wedge B)) \\
R(\neg((D \supset A) \vee (D \supset \neg C))) \\
R(\neg(D \supset A)) \\
R(\neg(D \supset \neg C)) \\
R(D) \\
R(\neg A) \\
R(D) \\
R(\neg\neg C) \\
R(C) \\
[A \vee \neg C]
\end{array}$$

At this point, $[A \vee \neg C]$ is simply propagated to the root of the tableau, and thus $A \vee \neg C$ is an interpolant of X.

(c) *definition*

A finite set S of (propositional) formulas is Craig consistent if $\langle S_1 \rangle \supset \neg\langle S_2 \rangle$ has no interpolant for some partition $S_1 \uplus S_2$ of S. Here $\langle S \rangle$ denotes the (generalized) conjuction of all formulas in S.

(d) *answers*

1	$\langle S_1 \rangle \supset \neg\langle S_2 \rangle$
2	A
3	$\neg A$
4	$\neg\neg Z \in S_2$
5	$S \cup \{Z\}$
6	$S \cup \{\beta_1\}$
7	$S \cup \{\beta_2\}$
8	$\langle S_1 \cup \{\beta_1\} \rangle \vee \langle S_1 \cup \{\beta_2\} \rangle$
9	$(S_1 \cup \{\beta_1\}) \uplus S_2$
10	$(S_1 \cup \{\beta_2\}) \uplus S_2$
11	$\equiv \langle S_1 \cup \{\beta_1\} \rangle \vee \langle S_1 \cup \{\beta_2\} \rangle \supset \gamma_1 \vee \gamma_2$
12	$\gamma_1 \vee \gamma_2$

answers

13 $\cdots \neg\langle S_2\rangle \equiv \neg\langle S_2 \cup \{\beta_1\}\rangle \wedge \neg\langle S_2 \cup \{\beta_2\}\rangle$. Since $S_1 \uplus (S_2 \cup \{\beta_1\})$ is a partition of $S \cup \{\beta_1\}$, it must have an interpolant, say δ_1. Since $S_1 \uplus (S_2 \cup \{\beta_2\})$ is a partition of $S \cup \{\beta_2\}$, it must have an interpolant, say δ_2. We have

$$\langle S_1\rangle \supset \delta_1 \qquad \delta_1 \supset \neg\langle S_2 \cup \{\beta_1\}\rangle$$
$$\langle S_1\rangle \supset \delta_2 \qquad \delta_2 \supset \neg\langle S_2 \cup \{\beta_2\}\rangle$$

Hence $\langle S_1\rangle \supset \delta_1 \wedge \delta_2 \supset \neg\langle S_2 \cup \{\beta_1\}\rangle \wedge \neg\langle S_2 \cup \{\beta_2\}\rangle \equiv \neg\langle S_2\rangle$. It follows that $\delta_1 \wedge \delta_2$ is an interpolant of $S_1 \uplus S_2$

(e) *proof*

For a proof by contradiction, suppose $X \supset Y$ has no interpolant. Let $S = \{X, \neg Y\}$ with partition $S_1 = \{X\}$ and $S_2 = \{\neg Y\}$. Any interpolant for $\langle S_1\rangle \supset \neg\langle S_2\rangle$ is an interpolant for $X \supset Y$, and thus does not exist. It follows that S is Craig consistent. Using the statement in item (d) and the Model Existence Theorem, it follows that $X \supset Y$ is no tautology.

3 (a) *definition*

The Herbrand expansion $\mathcal{E}(X, D)$ is defined recursively:

- if L is literal then $\mathcal{E}(L, D) = L$,
- $\mathcal{E}(\neg\neg Z, D) = \mathcal{E}(Z, D)$,
- $\mathcal{E}(\alpha, D) = \mathcal{E}(\alpha_1, D) \wedge \mathcal{E}(\alpha_2, D)$,
- $\mathcal{E}(\beta, D) = \mathcal{E}(\beta_1, D) \vee \mathcal{E}(\beta_2, D)$,
- $\mathcal{E}(\gamma, D) = \mathcal{E}(\gamma(t_1), D) \wedge \cdots \wedge \mathcal{E}(\gamma(t_n), D)$,
- $\mathcal{E}(\delta, D) = \mathcal{E}(\delta(t_1), D) \vee \cdots \vee \mathcal{E}(\delta(t_n), D)$.

(b) *definitions*

A validity functional form of a first-order sentence X is any sentence X' with the property that $\neg X'$ is a Skolemized version of $\neg X$.

A Herbrand expansion of X is a Herbrand expansion of Y over D (cf. item (a)), where Y is a validity functional form of X and D is any Herbrand domain for Y. (A Herbrand domain is any finite non-empty subset of the Herbrand universe, which is the set of all closed terms constructed from the functions symbols and constants in Y; if Y lacks constants, a new constant is added.)

(c) *statement*

Herbrand's theorem states that a first-order sentence X is valid if and only if some Herbrand expansion of X is a propositional tautology.

(d) *explanation*

First we transform the given sentence $X = (\forall x)(\exists y)[(\forall z)R(x,z) \supset R(x,y)]$ into a validity functional form Y. This is done by skolemizing its negation $\neg X$, which produces $\neg((\exists y)[(\forall z)R(c,z) \supset R(c,y)])$, and thus $Y = (\exists y)[(\forall z)R(c,z) \supset R(c,y)]$. The Herbrand universe consists of the single element c, and thus $D = \{c\}$ is the only Herbrand domain. Next we compute the Herbrand expansion of Y over D:

$$\begin{aligned}
\mathcal{E}(Y,D) &= \mathcal{E}((\forall z)R(c,z) \supset R(c,c), D) \\
&= \mathcal{E}(\neg(\forall z)R(c,z), D) \vee \mathcal{E}(R(c,c), D) \\
&= \mathcal{E}(\neg R(c,c), D) \vee R(c,c) \\
&= \neg R(c,c) \vee R(c,c)
\end{aligned}$$

Since $\neg R(c,c) \vee R(c,c)$ is a tautology, it follows from Herbrand's theorem that X is valid.

4

statement	true	false
(a)	☐	☒
(b)	☒	☐
(c)	☐	☒
(d)	☐	☒
(e)	☐	☒