Exercise 1. *(3.6.6. Bonus) The formulation (no common variables) makes one think immediately of Craig's Interpolation Theorem.*

By Craig's Interpolation theorem, $X \supset Y$ entails there is an interpolant of X and Y, that is, a formula Z such that $X \supset Z$ and $Z \supset Y$ with the propositional letters of Z contained in both X and Y. Thus, if X and Y have no propositional letters in common, then Z is a formula without propositional letters, hence it is either true or false. In the former case $Z \supset Y$ entails Y is a tautology,, whereas in the latter case $X \supset Z$ entails $\neg X$ is a tautology.

Exercise 2. *(3.6.7. Bonus) The formulation (compatibility) makes one think of closure operators (with which the set is compatible).*

W.l.o.g. we may assume $\mathcal{C}$ to be of finite character:

- *If the Propositional Consistency Property $\mathcal{C}$ is compatible with B, then its subset closure $\mathcal{C}'$ (see Exercise 3.6.1) is also compatible with B: If $S' \in \mathcal{C}'$ and $X \in B$, then $S' \subseteq S \in \mathcal{C}$ for some S, hence by assumption $S \cup \{X\} \in \mathcal{C}$. Hence $S' \cup \{X\} \in \mathcal{C}'$ by it being a subset of $S \cup \{X\}$.*

- *If the subset-closed Propositional Consistency Property $\mathcal{C}'$ is compatible with B, then its finite-character closure $\mathcal{C}''$ (see Exercise 3.6.3) is also compatible with B: If $S'' \in \mathcal{C}''$ and $X \in B$, then every finite subset S' of S'' is in $\mathcal{C}'$, and by assumption also $S' \cup \{X\} \in \mathcal{C}'$. From those being all finite subsets of $S'' \cup \{X\}$, we conclude that the latter is in $\mathcal{C}''$.*

Moreover by $\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{C}''$, if $S \in \mathcal{C}$ then $S \in \mathcal{C}'$ and $S \in \mathcal{C}''$, and it suffices to show that $S \cup B \in \mathcal{C}''$, i.e. that every finite subset of $S \cup B$ is in $\mathcal{C}'$. Such a subset can be written as $T \cup A$ with $T \subseteq S$ and $A \subseteq B$. Since $S \in \mathcal{C}'$ and $\mathcal{C}'$ is subset closed, $T \in \mathcal{C}'$. Since $\mathcal{C}'$ is compatible with B and $A \subseteq B$, an easy induction proof, by induction on the (finite!) cardinality of $A \subseteq B$, shows that then $T \cup A \in \mathcal{C}'$, from which we conclude.

Exercise 3. *(3.7.1) This follows from general fixed-point theory. That is, we proceed by a saturation process that is the same as for obtaining the transitive closure of a relation, i.e. the least relation extending a given one that is transitive, or the language generated by a grammar, i.e. the least language closed under the production rules of a grammar, namely closing under the generating rules/clauses.*

We spell out the details, first inductively defining the sets S^n of 'formulas obtained by a construction of depth n', then defining S^u of the (infinite) union of all the (finite S^n), and next showing that it meets the specification, i.e. that it extends *S, is* closed under the production rules of the grammar*, and that S^u is the* least *set having both properties.*

Define $S^0 = S$ and S^{n+1} is S^n to which each formula in the conclusion of one of the clauses is adjoined, if the formulas in its assumption(s) are in S^n. Let $S^u = \bigcup_n S^n$. The claim is that S^u fits the bill, i.e. is the least upward closed extension of S.

- *Since $S = S^0 \subseteq S^u$, so S^u is an* extension *of S;*

- *We next show S^u is* upward closed*. If formulas $X_1, \dots, X_k \in S^u$, then per construction, there is an n such that $X_1, \dots, X_k \in S^n$. Hence if these are in the assumption of one the clauses, then the conclusion of that clause is in S^{n+1}, hence in S^u again;*

- *Finally, we show S^u is* least *among the upward closed extensions of S. This follows from that for every n, $S^n \subseteq U$ if U is an arbitrary upward closed extension of S, by induction on n: That $S^0 \subseteq U$ follows from the latter being an extension of S. That $S^{n+1} \subseteq U$ if $S^n \subseteq U$ follows by U being closed under the clauses.*

Remark 1. *For people interested in more on this: fixed-point theory, lattice theory (Knaster–Tarski), and closure operators provide an abstract setting for the above. The above works since the clauses are* positive*, only expressing that* if *some elements are in the set* also *some other element must be in it; for* negative *clauses expressing that then some element* cannot *be in it, closures need not exist.*

That such an implicitly defined set (the least set such that ...) exists uniquely, is non-obvious in general; let X be the least set of non-zero real numbers such that $x \in X$ iff $x \notin X$. Intuitively, X is not well-defined due to the negativity *condition; it expresses that if some element is in the set, some other is* not*; contrast this to the* positive *conditions in our case.*

Several people only argued for existence *of* some *upward closed set S^u extending S. This is not enough, since for example the set of* all *formulas is also an upward closed set extending S, but typically distinct from S^u, so not the* least *upward closed set extending S.*

Exercise 4. *(3.7.2.(1,2)) (1) and (2) are defining properties of closure operators (see the above remark), known as* idempotence *and* monotonicity[1] *Again, both follow by general considerations, but we spell out the details.*

- *Idempotence follows by definition of S^u being the least upward closed extension of S, hence the* least *upward closed* extension *$(S^u)^u$ of S^u is just itself;*

- *Monotonicity follows since if $S_1 \subseteq S_2$ then S_2^u extends S_1 and is upward closed, hence $S_1^u \subseteq S_2^u$ by definition of S_1^u as the* least *set having both properties.*

Note that this proof is abstract *in the sense that it does not refer to the* concrete *conditions in Exercise 3.7.1, and only makes use of 'being a least extension that is closed under u'.*

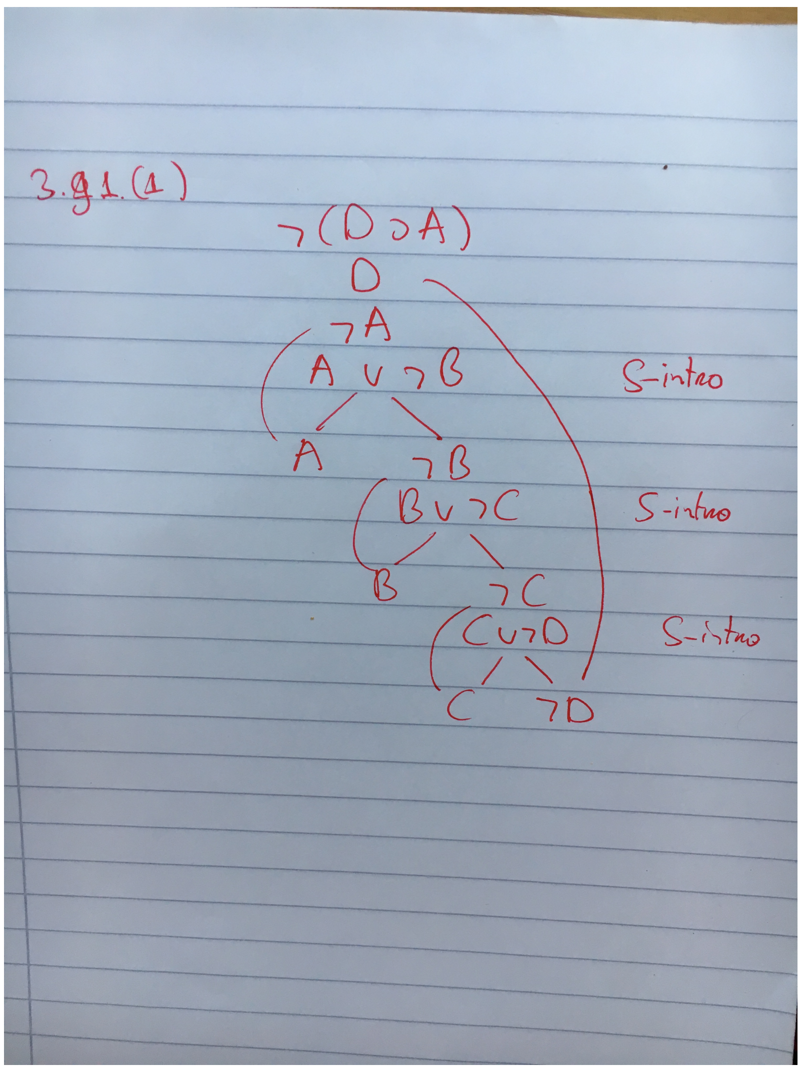
Exercise 5. *(3.7.4. Bonus) Sketch: Replay Theorem 3.7.3 and its proof with 'strict' inserted at appropriate places, defining a finite set S of propositional*

[1]The third defining property of closure operators is being *extensive*, i.e. $S \subseteq S^u$, which trivially holds.

formulas is strict *tableau consistent if there is no* strict *closed tableau for S, replacing the appeal to the Model Existence Theorem 3.6.2 by an appeal to its strict variant as in Exercise 3.7.3, and replacing the (implicit) appeal to Lemma 3.7.2, by an appeal to its strict variant. The reasoning in its proof refines that as given on page 64, now checking for strictness (non-reuse).*

Exercise 6. *(3.9.1)*

1. *Instead of proving this semantically (which is also easy), we use Theorem 3.9.4 to prove it syntactically, using a tableau.*



2. *This one we choose to prove semantically, by induction on n. If $n = 1$, then it trivial. Otherwise, $n > 1$, and we know by the IH that $S \vDash_p A_1 \supset A_{n-1}$. Since $A_{n-1} \supset A_n \in S$, we conclude to $S \vDash_p A_1 \supset A_n$ by transitivity of implication, which is easily checked by a truth table or by other means.*

Exercise 7. *(3.9.2) Left to you.*

Exercise 8. *(3.9.3) By definition of $\langle \ldots \rangle$, the former is equivalent to $(A_1 \wedge \ldots \wedge A_n) \supset X$ and by Theorem 3.9.4 and the deduction theorem (n times), the latter holds iff $A_1 \supset \ldots \supset A_n \supset X$ is a tautology. We conclude since $(X \wedge Y) \supset Z$ is equivalent to $X \supset Y \supset Z$ (n times again). Formally, to prove this needs induction on n, with the case $n = 2$ being the only interesting case.*

Exercise 9. *(4.1.1)*
A proof of $\{P \supset (Q \supset R), Q, P\} \vdash_{ph} R$ is:

1. $P \supset (Q \supset R)$ *(ass)*
2. Q *(ass)*
3. P *(ass)*
4. $Q \supset R$ *(MP 1,2)*
5. R *(MP 4,2)*

One application of the deduction theorem turns this (with some optimizations) into a proof of $\{P \supset (Q \supset R), Q\} \vdash_{ph} (P \supset R)$:

1. $P \supset (Q \supset R)$ *(ass)*
2. Q *(ass)*
3. $Q \supset (P \supset Q)$ *(K)*
4. $P \supset Q$ *(MP 3,2)*
5. $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$ *(S)*
6. $P \supset R$ *(MP 4,2)*

where K and S are names for Axiom Schemes 1 and 2, respectively; the idea is to make all proof steps 'parametrised over P' and in order to achieve that Ks and Ss are inserted appropriately.

Repeating this for the other two assumptions yields the desired proof. Since it is longish and writing all the formulas is tedious, and since the point of the exercise was to show the proof is *long (it has 19 steps compared to the 2, not counting assumptions, we started with; see the remark below), I only give the justifications.*

1. *(S)*

2. *(S)*
3. *(K)*
4. *(S)*
5. *(MP* 3, 4*)*
6. *(MP* 2, 5*)*
7. *(S)*
8. *(K)*
9. *(K)*
10. *(MP* 8, 9*)*
11. *(MP* 7, 10*)*
12. *(S)*
13. *(MP* 11, 12*)*
14. *(MP* 6, 13*)*
15. *(MP* 1, 14*)*
16. *(K)*
17. *(K)*
18. *(MP* 16, 17*)*
19. *(MP* 15, 18*)*

Remark 2. *The order of the assumptions in formulas is extremely important for the size of the proof. Swapping the final occurrences of P and Q in the formula, yields intuitively 'the same' conclusion namely that we can conclude R from P and Q, but it has a much simpler proof, namely that of the Example on page 80.*

In general, as already stated on the slides, repeating the construction in the deduction theorem n times may lead of a proof of size exponential in n. Hence this is not suitable to do by hand. But the important point is that it is a construction, *i.e. it can be* implemented, *which i.m.o. is a not-so-difficult and rewarding exercise.*

One think of K as erasing, *and of S as* distributing; *we will come back to this in the lecture(s) on the* Curry–Howard *isomorphism, in particular S and K can be viewed as* combinators *in* combinatory logic.

Exercise 10. *(4.1.2)*

Looking at the formula, we see that it's an implication with lhs the disjunction of $\neg\neg X$ and X, both of which imply X. This suggests to use Axiom Scheme 9 *'disjunction elimination', and indeed that works:*

1. $(\neg\neg X \supset X) \supset ((\bot \supset X) \supset ((\neg X \supset \bot) \supset X))$ *(Ax* 9*)*

2. $(\neg\neg X \supset X) \supset X$ *(Ax* 5*)*

3. $(\bot \supset X) \supset ((\neg X \supset \bot) \supset X)$ *(MP* 1, 2*)*

4. $\bot \supset X$ *(Ax* 3*)*

5. $(\neg X \supset \bot) \supset X$ *(MP* 3, 4*)*

Exercise 11. *(4.1.4) Left to you.*

Exercise 12. *(4.1.5)*

A standard trick to conclude Y *from some Axioms Scheme of shape* $\ldots(X \supset Y)$ *is to try to instantiate* X *with some appropriate known or simple tautology. In this case the hint suggests to use Axiom Scheme* 8 *with tautology* $\neg Z \supset \neg Z$*:*

1. $(\neg\neg Z \supset (Z \supset \neg\neg Z)) \supset (\neg Z \supset (Z \supset \neg\neg Z)) \supset ((\neg Z \supset \neg Z) \supset (Z \supset \neg\neg Z))$ *(*8*)*

2. $\neg\neg Z \supset (Z \supset \neg\neg Z)$ *(K)*

3. $(\neg Z \supset (Z \supset \neg\neg Z)) \supset ((\neg Z \supset \neg Z) \supset (Z \supset \neg\neg Z))$ *(MP* 1, 2*)*

4. $Z \supset (\neg Z \supset \neg\neg Z)$ *(Ax* 6*)*

5. $(Z \supset (\neg Z \supset \neg\neg Z)) \supset (\neg Z \supset (Z \supset \neg\neg Z))$ *(Example p.81)*

6. $\neg Z \supset (Z \supset \neg\neg Z)$ *(MP* 5, 4*)*

7. $(\neg Z \supset \neg Z) \supset (Z \supset \neg\neg Z)$ *(MP* 3, 6*)*

8. $(\neg Z \supset \neg Z)$ *(Example p.80)*

9. $(Z \supset \neg\neg Z)$ *(MP* 7, 8*)*

This is of course not a real proofs, since it makes use of tautologies derived before, but it could easily be expanded to a real proof.

Exercise 13. *(4.1.6)*

Exercise 14. *(4.1.7) To verify that the collection of all* X*-Hilbert consistent sets, i.e. sets* S *of formulas such that not* $S \vdash_{ph} X$*, is a propositional consistency property, we verify the* 5 *conditions of the latter as given in Definition 3.6.1 hold for the collection, in as far as not already given in the book. We proceed as on page 83, by proving contrapositives.*

1. *Suppose for some propositional letter* A*, both* $A \in S$ *and* $\neg A \in S$*; The by Axiom Scheme 6 and twice MP we infer* X*;*

2. *The case* $\bot$ *is analogous to the case* $\neg\top$ *in the book, but even simpler: we can use Axiom Scheme 3 and MP to infer* X*;*

3. *Suppose $S \cup \{Z\}$ is X-Hilbert-inconsistent. We show $S \cup \{\neg\neg Z\}$ is X-Hilbert inconsistent: We are given that $S \cup \{Z\} \vdash_{ph} X$, so by the deduction theorem $S \vdash_{ph} Z \supset X$. By Axiom Sxheme 5 we have $\neg\neg Z \supset Z$. From that we obtain $S \vdash_{ph} \neg\neg Z \supset X$ by transitivity $(\neg\neg Z \supset Z) \supset (Z \supset X) \supset (\neg\neg Z \supset X)$ and twice MP, from which we conclude, by deduction theorem again, to $S \vdash \{\neg\neg Z\} \vdash_{ph} X$ as desired. Note that transitivity $(X \supset Y) \supset (Y \supset Z) \supset (X \supset Z)$ follows from $\{X \supset Y, Y \supset Z, X\} \vdash_{ph} Z$, whose proof is just two MPs, by three applications of the deduction theorem.*

4. *Suppose $S \cup \{\alpha_1, \alpha_2\}$ is X-Hilbert-inconsistent. We show $S \cup \{\alpha\}$ is X-Hilbert inconsistent. The reasoning is analogous to the above, first obtaining $S \vdash_{ph} \alpha_1 \supset (\alpha_2 \supset X)$, and then combining that and Axiom Schemes 7 $\alpha \supset \alpha_1$ and 8 $\alpha \supset \alpha_2$ using 'binary' transitivity $(\alpha \supset \alpha_1) \supset (\alpha \supset \alpha_2) \rightarrow (\alpha_1 \supset \alpha_2 \supset X) \supset (\alpha \supset X)$, where the latter is obtained from $\{\alpha \supset \alpha_1, \alpha \supset \alpha_2, \alpha_1 \supset \alpha_2 \supset X, \alpha\} \vdash_{ph} X$ similar to transitivity in the previous item.*

5. *The proof for β-clauses is in the book.*

Remark 3. *Note the crucial role the deduction theorem plays in 'shifting' formulas 'in' and 'out' of the set of assumptions.*

Exercise 15. *(4.1.8) Too hard, so left to you. (But will come back to it later.)*

Exercise 16. *(4.5.2) The main point of this exercise is that the tableau is 'surprisingly' large, compared to the simplicity of the pigeon hole principle itself.*